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CONVEX BODIES WITH SIMILAR PROJECTIONS

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ABSTRACT. By examining an example constructed by Petty and McKinney, we show that there are pairs of centered and coaxial bodies of revolution in $\mathbb{E}^d$, $d \geq 3$, whose projections onto each two-dimensional subspace are similar, but which are not themselves even affinely equivalent.

1. INTRODUCTION

In [H], Hadwiger proved the following theorem: If K_1 and K_2 are convex bodies in $\mathbb{E}^d$, $2 \leq k \leq d-1$, and the projections of K_1 and K_2 onto each k-dimensional subspace are directly homothetic, then K_1 and K_2 must also be directly homothetic. (The case $d = 3$ was first published by Süss in [S] and Nakajima in [N].) Later, in [R], Rogers showed that the result remains true when projections are replaced by sections through some common interior point of K_1 and K_2. The two theorems raised questions which led to significant developments in the study of projections and sections of convex bodies. The result on sections began a string of papers on the so-called False Center Conjecture, culminating in the powerful theorem of Burton and Mani in [BM]. Also, Petty and McKinney [PM] found an example to show that certain generalizations of the two theorems in [R] are not possible.

The Petty-McKinney example demonstrates that the hypotheses in the above theorems that projections (or sections) are directly homothetic cannot be replaced by the assumption of similarity. There is certainly one clear difference between direct homothety and similarity with respect to projections; projections of directly homothetic convex bodies are directly homothetic, while a simplex and a rotation of it will generally not have similar projections. Nevertheless, the Petty-McKinney example is extremely surprising and deserves to be better known. In [PM] it is shown that there are pairs K_1, K_2 of centered (centrally symmetric with center at the origin) convex bodies in $\mathbb{E}^d$, $d \geq 3$, such that for each two-dimensional subspace S the projection $K_1|S$ of K_1 onto S is directly homothetic to a rotation of $K_2|S$ by $\pi/2$ about the origin, yet K_1 and

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K_2 are not directly homothetic. Moreover, Theorem 3.1 of [PM] characterizes all pairs K_1, K_2 with these properties. Corresponding examples for sections instead of projections follow immediately using polar duality.

It is worth noting that the Petty-McKinney example also serves to provide contrast to the famous uniqueness theorems of Alexandrov and Funk. In [A] and [F] it is proved that if $1 \leq k \leq d-1$ and two centered convex bodies in $\mathbb{E}^d$ are such that their projections onto (or sections by, respectively) each k-dimensional subspace have the same k-dimensional volume, then they must be equal. In particular, they must be equal if their projections or sections are congruent.

Like Roger's theorem, the Petty-McKinney example also raises some natural questions, and it is the purpose of this note to answer some of these. We show that although it is possible for pairs K_1, K_2 in the Petty-McKinney example to be similar (but not directly homothetic), there are pairs which are not even affinely equivalent. We also prove that such pairs are affinely equivalent if and only if they are similar and characterize when this can occur. It follows that direct homothety in Hadwiger's theorem cannot be replaced throughout by either similarity or affine equivalence. Again, polar duality yields the corresponding results for sections.

2. Definitions and preliminaries

We denote d-dimensional Euclidean space by $\mathbb{E}^d$ and its unit sphere and origin by S^{d-1} and $\underline{o}$, respectively. If S is a subspace, then $E|S$ is the orthogonal projection of the set E onto S.

A *convex body* is a compact convex set with nonempty interior. We say a convex body is *centered* if it is centrally symmetric, with center $\underline{o}$. If K is a convex body, we write h_K for its *support function* (see, for example, [BF, §15]). Suppose A is a nonsingular affine transformation of $\mathbb{E}^d$ with the transpose denoted by A^T. Then it follows easily from the definition of h_K that

$$h_{AK}(x) = h_K(A^T x) ,$$

for all $x \in \mathbb{E}^d$.

Consider pairs K_1, K_2 of convex bodies defined as follows. The support function h_{K_1} of K_1 is defined for nonzero $x \in \mathbb{E}^d$ by

$$h_{K_1}(x) = \|x\| \exp\left(\frac{x^T C x}{\|x\|^2}\right) ,$$

where C is any real symmetric matrix of order d, with eigenvalues $c_1, \dots, c_d$ satisfying the condition $\max|c_i - c_j| \leq \frac{1}{2}$. The support function h_{K_2} of K_2 is defined similarly, where the matrix C is replaced by $-C$ (whose eigenvalues satisfy the same condition). The authors of [PM] show that these are precisely the pairs of centered convex bodies in $\mathbb{E}^d$, $d \geq 3$, such that the projection of one onto each two-dimensional subspace is directly homothetic to a rotation by $\pi/2$ about the origin of the projection onto the same subspace of the other.

3. Results

Theorem. *The convex bodies K_1, K_2 of the Petty-McKinney example are affinely equivalent if and only if they are similar, and this occurs if and only if there is a* *constant* a *such that the eigenvalues* c_i *of the matrix* C, *arranged so that they increase with* i, *satisfy*

$$c_i + c_{d+1-i} = a\,, \tag{1}$$

for $i = 1, \ldots, d$.

Proof. Let K_1, K_2 be a pair of convex bodies in $\mathbb{E}^d$, $d \geq 3$, with support functions defined as in §2. By applying an orthogonal transformation, if necessary, we may assume C to be a diagonal matrix such that $c_1 \leq c_2 \leq \cdots \leq c_d$.

Suppose $AK_1 = K_2$, where A is a nonsingular affine map. We shall prove that K_1 and K_2 are similar and (1) holds. Since K_i is centered, $i = 1, 2$, A must actually be a linear map. For, let $[x, -x]$ be a chord of K_1 containing $\underline{o}$, and therefore bisected by $\underline{o}$. Then $[Ax, -Ax]$ is a chord of K_2 which is bisected by $A\underline{o}$, so K_2 is centrally symmetric about $A\underline{o}$. But $\underline{o}$ is the center of K_2, so $A\underline{o} = \underline{o}$.

We have $h_{K_2}(x) = h_{AK_1}(x) = h_{K_1}(A^T x)$, from which we obtain

$$\|A^T u\|^2 = \exp\left(-2\sum_{i=1}^{d} \frac{c_i (A^T u)_i^2}{\|A^T u\|^2} - 2\sum_{i=1}^{d} c_i u_i^2\right), \tag{2}$$

for all $u \in S^{d-1}$. Let us first set $u_1 = 2z/(1+z^2)$, $u_2 = (1-z^2)/(1+z^2)$, and $u_i = 0$ for $i = 3, \ldots, d$. We claim that both sides of (2) are then constant.

The substitution yields an equation of the form $p(z)/(1+z^2)^2 = e^{f(z)}$, where $p(z)$ is a polynomial of degree at most four. We rewrite this in the form

$$p(z) = (1+z^2)^2 e^{f(z)}\,,$$

which then holds for all real z. Further, $f(z)$ is a rational function whose denominator is nonzero for each real z. Therefore, both sides of the equation represent functions which are analytic in a domain in the complex plane $\mathbb{C}$ containing the real axis. Since $p(z)$ is a polynomial, we may take its domain to be the whole of $\mathbb{C}$, and then a standard uniqueness theorem (see, for example, [C, Theorem 1, p. 261]) implies that the last equation holds for all $z \in \mathbb{C}$. The exponential function has no zeros in $\mathbb{C}$, so the only zeros of the right-hand side are double zeros at $z = \pm i$. These must then be precisely the zeros of the left-hand side, implying that $p(z)$ is a constant multiple of $(1+z^2)^2$ and hence that $e^{f(z)}$ is constant. Therefore, both sides of (2) are constant, under the assumption that $u_i = 0$ for $i = 3, \ldots, d$.

This implies that the first two columns of the matrix A^T are orthogonal and the sum of the squares of the entries in each of these columns is the same.

The same conclusion can now be drawn for any pair of columns by replacing u_1 and u_2 by the appropriate pair of coordinates of u. It follows that A^T is an orthogonal matrix $W = (w_{ij})$ multiplied by a constant, b^{-1} say. (This means that A must be a similarity.) Substituting in (2) and using $\sum_{i=1}^{d} u_i^2 = 1$, we obtain

$$\sum_{i=1}^{d} c_i (Wu)_i^2 + \sum_{i=1}^{d} c_i u_i^2 = \log|b|\,. \tag{3}$$

Comparing coefficients, we see that

$$\sum_{i=1}^{d} c_i w_{ij} w_{ik} = 0 ,$$

while the orthogonality of W yields

$$\sum_{i=1}^{d} w_{ij} w_{ik} = 0 ,$$

whenever $1 \leq j \neq k \leq d$. Let y_j, z_j denote the vectors whose ith coordinates are $c_i w_{ij}$, w_{ij}, respectively. The last two equations imply that both y_j and z_j are orthogonal to z_k for $k \neq j$. The vectors z_k are just the columns of the matrix W, so for each j the z_k's such that $k \neq j$ span a $(d-1)$-dimensional subspace. It follows that $y_j = t_j z_j$ for some real t_j and all j. This means that

$$c_i w_{ij} = t_j w_{ij} , \tag{4}$$

for all i and j.

For each m, $1 \leq m \leq d$, define $I_m = \{i : c_i = c_m\}$, and $J_m = \{j : w_{ij} \neq 0 \text{ for some } i \in I_m\}$. Then if $j \notin J_m$, we have $w_{ij} = 0$ for all $i \in I_m$. Since $J_m \neq \varnothing$, by the orthogonality of W, we can choose a $p \in J_m$. Suppose that $w_{ip} \neq 0$ for some $i \notin I_m$. Then by (4) $c_i w_{ip} = t_p w_{ip}$. Also, there is an $i' \in I_m$ with $w_{i'p} \neq 0$. Using (4) again, $c_{i'} w_{i'p} = t_p w_{i'p}$, which gives $c_i = c_{i'} = c_m$, a contradiction. Therefore, $w_{ip} = 0$ for each $i \notin I_m$. Let e_p denote the unit vector in the pth coordinate direction. Then

$$\sum_{i \notin I_m} (We_p)_i^2 = 0 ,$$

so that

$$\sum_{i \in I_m} (We_p)_i^2 = \sum_{i=1}^{d} (We_p)_i^2 = 1 .$$

Substituting $u = e_p$ in the left-hand side of (3) then yields

$$\sum_{i \in I_m} c_i (We_p)_i^2 + c_p = c_m + c_p .$$

Consequently, for each m, $1 \leq m \leq d$, there is a p with $c_m + c_p = \log|b|$. The fact that the eigenvalues c_i increase with i now forces

$$c_i + c_{d+1-i} = \log|b| ,$$

for $i = 1, \ldots, d$, which means that (1) holds.

Suppose now that (1) is true, where $a = \log|b|$. Again, applying an orthogonal transformation, if necessary, we may assume that the matrix C is diagonal. Then, for $u \in S^{d-1}$,

$$\begin{aligned} h_{K_1}(u) &= \exp\left(\sum_{i=1}^{d} c_i u_i^2\right) = \exp\left(\log|b| - \sum_{i=1}^{d} c_{d+1-i} u_i^2\right) \\ &= |b| \exp\left(-\sum_{i=1}^{d} c_i u_{d+1-i}^2\right) = |b| h_{K_2}(Wu) , \end{aligned}$$

where W is the orthogonal matrix which interchanges the ith and $(d+1-i)$th coordinate axes for $i = 1, \ldots, d$. So K_1 and K_2 are similar. □

Corollary. *For $d \geq 3$, there are centered, coaxial convex bodies of revolution K_1 and K_2 in $\mathbb{E}^d$ with the property that, for each two-dimensional subspace S, $K_1|S$ and $K_2|S$ are similar but K_1 and K_2 are not affinely equivalent.*

Proof. Let the convex bodies K_1 and K_2 be as in the Petty-McKinney example, with C the diagonal matrix with eigenvalues $c_1 = \frac{1}{2}$ and $c_i = 1$ for $i = 2, \ldots, d$. The corollary follows immediately, since equation (1) fails. □

It is easy to see that the convex bodies K_1 and K_2 of the Petty-McKinney example are directly homothetic if and only if the eigenvalues c_i of the matrix C are all equal, that is, precisely when both bodies are centered balls. It is therefore possible for K_1 and K_2 to be similar but not directly homothetic. For example, take $d = 3$ and C to be the diagonal matrix with diagonal entries $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.

4. Further questions

The results above suggest the following natural problems.

Question 4.1. Suppose $2 < k \leq d-1$ and K_1 and K_2 are centered convex bodies in $\mathbb{E}^d$ with $K_1|S$ similar to $K_2|S$, for every k-dimensional subspace S. Is K_1 similar to K_2?

Question 4.2. Suppose $2 \leq k \leq d-1$ and K_1 and K_2 are arbitrary convex bodies in $\mathbb{E}^d$ such that $K_1|S$ is congruent to $K_2|S$, for every k-dimensional subspace S. Is K_1 a translate of $\pm K_2$?

In [G_1] and [G_2], Golubyatnikov proves that the answer to Question 4.2 is positive when $k = 2$ and none of the projections $K_1|S$ and $K_2|S$ has an extra symmetry with respect to rotations. In fact, in [G_2, Theorem 6] it is shown that if K_1 and K_2 are convex bodies in $\mathbb{E}^d$ such that $K_1|S$ is similar to $K_2|S$, for every 2-dimensional subspace S, and none of the projections $K_1|S$ and $K_2|S$ has an extra symmetry, then K_1 is homothetic to $\pm K_2$. This still leaves open the following question.

Question 4.3. Suppose K_1 and K_2 are centered convex bodies in $\mathbb{E}^d$ such that $K_1|S$ is similar to $K_2|S$, for every two-dimensional subspace S. Must K_1 and K_2 be a pair as in the Petty-McKinney example?

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