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Modulation Spaces, Wiener Amalgam Spaces, and Brownian Motions

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Abstract. We study the local-in-time regularity of the Brownian motion with respect to localized variants of modulation spaces $M_s^{p,q}$ and Wiener amalgam spaces $W_s^{p,q}$. We show that the periodic Brownian motion belongs locally in time to $M_s^{p,q}(\mathbb{T})$ and $W_s^{p,q}(\mathbb{T})$ for $(s-1)q < -1$, and the condition on the indices is optimal. Moreover, with the Wiener measure μ on $\mathbb{T}$, we show that $(M_s^{p,q}(\mathbb{T}), \mu)$ and $(W_s^{p,q}(\mathbb{T}), \mu)$ form abstract Wiener spaces for the same range of indices, yielding large deviation estimates. We also establish the endpoint regularity of the periodic Brownian motion with respect to a Besov-type space $\widehat{b}^s_{p,\infty}(\mathbb{T})$. Specifically, we prove that the Brownian motion belongs to $\widehat{b}^s_{p,\infty}(\mathbb{T})$ for $(s-1)p = -1$, and it obeys a large deviation estimate. Finally, we revisit the regularity of Brownian motion on usual local Besov spaces $B^s_{p,q}$, and indicate the endpoint large deviation estimates.

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1. Introduction

Modulation spaces were born during the early eighties in pioneering work of H. Feichtinger. In subsequent fruitful collaborations with K. Gröchenig [17], [18] they established

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the basic theory of these function spaces, in particular their invariance, continuity, embeddings, and convolution properties. In contrast with the Besov spaces, which are defined by a dyadic decomposition of the frequency space, modulation spaces (and Wiener amalgam spaces) arise from a uniform partition of the frequency space. Their appeal is due to the fact that they can effectively capture the time-frequency concentration of a distribution. Both modulation and Wiener amalgam spaces are modeled on Lebesgue or Fourier-Lebesgue spaces, but they are more flexible in that they allow a separate control of the local regularity and the decay at infinity of a function. The central idea in this time-frequency analysis is to start with a function that is smoothly localized by a so-called window function and then take its Fourier transform. The resulting short-time Fourier transform (STFT) is also known, depending on its specific application and up to some normalization, as the Gabor transform, ambiguity function, coherent state transform, or Wigner distribution. Modulation spaces thus capture the joint time-frequency concentration of a function by appropriate decay and integrability conditions on the STFT. They originate in the early foundations of quantum mechanics and information theory. Engineers and physicists have sensed a huge potential here before mathematicians did, and the past years are seeing a resurgence in the applications of time-frequency analysis to signal analysis, image processing, or information theory. Their rapid development was almost in sync with that of wavelet theory. Grossmann and Morlet [23], for example, have realized early on that both the wavelet transform and the STFT are special cases of square integrable representations. Unsurprisingly perhaps, since the STFT is a joint representation of a function in both time and frequency, uncertainty principles can be thought of as appropriate embeddings of modulation spaces; see Galperin and Gröchenig's article [20]. Returning to the original analogy with the Besov spaces and their wavelet bases, modulation spaces do not admit orthonormal bases of time-frequency shifts, but rather so-called frames of time-frequency shifts. Due to Daubechies' insightful work [16], the concept of frame has quickly become crucial in signal analysis. Within pure and applied mathematics, modulation spaces and Wiener amalgam spaces are nowadays present in investigations that concern problems in numerical analysis, operator algebras, localization operators, Fourier multipliers, pseudodifferential operators, Fourier integral operators, non-linear partial differential equations, and so on. The list is extensive, and we cannot hope to acknowledge here all those who made the theory of modulation spaces such a successful story. Simply to give a flavor of some of the recent works in partial differential equations employing these spaces, we mention the contribution of Bényi-Göchenig-Okoudjou-Rogers [2] on unimodular multipliers and the phase-space concentration of the solutions to the free Schrödinger and wave equations, the work on the well-posedness of non-linear (Schrödinger, Ginzburg-Landau, Klein-Gordon, KdV) equations with rough data by Bényi-Okoudjou [3] and Wang et all [52, 53, 54, 55], or the articles of Cordero-Nicola on Strichartz estimates and the Schrödinger equation with quadratic Hamiltonian [14, 15].

This article lies at the interface between time-frequency analysis, probability theory, and partial differential equations (PDEs). Its main aim is to serve as a first bridge between these areas by specifically pointing out the role that the spaces of time-frequency analysis (modulation and Wiener amalgam) can play here. A central question in applied mathematics and theoretical physics is how initial data are propagated by non-linear PDEs, and an important issue is the existence of an invariant measure for the flow. Following Lebowitz, Rose, and Speer [30], Bourgain [4, 5] constructed invariant Gibbs measures for some Hamiltonian PDEs on the one dimensional torus $\mathbb{T}$. Such Gibbs measures can be regarded as weighted Wiener measures on $\mathbb{T}$ (that is, Brownian motions on $\mathbb{T}$), which are supported on function spaces with low regularity. Therefore, in constructing a flow on the support of the Gibbs measure, one is often forced to go beyond the usual Sobolev spaces, and cross over in the more exotic realm of the variants of Fourier-Lebesgue spaces discussed above. For example, in the case of the derivative non-linear Schrödinger equation (DNLS), the flow is not well defined in the classical $H^s(\mathbb{T})$ space for $s < 1/2$, yet almost sure global well-posedness can be established by employing Fourier-Lebesgue spaces $\mathcal{F}L^{s,q}(\mathbb{T})$ for some s and q with $(s-1)q < -1$; see the recent work of Nahmod, Oh, Rey-Bellet, and Staffilani on DNLS [32]. See also the earlier works of Bourgain on mKdV [4], Zakharov system [5], and Oh [34, 37] for the use of Fourier-Lebesgue spaces $\mathcal{F}L^{s,\infty}(\mathbb{T})$.

We will study the Brownian motion, which is arguably the most accessible continuous-time stochastic process yet it plays a central role in both pure and applied mathematics. Its applications range from the study of continuous-time martingales, stochastic calculus to control theory and financial mathematics. It is well known that the integral of a Gaussian white noise is represented through Brownian motion, and white noise is supported on functions of low regularity. Thus, understanding the regularity of Brownian motion is a natural problem. In the context of the classical function spaces of PDEs (Sobolev and Besov), the local regularity of the Brownian motion is well understood; see the works of Ciesielski [11, 12] and Roynette [41]. These results are summarized in Subsection 2.1. An extension to the vector valued setting can be found in Hyntönen and Veraar's paper [24]. Some immediate applications of Brownian motion regularity to stochastic integrals and equations have appeared early on in the work of Ciesielski, Kerkyacharian, and Roynette [13]. Schilling [45] has extended the local and global regularity properties in [13], as well as the ones for sample paths of Feller processes. Interestingly, [45] presents a nice connection to the asymptotic behavior of the symbol of the pseudodifferential operator given in terms of the infinitesimal generator of the Feller process. For a survey of recent advances in stochastic calculus with respect to (fractional) Brownian motion and its connection to Malliavin calculus, see Nualart's article [33]. In this work, we establish the local regularity of the Brownian motion on appropriate modulation spaces and Wiener amalgam spaces, and prove that it obeys so called large deviation estimates. For the "end-point" results we appeal to a Besov-type space introduced by Oh in a series of works that were concerned with the invariance of the white noise for the KdV equation [35] and the stochastic KdV equation with additive space-time white noise [36]. Incidentally, our results also recover the regularity of the white noise by dropping one regularity from the one of the Brownian motion. We also revisit the local regularity of Brownian motion on the usual Besov spaces. A common thread throughout this work is the use of random Fourier series.

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1.1. **Function spaces of time-frequency analysis.** We start by recalling some basic definitions regarding the function spaces used throughout this work; see, for example, Gröchenig's book [21]. Given a (fixed, non-zero) window function $\phi \in \mathcal{S}(\mathbb{R}^d)$, the short-time Fourier transform (STFT) $V_\phi f$ of a tempered distribution u is

$$V_\phi u(x,\xi) = \langle u, M_\xi T_x \phi\rangle = \mathcal{F}(u\overline{T_x\phi})(\xi) = \int_{\mathbb{R}^d} e^{-iy\cdot\xi}\overline{\phi(y-x)}u(y)dy.$$

Here, $\mathcal{F}(g)(\xi) = \widehat{g}(\xi) = \int_{\mathbb{R}^d} g(x)e^{-ix\cdot\xi}dx$ denotes the Fourier transform of a distribution g, while, for $x, \xi \in \mathbb{R}^d$, $M_\xi g(x) = e^{ix\cdot\xi}g(x)$, $T_x g(y) = g(y-x)$ denote the modulation and translation operators, respectively. $\mathcal{F}^{-1}(g) = \widehat{g}$ will denote the inverse Fourier transform of g. For $s \in \mathbb{R}$, we let $\langle\xi\rangle^s = (1+|\xi|^2)^{s/2}$.

The (continuous weighted) *modulation space* $M^{p,q}_s(\mathbb{R}^d), 1 \leq p, q \leq \infty$, consists of all tempered distributions $u \in \mathcal{S}(\mathbb{R}^d)$ such that

$$V_\phi u(x,\xi)\langle\xi\rangle^s \in L^p_x L^q_\xi,$$

and we equip the space $M^{p,q}_s(\mathbb{R}^d)$ with the norm

$$\|u\|_{M^{p,q}_s(\mathbb{R}^d)} = \big\|\|V_\phi u(x,\xi)\langle\xi\rangle^s\|_{L^p_x(\mathbb{R}^d)}\big\|_{L^q_\xi(\mathbb{R}^d)} = \bigg(\int_{\mathbb{R}^d}\langle\xi\rangle^{sq}\bigg(\int_{\mathbb{R}^d}|V_\phi u(x,\xi)|^p\,dx\bigg)^{q/p}d\xi\bigg)^{1/q},$$

with the obvious modifications if $p = \infty$ or $q = \infty$. It is not hard to see that these spaces are Banach spaces and two different windows yield equivalent norms, Moreover, the classical Bessel potential spaces (or the usual L^2-based Sobolev spaces) are particular cases of modulation spaces, $M^{2,2}_s(\mathbb{R}^d) = H^s(\mathbb{R}^d)$, the duality and embeddings are natural $(M^{p,q}_s(\mathbb{R}^d))' = M^{p',q'}_{-s}(\mathbb{R}^d)$, $M^{p_1,q_1}_s(\mathbb{R}^d) \subset M^{p_2,q_2}_s(\mathbb{R}^d)$ for $p_1 \leq p_2$ and $q_1 \leq q_2$, while $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p,q}_s(\mathbb{R}^d)$. The complex interpolation of these spaces is equally natural: $(M^{p_1,q_1}_{s_1}, M^{p_2,q_2}_{s_2})_{[\theta]} = M^{p,q}_s$, where $1/p = (1-\theta)/p_1 + \theta/p_2, 1/q = (1-\theta)/q_1 + \theta/q_2$ and $s = (1-\theta)s_1 + \theta s_2$.

Closely related to these spaces are the (continuous weighted) *Wiener amalgam spaces* $W^{p,q}_s(\mathbb{R}^d)$ which are now equipped with the norm

$$\|u\|_{W^{p,q}_s(\mathbb{R}^d)} = \big\|\|V_\phi u(x,\xi)\langle\xi\rangle^s\|_{L^q_\xi(\mathbb{R}^d)}\big\|_{L^p_x(\mathbb{R}^d)} = \bigg(\int_{\mathbb{R}^d}\bigg(\int_{\mathbb{R}^d}\langle\xi\rangle^{sq}|V_\phi u(x,\xi)|^q\,d\xi\bigg)^{p/q}dx\bigg)^{1/p}.$$

It is clear then that, in fact, $M^{p,q}_0(\mathbb{R}^d) = \mathcal{F}W^{q,p}_0(\mathbb{R}^d)$. As such, completely analogous comments to the one following the definition of modulation spaces hold for Wiener amalgam spaces as well. For example, $W^{p_1,q_1}_s(\mathbb{R}^d) \subset W^{p_2,q_2}_s(\mathbb{R}^d)$ for $p_1 \leq p_2$ and $q_1 \leq q_2$, $\mathcal{S}(\mathbb{R}^d)$ is dense in $W^{p,q}_s(\mathbb{R}^d)$, and so on. Moreover, $M^{p,q}_s(\mathbb{R}^d) \subset W^{p,q}_s(\mathbb{R}^d)$ for $q \leq p$, while the reverse inclusion holds if $p \leq q$.

The (weighted) *Fourier-Lebesgue spaces* $\mathcal{F}L^{s,p}(\mathbb{R}^d)$ are defined via the norm

$$\|u\|_{\mathcal{F}L^{s,p}(\mathbb{R}^d)} = \|\langle\xi\rangle^s\widehat{u}(\xi)\|_{L^p_\xi(\mathbb{R}^d)} = \bigg(\int_{\mathbb{R}^d}\langle\xi\rangle^{sp}|\widehat{u}(\xi)|^p\,d\xi\bigg)^{1/p}. \tag{1.1}$$

For our purposes, we will use the following equivalent definitions for the norms of modulation and Wiener amalgam spaces. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\operatorname{supp}\psi \subset [-1,1]^d$ and $\sum_{n\in\mathbb{Z}^d}\psi(\xi-n) \equiv 1$. Then,

$$\|u\|_{M^{p,q}_s(\mathbb{R}^d)} = \big\|\langle n\rangle^s\|\psi(D-n)u\|_{L^p_x(\mathbb{R}^d)}\big\|_{\ell^q_n(\mathbb{Z}^d)} \tag{1.2}$$

and

$$\|u\|_{W_s^{p,q}(\mathbb{R}^d)} = \big\| \| \langle n \rangle^s \psi(D-n)u\|_{\ell_n^q(\mathbb{Z}^d)} \big\|_{L_x^p(\mathbb{R}^d)}. \tag{1.3}$$

Here, we denoted

$$\psi(D-n)u(x) = \mathcal{F}^{-1}(\psi(\cdot - n)\mathcal{F}u(\cdot))(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \psi(\xi - n)\widehat{u}(\xi)e^{i\cdot\xi}\, d\xi.$$

Thus, the modulation norm in (1.2) can be spelled out, up to a $(2\pi)^{-d}$ factor, as

$$\bigg(\sum_{n\in\mathbb{Z}^d} \bigg(\int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \psi(\xi - n)\widehat{u}(\xi)e^{ix\cdot\xi}\, d\xi \Big|^p dx \bigg)^{q/p} \bigg)^{1/q}.$$

One should contrast these definitions with the one of *Besov spaces*. Let $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\operatorname{supp}\varphi_0 \subset \{|\xi| \le 2\}$, $\operatorname{supp}\varphi \subset \{\frac{1}{2} \le |\xi| \le 2\}$, and $\varphi_0(\xi) + \sum_{j=1}^{\infty} \varphi(2^{-j}\xi) \equiv 1$. With $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, $j \ge 1$, we define the usual Besov space $B^s_{p,q}$ via the norm

$$\|u\|_{B^s_{p,q}(\mathbb{R}^d)} = \big\| 2^{jsq} \|\varphi_j(D)u\|_{L^p(\mathbb{R}^d)} \big\|_{\ell_j^q(\mathbb{N})} = \bigg(\sum_{j\ge 0} 2^{sjq} \|u * \widehat{\varphi}_j\|^q_{L^p(\mathbb{R}^d)} \bigg)^{1/q}. \tag{1.4}$$

Clearly, when $q = \infty$, the norm is modified into $\sup_{j\ge 0} 2^{sj}\|u * \widehat{\varphi}_j\|_{L^p(\mathbb{R}^d)}$. The embeddings between Besov, Sobolev and modulation spaces are well understood. For example, with some explicitly found indices $s_1(n,p,q), s_2(n,p,q)$, we have $B^s_{p,q}(\mathbb{R}^d) \hookrightarrow M^{p,q}(\mathbb{R}^d)$ if and only if $s \ge s_1$ and $M^{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,q}(\mathbb{R}^d)$ if and only if $s \le s_2$; see Sugimoto-Tomita [47], also Okoudjou [38], Toft [49], and Kobayashi-Sugimoto [27].

As stated in the introduction, our goal is to investigate the local-in-time regularity of the Brownian motion. Without loss of generality, we can restrict ourselves to study the periodic Brownian motion; see Subsection 2.2. In Subsection 2.1, we first summarize some of the known regularity results for Sobolev and Besov spaces. Then, in Subsection 2.2, we establish similar results for modulation and Wiener amalgam spaces on the torus. We also observe that the same results can be obtained on appropriately defined localized spaces, and that straightforward modifications allow us to extend them for the Brownian motion on the d-dimensional torus. Moreover, in Section 3, we will do much more by showing a nice connection to abstract Wiener spaces, and, in particular, their large deviation estimates. In Subsection 2.3, we reprove some known results for Besov spaces by working, as for the other spaces of time-frequency analysis, on the Fourier side (compared to previous proofs which were done on the physical side) and using a multilinear analysis argument. From this perspective, the use of Fourier-Wiener series (see (2.2) below) can be viewed as a unifying theme of this article. Since we are only concerned with local regularity of (periodic) Brownian motions, we need a "localized" version of the time-frequency spaces defined above. These localizations were first introduced and studied by Ruzhansky, Sugimoto, Toft, and Tomita in [42]. Naturally, the spaces can be localized either in space (by multiplying an object in the functional space by a smooth cut-off function) or in frequency (by multiplying the Fourier transformation of an element in the functional space by a smooth cut-off). Proposition 2.1 and Remark 4.2 in [42] show that the localized versions, say in space, of the $M^{p,q}_s$, $W^{p,q}_s$ and $\mathcal{F}L^{q,s}$ spaces coincide, with equivalence of norms. In particular, the modulation, Wiener amalgam, and Fourier-Lebesgue spaces on the torus are all the same. Specifically, if for some ψ with compact support in the discrete topology of $\mathbb{Z}^d$, we let

$$\|u\|_{M^{p,q}_s(\mathbb{T}^d)} = \big\| \langle n \rangle^s \|\psi(D-n)u\|_{L^p_x(\mathbb{T}^d)} \big\|_{\ell^q_n(\mathbb{Z}^d)} \tag{1.5}$$

and

$$\|u\|_{W_s^{p,q}(\mathbb{T}^d)} = \big\| \|\langle n\rangle^s \psi(D-n)u\|_{\ell^q_n(\mathbb{Z}^d)} \big\|_{L^p_x(\mathbb{T}^d)}, \tag{1.6}$$

then, for all $1 \le p, q \le \infty$, we have

$$M_s^{p,q}(\mathbb{T}^d) = W_s^{p,q}(\mathbb{T}^d) = \mathcal{F}L^{s,q}(\mathbb{T}^d) = \mathcal{F}\ell^q_s.$$

We will also use the *Fourier-Besov spaces* $\widehat{b}^s_{p,q}(\mathbb{T}^d)$. With the notation φ_j introduced above for the classical Besov spaces, the Fourier-Besov space norm is defined by:

$$\|u\|_{\widehat{b}^s_{p,q}(\mathbb{T}^d)} = \big\| \|\langle n\rangle^s \varphi_j(n)\widehat{u}(n)\|_{L^p(\mathbb{Z}^d)} \big\|_{\ell^q_j(\mathbb{N})} = \bigg(\sum_{j\ge 0} \bigg(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{sp} |\varphi_j(n)\widehat{u}(n)|^p \bigg)^{q/p} \bigg)^{1/q}. \tag{1.7}$$

Clearly, this norm is equivalent to

$$\bigg(\sum_{j\ge 0} \bigg(\sum_{|n|\sim 2^j} 2^{jsp} |\widehat{u}(n)|^p \bigg)^{q/p} \bigg)^{1/q}. \tag{1.8}$$

In particular, in our end-point regularity analysis we will need the space $\widehat{b}^s_{p,\infty}(\mathbb{T}^d)$ defined via

$$\|u\|_{\widehat{b}^s_{p,\infty}(\mathbb{T}^d)} = \sup_{j\ge 0} \|\langle n\rangle^s \varphi_j(n)\widehat{u}(n)\|_{L^p(\mathbb{Z}^d)} \sim \sup_{j\ge 0} \bigg(\sum_{|n|\sim 2^j} 2^{jsp} |\widehat{u}(n)|^p \bigg)^{1/p}, \tag{1.9}$$

where, for a given $n \in \mathbb{Z}^d$, $|n| \sim 2^j$ denoted $2^{j-1} < |n| \le 2^j$. This space is well-suited for the analysis of the white noise, since, in particular, for $sp \le -d$, it contains its full support. For $d = 1$, this analysis of the white noise was initiated by Oh in [35], while the natural extension to any dimension d can be found in Veraar's work [51]. Noting that the white noise is a derivative of the Brownian motion, some of the results in [51] can be easily deduced from the ones presented here; see Subsection 2.2. Lastly, note that Hausdorff-Young inequality immediately implies that $B^s_{p',\infty}(\mathbb{T}^d) \subset \widehat{b}^s_{p,\infty}(\mathbb{T}^d)$ for $p > 2$, and that the equality holds for $p = 2$, where $B^s_{p',\infty}(\mathbb{T}^d)$ is the usual periodic Besov space and $p' = p/(p-1)$ denotes the Hölder conjugate exponent of p. Note also that $\mathcal{F}L^{s,p}(\mathbb{T}^d) = \widehat{b}^s_{p,p}(\mathbb{T}^d)$ for $p \ge 1$.

1.2. **Brownian motion.** Let $T > 0$ and $(\Omega, \mathcal{F}, Pr)$ be a probability space. We define the $\mathbb{R}^n$*-valued Brownian motion* (or Wiener process) as a stochastic process $\beta : [0,T] \times \Omega \to \mathbb{R}^n$ that satisfies the following properties:

(i) $\beta(0) = 0$ almost surely (a.s. $\omega \in \Omega$)
(ii) $\beta(t)$ has independent increments, and $\beta(t) - \beta(t')$ has the normal distribution with mean 0 and variance $t - t'$ (for $0 \le t' \le t$).

Here, we abused the notation and wrote, for a given $t \in [0,T]$, $\beta(t) : \Omega \to \mathbb{R}^n$, $\beta(t)(\omega) = \beta(t,\omega)$. This further induces a mapping Φ_β from Ω into the collection of functions from $[0,T]$ to $\mathbb{R}^n$ by $(\Phi_\beta(\omega))(t) = \beta(t)(\omega)$. The law of the Brownian motion, that is, the pushforward measure $(\Phi_\beta)_*(Pr)$ on Borel sets in $(\mathbb{R}^n)^{[0,T]}$, is nothing but the classical Wiener measure. Replacing $\mathbb{R}^n$ with some other Banach space will take us into the realm of abstract Wiener spaces. It is worth noting that, while there are several technicalities involved in properly defining a Wiener measure in this general context, the same point of view as in the classical $\mathbb{R}^n$ case is employed.

2. Regularity of Brownian motion

In what follows, we first discuss some known results about the regularity of Brownian motion on function spaces that are often used in PDEs, that is, Sobolev and Besov spaces. All spaces are considered local in time.

2.1. **Modulus of continuity.** It is well-known that $\beta(t)$ is almost surely (a.s.) continuous. Moreover, we have Lévy's theorem for the modulus of continuity:

$$\limsup_{\substack{t-t'\le\varepsilon\downarrow 0\\ 0\le t'<t\le 1}} \frac{|\beta(t)-\beta(t')|}{\sqrt{-2\varepsilon\log\varepsilon}} = 1, \text{ a.s.} \tag{2.1}$$

It follows from (2.1) that the Brownian motion is a.s. locally Hölder continuous of order s for every $s<1/2$, and is a.s. nowhere locally Hölder continuous of order s for $s\ge 1/2$; see Revuz and Yor's book [40].

Regularity on Sobolev spaces $H^s(0,1)$: By definition of the Brownian motion, we have

$$\mathbb{E}\big[|\beta(t)-\beta(t')|^2\big] = \mathbb{E}\big[|\beta(t-t')|^2\big] = |t-t'|,$$

where $\mathbb{E}$ denotes expectation. Thus, we have

$$\begin{aligned}\mathbb{E}\big[\|\beta(t)\|^2_{\dot H^s(0,1)}\big] &= \mathbb{E}\bigg[\int_0^1\int_0^1 \frac{|\beta(t)-\beta(t')|^2}{|t-t'|^{1+2s}}dt'dt\bigg] = \int_0^1\int_0^1 \frac{1}{|t-t'|^{2s}}dt'dt\\ &= 2\int_0^1\int_0^t \frac{1}{(t-t')^{2s}}dt'dt<\infty,\end{aligned}$$

if and only if $s<\frac12$. Hence, the Brownian motion $\beta(t)$ belongs almost surely to H^s_{loc} for $s<\frac12$. Indeed, one can also show that $\beta(t)\notin H^s_{\text{loc}}$ a.s. for $s\ge\frac12$.

Regularity on Sobolev spaces $\mathcal{W}^{s,p}(0,1)$:

Recall that we have, for $p>0$,

$$\mathbb{E}\big[|\beta(t)-\beta(t')|^p\big] = C_p|t-t'|^{\frac p2},$$

since $\beta(t)-\beta(t')$ is a mean-zero Gaussian with variance $|t-t'|$. Then, by the characterization of $\dot{\mathcal{W}}^{s,p}$ via the L^p modulus of continuity as in Tartar's book [48], we have

$$\begin{aligned}\mathbb{E}\big[\|\beta(t)\|^p_{\dot{\mathcal{W}}^{s,p}(0,1)}\big] &= \mathbb{E}\bigg[\int_0^1\int_0^1 \frac{|\beta(t)-\beta(t')|^p}{|t-t'|^{1+sp}}dt'dt\bigg] \sim \int_0^1\int_0^1 |t-t'|^{-1-sp+\frac p2}dt'dt\\ &= 2\int_0^1\int_0^t |t-t'|^{-1-sp+\frac p2}dt'dt<\infty,\end{aligned}$$

if and only if $s<\frac12$. Hence, the Brownian motion $\beta(t)$ belongs almost surely to $\mathcal{W}^{s,p}_{\text{loc}}$ for $s<\frac12$.

Regularity on Besov spaces $B^s_{p,q}(0,1)$:

Ciesielski [11], [12] and Roynette [41] proved that if $s<\frac12$, then the Brownian motion $\beta(t)$ belongs a.s. to $B^s_{p,q}(0,1)$ for all $p,q\ge 1$, and that if $s>\frac12$, then $\beta(t)\notin B^s_{p,q}(0,1)$ a.s. for any $p,q\ge 1$. Regarding the endpoint regularity of the Brownian motion, it was also shown that for $s=\frac12$, the Brownian motion $\beta(t)\in B^{\frac12}_{p,q}(0,1)$ a.s. if and only if $1\le p<\infty$ and $q=\infty$. Moreover, if the latter holds, there exists $c_p>0$ such that $\|\beta(t)\|_{B^{\frac12}_{p,\infty}(0,1)}\ge c_p$ a.s. The proof is based on the Schauder basis representation (or Franklin-Wiener series) of the Brownian motion; see Kahane's book [26].

In Subsection 2.3, we will present an alternate proof of these results which is of interest in its own, by using random Fourier series (also referred to as Fourier-Wiener series; see (2.2) below) and a careful multilinear analysis. It is worth noting that the independence on the index $p < \infty$ of the regularity result on, say, $\mathcal{W}^{s,p}_{\text{loc}}$ is unsurprising from the perspective of random Fourier series. Without delving into technicalities, this follows essentially by observing that, if $F(t) = \sum_{n\geq 1} c_n g_n(\omega) e^{int}$ with g_n an independent identically distributed sequence of complex-valued Gaussian random variables, then $F \in L^p$ if and only if the sequence $(c_n) \in \ell^2$; see Paley and Zygmund [39], and also the recent paper by Ayache and Tzvetkov [1]. A more detailed discussion of the comparison between classical Besov spaces and Fourier-Besov spaces is given in Subsection 2.2 after the statement of Theorem 2.1. We now turn our attention to the regularity of the Brownian motion on modulation spaces. By contrast to the classical function spaces discussed above, modulation spaces do capture the dependence of regularity on one of its defining indexes.

2.2. **Fourier analytic representation.** We are interested in the local-in-time regularity on modulation spaces $M^{p,q}_s(\mathbb{T})$ of the periodic Brownian motion defined on the 1-dimensional torus. The case of the Brownian motion on the d-dimensional torus $\mathbb{T}^d$ is similar; see Remark 2.4. Moreover, we can obtain the regularity of the Brownian motion on the real line as long as we replace the periodic modulation spaces $M^{p,q}(\mathbb{T})$ with the localized versions $M^{p,q}_s(I)$, where $I \subset \mathbb{R}$ is any given bounded interval; see Remark 2.3 and Appendix B.

Without loss of generality, it suffices to consider the mean-zero complex-valued Brownian loop, that is, satisfying both $\beta(0) = \beta(2\pi)$ and $\int_0^{2\pi} \beta(t)dt = 0$. The case of a Brownian loop with non-zero mean follows easily from the mean-zero case through a translation by the mean of the Brownian motion. Furthermore, if we let $b(t)$ denote any (non-periodic) Brownian motion, and set $\beta(t) = b(t) - tb(2\pi)/2\pi$, then $\beta(t)$ is periodic, and $b(t)$ has the same regularity as $\beta(t)$.

Since, for almost every $\omega \in \Omega$, the Brownian motion $\beta(t)$ represents a continuous (periodic) function, to simplify the notation, we will simply write u in the following to denote this function (for a fixed ω). It is well known [26] that u can be represented through a *Fourier-Wiener series*[1] as

$$u(t) = u(t;\omega) = \sum_{n\neq 0} \frac{g_n(\omega)}{n} e^{int}, \tag{2.2}$$

where $\{g_n\}$ is a family of independent standard complex-valued Gaussian random variables, that is, $\operatorname{Re} g_n$ and $\operatorname{Im} g_n$ are independent standard real-valued Gaussian random variables. Note that we are missing the linear term $g_0(\omega)t$ in the Fourier-Wiener series representation since we are only considering the (mean zero) periodic case. For the convenience of the reader, we present a detailed derivation of (2.2) in Appendix A.

With the notation above, our result can be stated as follows.

Theorem 2.1. *Let $1 \leq p, q \leq \infty$.*

(a) *If $q < \infty$, then $u \in M^{p,q}_s(\mathbb{T})$ a.s. for $(s-1)q < -1$, and $u \notin M^{p,q}_s(\mathbb{T})$ a.s. for $(s-1)q \geq -1$.*

(b) *If $q = \infty$, then $u \in M^{p,\infty}_s(\mathbb{T})$ a.s. for $s < 1$, and $u \notin M^{p,\infty}_s(\mathbb{T})$ a.s. for $s \geq 1$.*

Moreover,

[1] Henceforth, we drop a factor of 2π when it plays no role.

(c) *If $q < \infty$, then $u \in \widehat{b}^s_{p,q}(\mathbb{T})$ a.s. for $(s-1)p < -1$, and $u \notin \widehat{b}^s_{p,q}(\mathbb{T})$ a.s. for $(s-1)p \geq -1$.*
(d) *If $q = \infty$ and $p < \infty$, then $u \in \widehat{b}^s_{p,\infty}(\mathbb{T})$ a.s. for $(s-1)p \leq -1$, and $u \notin \widehat{b}^s_{p,\infty}(\mathbb{T})$ a.s. for $(s-1)p > -1$.*

Note that, in our statements regarding the Fourier-Besov spaces $\widehat{b}^s_{p,q}(\mathbb{T})$, the case $p = q = \infty$ is already addressed by part (b), since $\widehat{b}^s_{\infty,\infty}(\mathbb{T}) = \mathcal{F}L^{s,\infty}(\mathbb{T})$. We think of the case in which $(s-1)p = -1$ as an "end-point" that makes the transition of regularity from $\mathcal{F}L^{s,p}(\mathbb{T})$ to $\widehat{b}^s_{p,\infty}(\mathbb{T})$. The case when $(s-1)p = -1$ and $p = 2$ (that is, $s = 1/2$) corresponds to the end-point case for the usual Besov spaces.

It is also worthwhile to note that the Fourier-Besov spaces $\widehat{b}^s_{p,q}(\mathbb{T})$ are more finely tuned for this analysis than the regular Besov spaces because of their sensitivity to the value of p under randomization. Roughly speaking, this is implied by the definitions of the two spaces: the $\widehat{b}^s_{p,q}(\mathbb{T})$ spaces use the L^p_n norm on the Fourier side for each dyadic block, while the $B^s_{p,q}(\mathbb{T})$ spaces use the L^p_t norm on the physical side for each dyadic block. Now, in the case of regular Besov spaces, the L^p_t norms are equivalent under the expectation, that is, the L^p_t spaces are equivalent under randomization due to a Khintchine type argument or Paley-Zygmund's theorem; see [26]. In other words, the L^p_t part is "insensitive" to a finite p under randomization.

Remark 2.2. It is well-known that the Gaussian part of the Gibbs measure for the Benjamin-Ono equation corresponds to a (periodic) fractional Brownian motion having the Fourier-Wiener series representation $\sum_{n\neq 0} \frac{g_n(\omega)}{|n|^{1/2}} e^{int}$; see [50]. In general, the regularity of any (mean zero periodic) fractional Brownian motion represented by $\sum_{n\neq 0} \frac{g_n(\omega)}{|n|^{\alpha}} e^{int}$ is described by an analogous statement to the one of Theorem 2.1. In this case, the only modifications are that for the $M^{p,q}_s(\mathbb{T})$ regularity we require the condition $(s-\alpha)q < -1$, while for $\widehat{b}^s_{p,q}(\mathbb{T})$ we need $(s-\alpha)p < -1$. These modifications are straightforward consequences of the summability conditions appearing in the proof of Theorem 2.1, and thus we omit the details. Note also that when $\alpha = 0$ we recover the regularity of the mean zero Gaussian white noise on $\mathbb{T}$.

Remark 2.3. We can define the local-in-time versions of the functions spaces above in the following way. Given an interval $I \subset \mathbb{R}$, we let $M^{p,q}_s(I)$ denote the restriction of $M^{p,q}_s(\mathbb{R})$ onto I via

$$\|u\|_{M^{p,q}_s(I)} = \inf\big\{\|v\|_{M^{p,q}_s(\mathbb{R})} : v = u \text{ on } I\big\}.$$

The local-in-time versions of other function spaces can be defined in an analogous manner. With this notation, we can show that, given a bounded interval $I \subset \mathbb{R}$, *Theorem 2.1 holds with the same conditions on the indices for the Brownian motion $b(t)$ (defined on the unbounded domain $\mathbb{R}$) on the spaces $M^{p,q}_s(I)$, $W^{p,q}_s(I)$, and $\widehat{b}^s_{p,q}(I)$.* Clearly, $b(t)$ is unbounded in any of $M^{p,q}_s(\mathbb{R})$, $W^{p,q}_s(\mathbb{R})$, or $\widehat{b}^s_{p,q}(\mathbb{R})$ a.s., since $b(t)$ represents a.s. an unbounded function on $\mathbb{R}$. Namely, this unboundedness of the norms on $\mathbb{R}$ comes from the integrability condition rather than the differentiability condition. The "equivalence" of the periodic function spaces (defined on $\mathbb{T}$) and their local-in-time versions (defined on some bounded interval I) is proved in detail in Appendix B.

Remark 2.4. *Theorem 2.1 can also be extended to the Brownian motion on the d-dimensional torus $\mathbb{T}^d$.* We simply need to modify the corresponding statements by imposing the inequality $(s-1)q < -d$ when considering the regularity on $M_s^{p,q}(\mathbb{T}^d)$, or $(s-1)p < -d$ when considering the $\widehat{b}^s_{p,q}(\mathbb{T}^d)$ regularity. Let us briefly indicate how these conditions arise. The periodic Brownian motion on $\mathbb{T}^d$ can be represented by

$$u(t) = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \frac{g_n(\omega)}{|n|}\, e^{in\cdot t}, \tag{2.3}$$

where $\{g_n\}_{n\in\mathbb{Z}^d\setminus\{0\}}$ is a family of independent standard complex-valued Gaussian random variables; see [6, 7]. Note that (2.3) corresponds to a typical element in the support of the d-dimensional analogue of the Wiener measure μ defined in (3.3) and (3.5). Thus, in the proof of Theorem 2.1, say in (2.4), we only need to change our summation $\sum \langle n\rangle^{(s-1)q}$ over the lattice $\mathbb{Z}$ (which requires $(s-1)q < -1$ for convergence) to that over $\mathbb{Z}^d$ to recover the correct condition $(s-1)q < -d$; see also Remark 3.5. We note also that the regularity of the white noise $W(t)$ on $\mathbb{T}^d$ in [51] can be deduced from the d-dimensional version of our Theorem 2.1 by simply dropping one regularity from that of the Brownian motion $u(t)$. For example, we have that if $q<\infty$, then $W \in \widehat{b}^s_{p,q}(\mathbb{T}^d)$ a.s. for $sp < -d$, and $W \notin \widehat{b}^s_{p,q}(\mathbb{T}^d)$ a.s. for $sp \geq -d$; see also [35], and Remark 2.2. Finally, it is worth recalling that Brownian motion in higher dimensions is commonly referred to as a Gaussian Free Field (GFF). The physics literature uses the names massless free field or Euclidean bosonic massless free field. For a nice overview of GFF and its applications, we refer the reader to Sheffield's article [44]; see also Remark A.1.

Theorem 2.1 states that the Brownian motion belongs a.s. to $M_s^{p,q}(\mathbb{T})$, $W_s^{p,q}(\mathbb{T})$, and $\mathcal{F}L^{s,q}(\mathbb{T})$ for $(s-1)q < -1$, and to $\widehat{b}^s_{p,q}(\mathbb{T})$ for $(s-1)p<-1$, and to $\widehat{b}^s_{p,\infty}(\mathbb{T})$ for $(s-1)p \leq -1$. However, in applications, it is often very important to know how large the estimate on the norm is likely to be; see the works by Bourgain [4, 5, 6, 7], Burq and Tzvetkov [8, 10], and the second author [32, 34, 35, 37]. The following theorem provides us with the desirable "large deviation estimates".

Theorem 2.5. *There exists $c > 0$ such that for (sufficiently large) $K > 0$, the following holds:*

(i) *If $(s-1)q < -1$, then $Pr\big(\|u(\omega)\|_{M_s^{p,q}(\mathbb{T})} > K\big) < e^{-cK^2}$.*
(ii) *If $(s-1)p < -1$, then $Pr\big(\|u(\omega)\|_{\widehat{b}^s_{p,q}(\mathbb{T})} > K\big) < e^{-cK^2}$.*
(iii) *If $(s-1)p = -1$ (and $q=\infty$), then $Pr\big(\|u(\omega)\|_{\widehat{b}^s_{p,\infty}(\mathbb{T})} > K\big) < e^{-cK^2}$.*

Remark 2.6. We note that the same estimates in Theorem 2.5 hold for all $K>0$ as long as we replace the right-hand side by Ce^{-cK^2} for an appropriate C. If we wish to "normalize" the constant C so that it equals 1, we must select $K>0$ sufficiently large so that $Ce^{-\frac{c}{2}K^2} < 1$.

The proofs of parts (i) and (ii) in Theorem 2.5 rely on the theory of abstract Wiener spaces and Fernique's theorem. A detailed discussion of these proofs and the afferent technicalities is given in Section 3.

Proof of Theorem 2.1. We begin by showing statements (a) and (b). Recall that we have $M_s^{p,q}(\mathbb{T}) = W_s^{p,q}(\mathbb{T}) = \mathcal{F}L^{s,q}(\mathbb{T})$, where we defined the latter space via the norm:

$$\|u\|_{\mathcal{F}L^{s,q}(\mathbb{T})} = \|\langle n\rangle^s \widehat{u}(n)\|_{\ell^q_n}.$$

We easily see that, with $\mathbb{E}$ denoting expectation, we have

$$\mathbb{E}\big[\|u\|^q_{\mathcal{F}L^{s,q}(\mathbb{T})}\big] = \sum_{n\neq 0}\langle n\rangle^{sq}|n|^{-q}\mathbb{E}\big[|g_n|^q\big] \sim \sum_{n\neq 0}\langle n\rangle^{(s-1)q} < \infty \tag{2.4}$$

if and only if $(s-1)q < -1$.

Let us now define

$$X^{(q)}_j(\omega) := 2^{-j}\sum_{|n|\sim 2^j}|g_n(\omega)|^q. \tag{2.5}$$

Then, $X^{(q)}_j \to c_q := \mathbb{E}|g_1|^q$ almost surely, since $Y^{(q)}_j := 2^{-j}\sum_{1\leq |n|\leq 2^{j-1}}|g_n|^q \to c_q$ by the strong law of large numbers, and $X^{(q)}_j = 2Y^{(q)}_{j+1} - Y^{(q)}_j$.

Suppose now that $(s-1)q \geq -1$. Then, we have

$$\begin{aligned}\|u\|^q_{\mathcal{F}L^{s,q}(\mathbb{T})} &= \sum_{n\neq 0}\langle n\rangle^{sq}|n|^{-q}|g_n(\omega)|^q \sim \sum_{j=0}^\infty\sum_{|n|\sim 2^j}\langle n\rangle^{(s-1)q}|g_n(\omega)|^q \\ &\geq \sum_{j=0}^\infty\sum_{|n|\sim 2^j}\langle n\rangle^{-1}|g_n(\omega)|^q \sim \sum_{j=0}^\infty X^{(q)}_j(\omega) = \infty, \text{ a.s.}\end{aligned} \tag{2.6}$$

Hence, when $q < \infty$, $u \in \mathcal{F}L^{s,q}(\mathbb{T})$ a.s. for $(s-1)q < -1$, and $u \notin \mathcal{F}L^{s,q}(\mathbb{T})$ a.s. for $(s-1)q \geq -1$.

Now, let $q = \infty$. Then, we have

$$\|u\|_{\mathcal{F}L^{s,\infty}(\mathbb{T})} = \sup_{n\neq 0}\langle n\rangle^s|n|^{-1}|g_n(\omega)| \sim \sup_{n\neq 0}\langle n\rangle^{s-1}|g_n(\omega)| < \infty,$$

a.s. for $s < 1$ (i.e. "$(s-1)\cdot\infty < -1$"), since $\lim_{n\to\infty} n^{-\varepsilon}|g_n(\omega)| = 0$ for any $\varepsilon > 0$. When $s \geq 1$, the continuity from below of the probability measure gives

$$Pr(\|u\|_{\mathcal{F}L^{s,\infty}(\mathbb{T})} < \infty) \leq Pr(\sup_n |g_n(\omega)| < \infty) = \lim_{K\to\infty} Pr(\sup_n |g_n(\omega)| < K) = 0.$$

Hence, $u \in \mathcal{F}L^{s,\infty}(\mathbb{T})$ a.s. for $s < 1$, and $u \notin \mathcal{F}L^{s,\infty}(\mathbb{T})$ a.s. for $s \geq 1$.

Regarding (c) and (d), recall that we defined $\widehat{b}^s_{p,q}(\mathbb{T})$ via the norm in (1.7) and (1.8). First, suppose $(s-1)p < -1$. Then, for $1 \leq p < \infty$ and $q \geq 1$, we have

$$\begin{aligned}\mathbb{E}\big[\|u\|^p_{\widehat{b}^s_{p,q}(\mathbb{T})}\big] &\leq \mathbb{E}\big[\|u\|^p_{\widehat{b}^s_{p,1}(\mathbb{T})}\big] \leq \bigg(\sum_{j=0}^\infty\Big(\sum_{|n|\sim 2^j}\langle n\rangle^{sp}|n|^{-p}\mathbb{E}[|g_n|^p]\Big)^{\frac{1}{p}}\bigg)^p \\ &\sim \bigg(\sum_{j=0}^\infty\Big(\sum_{|n|\sim 2^j}\langle n\rangle^{(s-1)p}\Big)^{\frac{1}{p}}\bigg)^p \sim \Big(\sum_{j=0}^\infty 2^{\frac{(s-1)p+1}{p}j}\Big)^p < \infty.\end{aligned} \tag{2.7}$$

Also, when $p, q < \infty$, $\|u\|_{\widehat{b}^s_{p,q}(\mathbb{T})} = \infty$ a.s. for $(s-1)p \geq -1$ by modifying the argument in (2.6). When $p = \infty$ and $s < 1$, we have $\|u\|_{\widehat{b}^s_{\infty,q}(\mathbb{T})} \leq \|u\|_{\widehat{b}^s_{r,q}(\mathbb{T})}$ for $r < \infty$. Moreover, we can take r to be sufficiently large such that $(s-1)r < -1$. Then, it follows from (2.7) that $\|u\|_{\widehat{b}^s_{\infty,q}(\mathbb{T})} < \infty$ a.s. For $s \geq 1$, we have

$$\|u\|_{\widehat{b}^s_{\infty,q}(\mathbb{T})} \sim \big\|\sup_{|n|\sim 2^j}\langle n\rangle^{s-1}|g_n|\big\|_{\ell^q_j} \geq \Big\|2^{-j}\sum_{|n|\sim 2^j}|g_n|\Big\|_{\ell^q_j} = \infty, \text{ a.s.},$$

as in (2.6). Hence, $u \in \widehat{b}^s_{p,q}(\mathbb{T})$ a.s. for $(s-1)p < -1$, while $u \notin \widehat{b}^s_{p,q}(\mathbb{T})$ a.s. for $(s-1)p \geq -1$ and $q < \infty$.

Finally, we consider the case $q = \infty$ and $p < \infty$. In the endpoint case $(s-1)p = -1$, we have

$$\begin{aligned} \|u\|^p_{\widehat{b}^s_{p,\infty}(\mathbb{T})} &\sim \sup_j \sum_{|n|\sim 2^j} \langle n\rangle^{(s-1)p} |g_n(\omega)|^p \\ &\sim \sup_j 2^{-j} \sum_{|n|\sim 2^j} |g_n(\omega)|^p = \sup_j X_j^{(p)}(\omega) < \infty, \text{ a.s.}, \end{aligned} \tag{2.8}$$

where $X_j^{(p)}$ is defined in (2.5). When $(s-1)p > -1$, a similar computation along with the convergence of $X_j^{(p)}$ shows that $u \notin \widehat{b}^s_{p,\infty}(\mathbb{T})$ a.s. $\square$

2.3. **Alternate proof for the Besov spaces.** We close this section by providing a new (alternate) proof, via Fourier-Wiener series (2.2), of the regularity results on Besov spaces that we exposed at the end of Subsection 2.1. We decided to include this proof because random Fourier series are a unifying theme of this paper. Furthermore, our proof, which is done on the Fourier side, seems to complement nicely the existing one using Franklin-Wiener series on the physical side [41]. For notational simplicity we write $B^s_{p,q}$ for $B^s_{p,q}(\mathbb{T})$.

We begin by recalling the general Gaussian bound

$$\Big\| \sum_n c_n g_n(\omega) \Big\|_{L^p(\Omega)} \leq C\sqrt{p} \|c_n\|_{l^2_n}; \tag{2.9}$$

see the works of Burq and Tzvetkov [9], and Tzvetkov [50]. Also, see Lemma 3.11 below. Then, we have, for $1 \leq p < \infty$ and $1 \leq q \leq \infty$, (recall $t \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$),

$$\begin{aligned} \mathbb{E}\big[\|u\|_{B^s_{p,q}}\big] &= \mathbb{E}\bigg\| \Big\| \sum_{|n|\sim 2^j} \langle n\rangle^s |n|^{-1} g_n e^{int} \Big\|_{L^p_t} \bigg\|_{\ell^q_j} \leq \bigg\| \Big\| \Big\| \sum_{|n|\sim 2^j} \langle n\rangle^s |n|^{-1} g_n(\omega) e^{int} \Big\|_{L^p(\Omega)} \Big\|_{L^p_t} \bigg\|_{\ell^1_j} \\ &\lesssim \sum_{j=0}^\infty \Big(\sum_{|n|\sim 2^j} \langle n\rangle^{2(s-1)} \Big)^{\frac{1}{2}} < \infty \end{aligned}$$

for $2(s-1) < -1$, i.e. $s < \frac{1}{2}$. When $s < \frac{1}{2}$ and $p = \infty$, Sobolev's inequality gives $\|\langle\partial_t\rangle^s u\|_{L^\infty} \lesssim \|\langle\partial_t\rangle^{s+\varepsilon} u\|_{L^r}$, for small $\varepsilon > 0$ and large r such that $s + \varepsilon < \frac{1}{2}$ and $\varepsilon r > 1$. Then, the above computation shows that $\|u\|_{B^s_{\infty,q}} \lesssim \|u\|_{B^{s+\varepsilon}_{r,q}} < \infty$ a.s. for $s < \frac{1}{2}$.

Suppose $2 \leq p \leq \infty$. Then, for $q < \infty$ and $s \geq \frac{1}{2}$, we have

$$\|u\|_{B^s_{p,q}} \geq \|u\|_{B^s_{2,q}} \sim \bigg\| \Big(\sum_{|n|\sim 2^j} 2^{2(s-1)j} |g_n|^2 \Big)^{\frac{1}{2}} \bigg\|_{\ell^q_j} \sim \big\| \big(2^{(2s-1)j} X_j^{(2)}\big)^{\frac{1}{2}} \big\|_{\ell^q_j} = \infty, \text{ a.s.} \tag{2.10}$$

since $X_j^{(2)}$ defined in (2.5) converges to $c_2 > 0$ a.s. It also follows from (2.10) that $\|u\|_{B^s_{p,\infty}} = \infty$ a.s. for $s > \frac{1}{2}$ when $q = \infty$. Now, let $1 \leq p < 2$, $q < \infty$, and $s \geq \frac{1}{2}$. Then, we have

$$\|u\|^q_{B^s_{p,q}} \geq \|u\|^q_{B^{\frac{1}{2}}_{1,q}} \sim \sum_j Z_j^{(q)}(\omega),$$

where $Z_j^{(q)}(\omega) = \|\widetilde{X}_j(t,\omega)\|^q_{L^1_t}$ and

$$\widetilde{X}_j(t,\omega) := 2^{-\frac{j}{2}} \sum_{|n|\sim 2^j} g_n(\omega) e^{int}. \tag{2.11}$$

Note that for each $t \in \mathbb{T}$, $\widetilde{X}_j(t,\omega)$ is a standard complex Gaussian random variable. Thus, we have $\mathbb{E}\big[\|\widetilde{X}_j(t,\omega)\|_{L^1_t}\big] = \big\|\mathbb{E}\,|\widetilde{X}_j(t,\omega)|\big\|_{L^1_t} = 2\pi c_1 > 0$. In particular, we have

$$\big(\mathbb{E}\big[Z_j^{(q)}\big]\big)^{\frac{1}{q}} \geq \mathbb{E}\big[\|\widetilde{X}_j(t,\omega)\|_{L^1_t}\big] = 2\pi c_1 \tag{2.12}$$

Also, by (2.9), we have $\mathbb{E}\big[|Z_j^{(q)}|^2\big] \leq \big\|\|\widetilde{X}_j(t,\omega)\|_{L^{2q}_\omega}\big\|^{2q}_{L^1_t} \leq C(2q)^q < \infty$. Hence, by Kolmogorov's strong law of large numbers, we have $\frac{S_m - \mathbb{E}[S_m]}{m+1} \to 0$ a.s., where $S_m = \sum_{j=0}^m Z_j^{(q)}$. It follows from (2.12) that $\frac{\mathbb{E}[S_m]}{m+1} \geq (2\pi c_1)^q$. This implies that $S_m(\omega) \to \infty$ a.s. in $\omega \in \Omega$. Hence, we have

$$\|u\|_{B^s_{p,q}} \geq \|u\|_{B^{\frac{1}{2}}_{1,q}} = \Big(\lim_{m\to\infty} S_m\Big)^{\frac{1}{q}} = \infty, \text{ a.s.}$$

For $q = \infty$ and $s > \frac{1}{2}$, we have

$$\|u\|_{B^s_{p,\infty}} \geq \|u\|_{B^s_{1,\infty}} \sim \sup_j 2^{(s-\frac{1}{2})j} Z_j^{(1)}(\omega), \tag{2.13}$$

where $Z_j^{(1)}(\omega) = \|\widetilde{X}_j(t,\omega)\|_{L^1_t}$. Note that $\mathbb{E}\big[Z_j^{(1)}\big] = 2\pi c_1$ and $\mathbb{E}\big[|Z_j^{(1)}|^2\big] \leq C < \infty$ for all j. This implies that there exist $\delta, \varepsilon > 0$ and Ω_j, $j = 0, 1, \dots$, such that $Z_j^{(1)}(\omega) > \delta$ for $\omega \in \Omega_j$ and $Pr(\Omega_j) > \varepsilon$. In particular, we have $\sum_j Pr(\Omega_j) = \infty$. Then, by the Borel zero-one law,

$$Pr([Z_j^{(1)}(\omega) > \delta, \text{ infinitely often}]) = 1. \tag{2.14}$$

From (2.13) and (2.14), $\|u\|_{B^s_{p,\infty}} = \infty$ a.s. for $s > \frac{1}{2}$.

Now, consider $p = q = \infty$ and $s = \frac{1}{2}$. We have

$$\|u\|_{B^{\frac{1}{2}}_{\infty,\infty}} = \sup_j \|\widetilde{X}_j(t,\omega)\|_{L^\infty_t} \geq \sup_j |\widetilde{X}_j(t_j^*,\omega)|,$$

where $\widetilde{X}_j$ is defined in (2.11) and t_j^*'s are points in $\mathbb{T}$. Recall that $\{\widetilde{X}_j(t_j^*)\}_{j=0}^\infty$ is a family of independent standard complex-valued Gaussian random variables. Hence, $\sup_j |\widetilde{X}_j(t_j^*,\omega)| = \infty$ a.s. and thus $\|u\|_{B^{\frac{1}{2}}_{\infty,\infty}} = \infty$ a.s.

Finally, we consider the case $p < \infty$, $q = \infty$ and $s = \frac{1}{2}$. First, assume $p \leq 2$. Then, we have

$$\|u\|^2_{B^{\frac{1}{2}}_{p,\infty}} \leq \|u\|^2_{B^{\frac{1}{2}}_{2,\infty}} = \sup_j X_j^{(2)} < \infty, \text{ a.s.}$$

since $X_j^{(2)}$ defined in (2.5) converges to c_2 a.s. In the following, we consider

$$\|u\|^p_{B^{\frac{1}{2}}_{p,\infty}} \sim \sup_j \Big\|2^{-\frac{j}{2}} \sum_{|n|\sim 2^j} g_n e^{int}\Big\|^p_{L^p_t} \tag{2.15}$$

only for $p = 2k$ with $k = 2, 3, \dots$ since $\|u\|_{B^{\frac{1}{2}}_{p,\infty}} \leq \|u\|_{B^{\frac{1}{2}}_{2k,\infty}}$ for $p \leq 2k$.

When $p = 4$, we have

$$\Big\|2^{-\frac{j}{2}} \sum_{|n|\sim 2^j} g_n e^{int}\Big\|_{L^4_t}^4 = 2\cdot 2^{-2j} \sum_{|n_1|,|n_2|\sim 2^j} |g_{n_1}|^2|g_{n_2}|^2 + 2^{-2j} \sum_{\substack{|n_\alpha|,|m_\beta|\sim 2^j\\ n_\alpha\neq m_\beta\\ n_1+n_2=m_1+m_2}} g_{n_1}g_{n_2}\overline{g}_{m_1}\overline{g}_{m_2}$$
$$- 2^{-2j} \sum_{|n|\sim 2^j} |g_n|^4 =: \mathrm{I}_j^{(2)} + \mathrm{II}_j^{(2)} + \mathrm{III}_j^{(2)}. \tag{2.16}$$

Note that $\mathrm{I}_j^{(2)} = \big(X_j^{(2)}\big)^2 \to c_2^2$ and $\mathrm{III}_j^{(2)} = 2^{-j}X_j^{(4)} \to 0$ a.s. by the strong law of large numbers, where $X_j^{(p)}$ is defined in (2.5). Hence, it suffices to prove that $\sup_j |\mathrm{II}_j^{(2)}| < \infty$ a.s. By independence of $\mathrm{II}_j^{(2)}$ and the Borel zero-one law, it suffices to show that

$$\sum_j Pr(|\mathrm{II}_j^{(2)}| > K) < \infty \tag{2.17}$$

for some $K > 0$. By Chebyshev's inequality, we have

$$Pr(|\mathrm{II}_j^{(2)}| > K) \le K^{-2}\mathbb{E}\big[|\mathrm{II}_j^{(2)}|^2\big] \le C_2K^{-2}2^{-j}. \tag{2.18}$$

Hence, (2.17) follows, and thus we have $\|u\|_{B^{\frac{1}{2}}_{4,\infty}} < \infty$ a.s.

In order to estimate (2.15) for the general case $p = 2k$, we use an induction argument and assume the existence of estimates for $p = 2,\dots,2(k-1)$. We have

$$\Big\|2^{-\frac{j}{2}} \sum_{|n|\sim 2^j} g_n e^{int}\Big\|_{L^{2k}_t}^{2k} = k!\cdot 2^{-kj} \sum_{|n_\alpha|\sim 2^j} \prod_{j=1}^{k} |g_{n_\alpha}|^2 + 2^{-kj} \sum_{*} \prod_{\alpha=1}^{k} g_{n_\alpha} \prod_{\beta=1}^{k} \overline{g}_{m_\beta}$$
$$+ \text{ error terms} =: \mathrm{I}_j^{(k)} + \mathrm{II}_j^{(k)} + \text{ error terms}, \tag{2.19}$$

where $* = \{n_\alpha, m_\beta : \alpha,\beta = 1,\dots,k,\ |n_\alpha|,|m_\beta| \sim 2^j,\ n_\alpha \neq m_\beta\ \sum n_\alpha = \sum m_\beta\}$. Note that $\mathrm{I}_j^{(k)}$ consists of the terms for which n_α's and m_β's form exactly k pairs (including higher multiplicity) and that $\mathrm{II}_j^{(k)}$ consists of the terms with no pair. There are two types of error terms, which we call of type (i) and type (ii):

(i) $\text{error}_j^{(k)}$(i): $\{n_\alpha\}_{\alpha=1}^k = \{m_\beta\}_{\beta=1}^k$, and there exists at least one pair α, $\widetilde{\alpha}$ with $\alpha \neq \widetilde{\alpha}$ such that $n_\alpha = n_{\widetilde{\alpha}}$, i.e., n_α's and m_β's form exactly k pairs, but there exists at least "four of a kind" (or higher order): $n_{\alpha_1} = n_{\alpha_2} = m_{\beta_1} = m_{\beta_2}$ with $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$.

(ii) $\text{error}_j^{(k)}$(ii): $\{n_\alpha\}_{\alpha=1}^k \neq \{m_\beta\}_{\beta=1}^k$, but there exists at least one pair α, β such that $n_\alpha = m_\beta$. i.e. n_α's and m_β's form exactly ℓ pairs for some $1 \le \ell \le k-1$.

As before, by the strong law of large numbers, $\mathrm{I}_j^{(k)} = \big(X_j^{(2)}\big)^k \to c_2^k$ a.s. and the error terms of type (i) go to 0 a.s. since each of them can be written as

$$2^{-(k-L)j} \prod_{\ell\,:\,k=\sum_{\ell=1}^L k_\ell} X_j^{(2k_\ell)}, \quad L < k,$$

where each $X_j^{(2k_\ell)} \to c_{2k_\ell}$ a.s. Hence, we have $Pr(\sup_j |\text{error}_j^{(k)}(\text{i})| < \infty) = 1$. As for the error terms of type (ii), the worst ones can be written as

$$2^{-kj} \sum_n |g_n|^2 \sum_{**} \prod_{\alpha=1}^{k-1} g_{n_\alpha} \prod_{\beta=1}^{k-1} \overline{g}_{m_\beta}$$

where $** = \{n_\alpha, m_\beta : \alpha, \beta = 1, \dots, k-1,\, |n_\alpha|, |m_\beta| \sim 2^j,\; n_\alpha \neq m_\beta \; \sum n_\alpha = \sum m_\beta\}$. It is basically a product of $X_j^{(2)}$ (which converges to c_2 a.s.) and $\text{II}_j^{(k-1)}$, i.e., the $(k-1)$-fold products over frequencies $\{n_\alpha\}_{\alpha=1}^{k-1}$ and $\{m_\beta\}_{\beta=1}^{k-1}$ containing no pair, which appeared at the $k-1$ inductive step. All the other error terms of type (ii) can be basically written as $\text{error}_j^{(\ell)}(\text{i}) \cdot \text{II}_j^{(k-\ell)}$ for some $\ell = 1, \dots, k-1$. Hence, we have $Pr(\sup_j |\text{error}_j^{(k)}(\text{ii})| < \infty) = 1$.

Now, it remains to estimate $\text{II}_j^{(k)}$. As before, it suffices to show that

$$\sum_j Pr(|\text{II}_j^{(k)}| > K) < \infty \tag{2.20}$$

for some $K > 0$. By Chebyshev's inequality, we have

$$Pr(|\text{II}_j^{(k)}| > K) \leq K^{-2}\mathbb{E}\big[|\text{II}_j^{(k)}|^2\big] \leq C_k K^{-2} 2^{-j}.$$

Hence, (2.20) follows, and thus we have $\|u\|_{B^{\frac{1}{2}}_{2k,\infty}} < \infty$ a.s. This completes the proof.

3. Large deviation estimates

3.1. **Abstract Wiener spaces and Fernique's theorem.** Let B denote any of the spaces $M_s^{p,q}(\mathbb{T})$, $W_s^{p,q}(T)$, $\mathcal{F}L^{s,q}(\mathbb{T})$, or $\widehat{b}^s_{p,q}(\mathbb{T})$, and, as before, let u be the mean zero complex-valued Brownian loop on $\mathbb{T}$. While the previous section was concerned with the question of B-regularity, i.e., whether or not u is in B, this section will be concerned with the complementary topic of large deviation estimates on B. Specifically, we will establish estimates of the form

$$Pr\big(\|u(\omega)\|_B > K\big) < e^{-cK^2}, \tag{3.1}$$

for large $K > 0$, and some constant $c = c(B) > 0$. As we shall see, the theory of abstract Wiener spaces and Fernique's theorem play a crucial role in establishing estimates such as (3.1) on all non-endpoint cases, see Proposition 3.6 below. In Subsection 3.2, we prove that the large deviation estimate still holds for $\widehat{b}^s_{p,\infty}(\mathbb{T})$ in the endpoint case $(s-1)p = -1$ even though Fernique's theorem is not applicable. We also discuss, in Subsection 3.3, the issue of deviation estimates in the endpoint case of regular Besov spaces $B^{\frac{1}{2}}_{p,\infty}, 1 \leq p < \infty$. An alternate point of view can be found in [24]. For non-endpoint deviation estimates on $B^s_{p,q}$, the reader is referred to Roynette's work [41].

Recall that if u is the mean zero complex-valued Brownian loop on $\mathbb{T}$, then we can expand it in its Fourier-Wiener series as

$$u(x, \omega) = \sum_{n \neq 0} \frac{g_n(\omega)}{n} e^{inx}, \quad x \in \mathbb{T}, \tag{3.2}$$

where $\{g_n(\omega)\}_{n\neq 0}$ is a family of independent standard complex-valued Gaussian random variables. This induces a probability measure on the periodic functions on $\mathbb{T}$, namely the mean zero *Wiener measure* on $\mathbb{T}$, which can be formally written as

$$d\mu = Z^{-1} \exp\bigg(-\frac{1}{2}\int_{\mathbb{T}} |u_x|^2 dx\bigg) \prod_{x\in\mathbb{T}} du(x), \quad u \text{ mean } 0. \tag{3.3}$$

In the following, we use the theory of abstract Wiener spaces to provide the precise meaning of expression (3.3). Let $u(x) = \sum_{n\neq 0} \widehat{u}_n e^{inx}$ denote any periodic function on $\mathbb{T}$ with mean 0. We define the Gaussian measure μ_N on $\mathbb{C}^{2N}$ with the density

$$d\mu_N = Z_N^{-1} \exp\bigg(-\frac{1}{2}\sum_{0<|n|\leq N} |n|^2|\widehat{u}_n|^2\bigg) \prod_{0<|n|\leq N} d\widehat{u}_n, \tag{3.4}$$

where $d\widehat{u}_n$ denotes the complex Lebesgue measure on $\mathbb{C}$ and

$$Z_N = \int_{\mathbb{C}^{2N}} \exp\bigg(-\frac{1}{2}\sum_{0<|n|\leq N} |n|^2|\widehat{u}_n|^2\bigg) \prod_{0<|n|\leq N} d\widehat{u}_n.$$

In our definition above, we have abused the notation and we denoted *any* generic periodic function on $\mathbb{T}$ (not just the Brownian motion) by the letter u. The context, however, will make it clear when we refer specifically to Brownian motion.

Note that the measure μ_N is the induced probability measure on $\mathbb{C}^{2N}$ (that is, the $2N$ dimensional complex Gaussian or $4N$ dimensional real Gaussian measure) under the map $\omega \mapsto \{g_n(\omega)/|n|\}_{0<|n|\leq N}$. Indeed, if we replace $\widehat{u}_n$ by $g_n/|n|$ in (3.4), we have

$$d\mu_N = \widetilde{Z_N}^{-1} \prod_{0<|n|\leq N} \exp\bigg(-\frac{1}{2}|g_n|^2\bigg) dg_n,$$

where

$$\widetilde{Z_N} = \prod_{0<|n|\leq N} \int_{\mathbb{C}} \exp\bigg(-\frac{1}{2}|g_n|^2\bigg) dg_n = (2\pi)^{2N}.$$

We would like to define the mean zero Wiener measure in (3.3) as a limit of the finite dimensional Gaussian measures μ_N as $N\to\infty$, i.e., we would like to define the Wiener measure μ in (3.3) by

$$d\mu = Z^{-1} \exp\bigg(-\frac{1}{2}\sum_{n\neq 0} |n|^2|\widehat{u}_n|^2\bigg) \prod_{n\neq 0} d\widehat{u}_n, \tag{3.5}$$

where

$$Z = \int \exp\bigg(-\frac{1}{2}\sum_{n\neq 0} |n|^2|\widehat{u}_n|^2\bigg) \prod_{n\neq 0} d\widehat{u}_n.$$

Note that the expression in the exponent in (3.5) can be written as

$$-\frac{1}{2}\sum_{n\neq 0} |n|^2|\widehat{u}_n|^2 = -\frac{1}{2}\|u\|_{\dot{H}^1} = -\frac{1}{2}\Big\langle |\partial_x|^{2-2s}|\partial_x|^s u, |\partial_x|^s u\Big\rangle_{L^2} = -\frac{1}{2}\langle B_s^{-1}u, u\rangle_{\dot{H}^s},$$

where $B_s = |\partial_x|^{2s-2}$.

It follows from the theory of Gaussian measures on Hilbert spaces that (3.5) defines a countably additive measure on $\dot{H}^s$ if and only if B_s is of trace class, i.e., if $\sum_{n\neq 0}|n|^{2s-2} < \infty$, which is equivalent to $s<\frac{1}{2}$; see Zhidkov's work [56]. This makes the Sobolev space H^s, $s<\frac{1}{2}$, a strong and natural candidate for the study of Brownian motion. Unfortunately, the spaces under consideration are not Hilbert spaces in general. To deal with this issue, the concept of abstract Wiener space comes to the rescue, since, roughly speaking, it provides us with a larger (Hilbert or Banach) space, as an extension of $\dot{H}^1$, on which μ can be realized as a countably additive probability measure.

In the following, we recall first some basic definitions from Kuo's monograph [28]. Given a real separable Hilbert space H with norm $\|\cdot\|_H$, let $\mathcal{F}$ denote the set of finite dimensional orthogonal projections $\mathbb{P}$ of H. Then, a *cylinder set* E is defined by $E = \{u \in H : \mathbb{P}u \in F\}$ where $\mathbb{P} \in \mathcal{F}$ and F is a Borel subset of $\mathbb{P}H$. We let $\mathcal{R}$ denote the collection of all such cylinder sets. Note that $\mathcal{R}$ is a field but not a σ-field. Then, the Gaussian measure μ on H is defined by

$$\mu(E) = (2\pi)^{-\frac{n}{2}} \int_F e^{-\frac{1}{2}\|u\|_H^2} du$$

for $E \in \mathcal{R}$, where $n = \dim \mathbb{P}H$ and du is the Lebesgue measure on $\mathbb{P}H$. It is known that μ is finitely additive but not countably additive on $\mathcal{R}$.

A seminorm $|||\cdot|||$ in H is called *measurable* if, for every $\varepsilon > 0$, there exists $\mathbb{P}_\varepsilon \in \mathcal{F}$ such that

$$\mu(|||\mathbb{P}u||| > \varepsilon) < \varepsilon \tag{3.6}$$

for $\mathbb{P} \in \mathcal{F}$ orthogonal to $\mathbb{P}_\varepsilon$. Any measurable seminorm is weaker than the norm of H, and H is not complete with respect to $|||\cdot|||$ unless H is finite dimensional. Let B be the completion of H with respect to $|||\cdot|||$ and denote by i the inclusion map of H into B. The triple (i, H, B) is called an *abstract Wiener space*. (The pair (B, μ) is often called an abstract Wiener space as well.)

Now, regarding $v \in B^*$ as an element of $H^* \equiv H$ by restriction, we embed B^* in H. Define the extension of μ onto B (which we still denote by μ) as follows. For a Borel set $F \subset \mathbb{R}^n$, set

$$\mu(\{u \in B : ((u, v_1), \cdots, (u, v_n)) \in F\}) := \mu(\{u \in H : (\langle u, v_1\rangle_H, \cdots, \langle u, v_n\rangle_H) \in F\}),$$

where v_j's are in B^* and $(\cdot, \cdot)$ denote the dual pairing between B and B^*. Let $\mathcal{R}_B$ denotes the collection of cylinder sets $\{u \in B : ((u, v_1), \cdots, (u, v_n)) \in F\}$ in B.

Proposition 3.1 (Gross [22]). *μ is countably additive in the σ-field generated by $\mathcal{R}_B$.*

In the context of our paper, let $H = \dot{H}^1(\mathbb{T})$. Then, we have

Theorem 3.2. *The seminorms $\|\cdot\|_{M_s^{p,q}(\mathbb{T})}$, $\|\cdot\|_{W_s^{p,q}(\mathbb{T})}$, and $\|\cdot\|_{\mathcal{F}L^{s,q}(\mathbb{T})}$ are measurable for $(s-1)q < -1$. Also, the seminorm $\|\cdot\|_{\widehat{b}^s_{p,q}(\mathbb{T})}$ is measurable for $(s-1)p < -1$.*

The proof of Theorem 3.2 follows closely the ideas from [35, Proposition 3.4]. For completeness, we present it in detail at the end of this subsection.

Corollary 3.3. *Let μ be the mean zero Wiener measure on $\mathbb{T}$. Then, $(M_s^{p,q}(\mathbb{T}), \mu)$, $(W_s^{p,q}(\mathbb{T}), \mu)$, and $(\mathcal{F}L^{s,q}(\mathbb{T}), \mu)$ are abstract Wiener spaces for $(s-1)q < -1$. Also, $(\widehat{b}^s_{p,q}(\mathbb{T}), \mu)$ is an abstract Wiener space for $(s-1)p < -1$.*

Remark 3.4. As we shall see later, condition (3.6) is not satisfied for the endpoint case $\widehat{b}^s_{p,\infty}(\mathbb{T})$ with $(s-1)p = -1$. Nevertheless, we can still establish a large deviation estimate using a different approach.

Remark 3.5. By making an analogous argument, we can define the mean zero Wiener measure on $\mathbb{T}^d$, where a typical element on the support is represented by (2.3). Then, Theorem 3.2 and Corollary 3.3 can be extended to $\mathbb{T}^d$. One simply needs to modify the conditions on the indices to read either $(s-1)q < -d$ or $(s-1)p < -d$ depending on the space considered, that is, modulation or Fourier-Besov, respectively.

Given an abstract Wiener space (B, μ), we have the following integrability result due to Fernique [19].

Proposition 3.6 (Theorem 3.1 in [28]). *Let (B, μ) be an abstract Wiener space. Then, there exists $c > 0$ such that $\int_B e^{c\|u\|_B^2} \mu(du) < \infty$. In particular, this implies the following large deviation estimate: there exists $c' > 0$ such that*

$$\mu(\|u\|_B \geq K) \leq e^{-c'K^2}, \tag{3.7}$$

for sufficiently large $K > 0$.

From Theorem 3.2 and Proposition 3.6, we obtain the following corollary.

Corollary 3.7. *Let μ be the mean zero Wiener measure on $\mathbb{T}$. Then, the large deviation estimate (3.7) holds for $B = M_s^{p,q}(\mathbb{T})$, $W_s^{p,q}(\mathbb{T})$, and $\mathcal{F}L^{s,q}(\mathbb{T})$ with $(s-1)q < -1$, Also, (3.7) holds for $B = \widehat{b}^s_{p,q}(\mathbb{T})$ with $(s-1)p < -1$.*

While Proposition 3.6 is not applicable to the endpoint case $\widehat{b}^s_{p,\infty}(\mathbb{T})$ with $(s-1)p = -1$ (see Remark 3.4), we can still prove the following result.

Theorem 3.8. *Let $(s-1)p = -1$. Then,*

$$\mu(\|u\|_{\widehat{b}^s_{p,\infty}(\mathbb{T})} \geq K) \leq e^{-cK^2}, \tag{3.8}$$

for sufficiently large $K > 0$.

We prove Theorem 3.8 in Subsection 3.2. Theorem 3.8 also holds for the endpoint case of the usual Besov spaces $B^s_{p,\infty}$, with $s = \frac{1}{2}$ and $p < \infty$. However, the proof becomes rather cumbersome for large values of p. Therefore, we will only map the proof of the large deviation estimates for the Besov spaces with $p \leq 4$; see Subsection 3.3.

For the proof of Theorem 3.2, we will need the following lemma from [37], which we now recall.

Lemma 3.9 (Lemma 4.7 in [37]). *Let $\{g_n\}$ be a sequence of independent standard complex-valued Gaussian random variables. Then, for M dyadic and $\delta < \frac{1}{2}$, we have*

$$\lim_{M \to \infty} M^{2\delta} \frac{\max_{|n| \sim M} |g_n|^2}{\sum_{|n| \sim M} |g_n|^2} = 0 \quad a.s.$$

With these preliminaries, we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. First, note that $\|u\|_{M_s^{p,q}(\mathbb{T})} = \|u\|_{W_s^{p,q}(\mathbb{T})} = \|u\|_{\mathcal{F}L^{s,q}(\mathbb{T})} = \|u\|_{\widehat{b}^s_{q,q}(\mathbb{T})}$. Hence, it suffices to prove the result for $\widehat{b}^s_{p,q}(\mathbb{T})$ with $(s-1)p < -1$ and any $q \in [1, \infty]$. In view of (3.6), it suffices to show that for given $\varepsilon > 0$, there exists large M_0 such that

$$\mu\big(\|\mathbb{P}_{>M_0} u\|_{\widehat{b}^s_{p,q}(\mathbb{T})} > \varepsilon\big) < \varepsilon, \tag{3.9}$$

where $\mathbb{P}_{>M_0}$ is the Dirichlet projection onto the frequencies $|n| > M_0$.

Since $\widehat{b}^s_{p,1}(\mathbb{T}) \subset \widehat{b}^s_{p,q}(\mathbb{T})$, it suffices to prove (3.9) for $q = 1$. If $p < 2$ with $(s-1)p < -1$, then by Hölder inequality, we have

$$\begin{aligned} \|\langle n \rangle^s \widehat{u}(n)\|_{\ell^p_{|n|\sim 2^j}} &\leq \|\langle n \rangle^{-\frac{2-p}{2p}}\|_{\ell^{\frac{2p}{2-p}}_{|n|\sim 2^j}} \|\langle n \rangle^{s+\frac{2-p}{2p}} \widehat{u}(n)\|_{\ell^2_{|n|\sim 2^j}} \\ &\sim \|\langle n \rangle^{s+\frac{2-p}{2p}} \widehat{u}(n)\|_{\ell^2_{|n|\sim 2^j}}, \end{aligned}$$

where $(s+\frac{2-p}{2p}-1)\cdot 2<-1$. Hence, it suffices to prove (3.9) for $2\leq p\leq\infty$ and $q=1$ with $(s-1)p<-1$.

Now, we consider the case $2\leq p<\infty$. In the following, we assume that u is of the form (3.2). Fix $K>1$ and $\delta\in(0,\frac{1}{2})$ (to be chosen later.) Then, by Lemma 3.9 and Egoroff's theorem, there exists a set E such that $\mu(E^c)<\frac{1}{2}\varepsilon$ and the convergence in Lemma 3.9 is uniform on E, i.e. we can choose dyadic M_0 large enough such that

$$\frac{\|\{g_n(\omega)\}_{|n|\sim M}\|_{\ell^\infty_n}}{\|\{g_n(\omega)\}_{|n|\sim M}\|_{\ell^2_n}}\leq M^{-\delta}, \tag{3.10}$$

for all $\omega\in E$ and dyadic $M>M_0$. In the following, we will work only on E and drop '$\cap E$' for notational simplicity. However, it should be understood that all the events are under the intersection with E so that (3.10) holds.

Let $\{\sigma_j\}_{j\geq 1}$ be a sequence of positive numbers such that $\sum\sigma_j=1$, and let $M_j=M_0 2^j$ dyadic. Note that $\sigma_j=C2^{-\lambda j}=CM_0^\lambda M_j^{-\lambda}$ for some small $\lambda>0$ (to be determined later.) Then, from (3.2), we have

$$\mu\big(\|\mathbb{P}_{>M_0}u(\omega)\|_{\widehat{b}^s_{p,1}(\mathbb{T})}>\varepsilon\big)\leq\sum_{j=1}^\infty\mu\big(\|\{\langle n\rangle^s n^{-1}g_n(\omega)\}_{|n|\sim M_j}\|_{\ell^p_n}>\sigma_j\varepsilon\big). \tag{3.11}$$

By interpolation and (3.10), we have

$$\begin{aligned}\|\{\langle n\rangle^s n^{-1}g_n\}_{|n|\sim M_j}\|_{\ell^p_n}&\sim M_j^{s-1}\|\{g_n\}_{|n|\sim M_j}\|_{\ell^p_n}\leq M_j^{s-1}\|\{g_n\}_{|n|\sim M_j}\|_{\ell^2_n}^{\frac{2}{p}}\|\{g_n\}_{|n|\sim M_j}\|_{\ell^\infty_n}^{\frac{p-2}{p}}\\&\leq M_j^{s-1}\|\{g_n\}_{|n|\sim M}\|_{\ell^2_n}\bigg(\frac{\|\{g_n\}_{|n|\sim M_j}\|_{\ell^\infty_n}}{\|\{g_n\}_{|n|\sim M_j}\|_{\ell^2_n}}\bigg)^{\frac{p-2}{p}}\leq M_j^{s-1-\delta\frac{p-2}{p}}\|\{g_n\}_{|n|\sim M_j}\|_{\ell^2_n}.\end{aligned}$$

Thus, if we have $\|\{\langle n\rangle^s n^{-1}g_n\}_{|n|\sim M_j}\|_{\ell^p_n}>\sigma_j\varepsilon$, then we have $\|\{g_n\}_{|n|\sim M_j}\|_{\ell^2_n}\gtrsim R_j$ where $R_j:=\sigma_j\varepsilon M_j^\alpha$ with $\alpha:=-s+1+\delta\frac{p-2}{p}$. With $p=2+\theta$, we have $\alpha=\frac{-(s-1)p+\delta\theta}{2+\theta}>\frac{1}{2}$ by taking δ sufficiently close to $\frac{1}{2}$ since $-(s-1)p>1$. Then, by taking $\lambda>0$ sufficiently small, $R_j=\sigma_j\varepsilon M_j^\alpha=C\varepsilon M_0^\lambda M_j^{\alpha-\lambda}\gtrsim C\varepsilon M_0^\lambda M_j^{\frac{1}{2}+}$. By a direct computation in the polar coordinates, we have

$$\mu\big(\|\{g_n\}_{|n|\sim M_j}\|_{\ell^2_n}\gtrsim R_j\big)\sim\int_{B^c(0,R_j)}e^{-\frac{1}{2}|g_n|^2}\prod_{|n|\sim M_j}dg_n\lesssim\int_{R_j}^\infty e^{-\frac{1}{2}r^2}r^{2\cdot\#\{|n|\sim M_j\}-1}dr.$$

Note that, in the inequality, we have dropped the implicit constant $\sigma(S^{2\cdot\#\{|n|\sim M_j\}-1})$, a surface measure of the $2\cdot\#\{|n|\sim M_j\}-1$ dimensional unit sphere, since $\sigma(S^n)=2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})\lesssim 1$. By the change of variable $t=M_j^{-\frac{1}{2}}r$, we have $r^{2\cdot\#\{|n|\sim M_j\}-2}\lesssim r^{4M_j}\sim M_j^{2M_j}t^{4M_j}$. Since $t>M_j^{-\frac{1}{2}}R_j=C\varepsilon M_0^\lambda M_j^{0+}$, we have $M_j^{2M_j}=e^{2M_j\ln M_j}<e^{\frac{1}{8}M_jt^2}$ and $t^{4M_j}<e^{\frac{1}{8}M_jt^2}$ for M_0 sufficiently large. Thus, we have $r^{2\cdot\#\{|n|\sim M_j\}-2}<e^{\frac{1}{4}M_jt^2}=e^{\frac{1}{4}r^2}$ for $r>R$. Hence, we have

$$\mu\big(\|\{g_n\}_{|n|\sim M_j}\|_{\ell^2_n}\gtrsim R_j\big)\leq C\int_{R_j}^\infty e^{-\frac{1}{4}r^2}rdr\leq e^{-cR_j^2}=e^{-cC^2M_0^{2\lambda}M_j^{1+}\varepsilon^2}. \tag{3.12}$$

From (3.11) and (3.12), we have

$$\mu\big(\|\mathbb{P}_{>M_0}u\|_{\widehat{b}^s_{p,1}(\mathbb{T})}>\varepsilon\big)\le\sum_{j=1}^{\infty}e^{-cC^2M_0^{1+2\lambda+}(2^j)^{1+}\varepsilon^2}\le\tfrac{1}{2}\varepsilon$$

by choosing M_0 sufficiently large as long as $(s-1)p<-1$.

When $p=\infty$, we have $s<1$. By repeating the computation with (3.10), we see that if we have $\|\{\langle n\rangle^s n^{-1}g_n\}_{|n|\sim M_j}\|_{\ell^\infty_n}>\sigma_j\varepsilon$, then we have $\|\{g_n\}_{|n|\sim M_j}\|_{\ell^2_n}\gtrsim R_j$ where $R_j:=\sigma_j\varepsilon M_j^\alpha$ with $\alpha:=-s+1+\delta$. Since $-s+1>0$, we have $\alpha>\frac{1}{2}$ by taking δ sufficiently close to $\frac{1}{2}$. The rest follows exactly as before. $\square$

3.2. **Large deviation estimates for $\widehat{b}^s_{p,\infty}(\mathbb{T})$ at the endpoint** $(s-1)p=-1$**.** Now, we show that the condition (3.6) actually fails for $\widehat{b}^s_{p,\infty}(\mathbb{T})$ for the endpoint case $(s-1)p=-1$; see Remark 3.4. By the strong law of large numbers, $X_j^{(p)}$ defined in (2.5) converges a.s. to $c_p>0$. Then, by Egoroff's theorem, there exists E with $Pr(E)\ge\frac{1}{2}$ such that X_j converges uniformly to c_p on E. Thus, given $\delta>0$, there exists $J_0\in\mathbb{N}$ such that $Pr(\{\omega:\sup_{j\ge J}X_j^{(p)}(\omega)>c_p-\delta)\ge\frac{1}{2}$ for any $J\ge J_0$. In view of (2.8), this shows that the condition (3.6) does not hold once $\varepsilon<c_p$. In particular, Proposition 3.6 does not hold automatically.

The remainder of this subsection is dedicated to the proof of Theorem 3.8 via a direct approach that bypasses the assumption of abstract Wiener space. Specifically, we establish that, for some $c=c(p)$ and all sufficiently large $K\ge K_p$, the large deviation estimate

$$Pr(\|u\|_{\widehat{b}^s_{p,\infty}(\mathbb{T})}>K)<e^{-cK^2},\tag{3.13}$$

also holds in the endpoint case $(s-1)p=-1$.

Let us first consider the case $p\le 2$. By Hölder inequality with $\frac{1}{r}=\frac{1}{p}-\frac{1}{2}$, we have

$$\big\|2^{-\frac{j}{p}}|g_n|\big\|_{\ell^p_{|n|\sim 2^j}}\le\|2^{-\frac{j}{r}}\|_{\ell^r_{|n|\sim 2^j}}\big\|2^{-\frac{j}{2}}|g_n|\big\|_{\ell^2_{|n|\sim 2^j}}\sim\big\|2^{-\frac{j}{2}}|g_n|\big\|_{\ell^2_{|n|\sim 2^j}}.\tag{3.14}$$

i.e., we have $\|u\|_{\widehat{b}^s_{p,\infty}(\mathbb{T})}\lesssim\|u\|_{\widehat{b}^{\frac{1}{2}}_{2,\infty}(\mathbb{T})}$. Hence, we only need to consider the case $p=2$ and $s=\frac{1}{2}$. By definition of the norm, we have

$$Pr(\|u\|_{\widehat{b}^{\frac{1}{2}}_{2,\infty}(\mathbb{T})}>K)\le\sum_{j=0}^{\infty}Pr\big(2^{-j}\sum_{|n|\sim 2^j}|g_n|^2>K^2\big).\tag{3.15}$$

Let us now recall the so called *Cramér condition*: a sequence $\{\xi_n\}$ of independent identically distributed (i.i.d.) random variables is said to satisfy Cramér's condition if there exists $\lambda>0$ such that

$$\varphi(\lambda)=\mathbb{E}\big[e^{\lambda|\xi_1|}\big]<\infty.$$

If the condition holds, then we can define *the Cramér transform*

$$H(a)=\sup_{\lambda>0}\{a\lambda-\psi(\lambda)\},$$

with $\psi(\lambda)=\ln\varphi(\lambda)$, and, for $a>\mathbb{E}\big[\xi_1\big]$, we have

$$Pr\Big(\frac{1}{N}\sum_{n=1}^{N}\xi_n\ge a\Big)\le e^{-nH(a)};\tag{3.16}$$

see Shiryaev's book [46]. With $\xi_n = |g_n|^p$ (note that g_n is complex-valued), we see that Cramér's condition is satisfied for $p = 2$. Indeed, when $p = 2$, we have

$$\mathbb{E}\big[e^{\lambda|g_n|^2}\big] = \frac{1}{2\pi}\int e^{(\lambda-\frac{1}{2})|g_n|^2}dg_n = \frac{1}{2\pi(1-2\lambda)}$$

for $\lambda < \frac{1}{2}$. Then, $H(a) = \sup_{\lambda>0}\{a\lambda + \ln(1-2\lambda) + \ln(2\pi)\}$ has the maximum value $\frac{a-2}{2} + \ln\frac{2}{a} + \ln(2\pi)$ at $\lambda = \frac{a-2}{2a}$. Then, from (3.16), we have

$$Pr\big(2^{-j}\sum_{|n|\sim 2^j}|g_n|^2 > K^2\big) < e^{-c2^jK^2}. \tag{3.17}$$

Hence, we have $Pr(\|u\|_{\widehat{b}^{\frac{1}{2}}_{2,\infty}(\mathbb{T})} > K) < e^{-cK^2}$ in view of (3.15). This proves (3.13) for $p \le 2$.

Note that the Cramér condition no longer holds for $p > 2$. Thus, we need another approach. There are known large deviation results even when the Cramér's condition fails; see, for example, Saulis and Nakas' work [43]. However, they do not seem to be directly applicable to obtain (3.13). Instead, we use the hypercontractivity of the Ornstein-Uhlenbeck semigroup related to products of Gaussian random variables. For the following discussion, see the works of Kuo [29], Ledoux-Talagrand [31], and Janson [25]. A nice summary is given by Tzvetkov in [50, Section 3].

In our discussion, we will use the Hermite polynomials $H_n(x)$. They are defined by

$$e^{tx-\frac{1}{2}t^2} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n.$$

The first three Hermite polynomials are: $H_0(x) = 1$, $H_1(x) = x$, and $H_2(x) = x^2 - 1$.

Now, consider the Hilbert space $H = L^2(\mathbb{R}^m, \mu_m)$ with $d\mu_m = (2\pi)^{-\frac{m}{2}}\exp(-|x|^2/2)dx$, $x = (x_1, \dots, x_m) \in \mathbb{R}^m$. We define a *homogeneous Wiener chaos of order* n to be an element of the form $\prod_{j=1}^m H_{n_j}(x_j)$, $n = n_1 + \cdots + n_m$. Consider the Hartree-Fock operator $L = \Delta - x\cdot\nabla$, which is the generator for the Ornstein-Uhlenbeck semigroup. Then, by the hypercontractivity of the Ornstein-Uhlenbeck semigroup $S(t) = e^{Lt}$, we have the following

Lemma 3.10. *Fix $q \ge 2$. Then, for every $u \in H$ and $t \ge \frac{1}{2}\log(q-1)$, we have*

$$\|S(t)u\|_{L^q(\mathbb{R}^m,\mu_m)} \le \|u\|_{L^2(\mathbb{R}^m,\mu_m)}. \tag{3.18}$$

Note that (3.18) holds, independent of the dimension m. It is known that the eigenfunction of L with eigenvalue $-n$ is precisely the homogeneous Wiener chaos of order n. Thus, we have

Lemma 3.11. *Let $F(x)$ be a linear combination of homogeneous chaoses of order n. Then, for $q \ge 2$, we have*

$$\|F(x)\|_{L^q(\mathbb{R}^m,\mu_m)} \le (q-1)^{\frac{n}{2}}\|F(x)\|_{L^2(\mathbb{R}^m,\mu_m)}. \tag{3.19}$$

The proof is basically the same as in [50, Propositions 3.3–3.5]. We only have to note that $F(x)$ is an eigenfunction of $S(t) = e^{Lt}$ with eigenvalue e^{-nt}. Then, (3.19) follows from (3.18) by evaluating (3.18) at time $t = \frac{1}{2}\log(q-1)$.

Denote now by $\mathcal{K}_n$ the collection of the homogeneous chaoses of order n. Given a homogeneous polynomial $P_n(x) = P_n(x_1, \dots, x_m)$ of degree n, we define *the Wick ordered monomial* $:P_n(x):$ to be its projection onto $\mathcal{K}_n$. In particular, we have $: x_j^n := H_n(x_j)$ and $: \prod_{j=1}^m x_j^{n_j} := \prod_{j=1}^m H_{n_j}(x_j)$ with $n = n_1 + \cdots + n_m$.

Since the Fourier coefficients of Brownian motion involve complex Gaussian random variables, let us consider the Wick ordering on them as well. Let g denote a standard complex-valued Gaussian random variable. Then, g can be written as $g = x + iy$, where x and y are independent standard real-valued Gaussian random variables. Note that the variance of g is $\text{Var}(g) = 2$. Next, we investigate the Wick ordering on $|g|^{2n}$ for $n \in \mathbb{N}$, that is, the projection of $|g|^{2n}$ onto $\mathcal{K}_{2n}$.

When $n = 1$, $|g|^2 = x^2 + y^2$ is Wick-ordered into

$$: |g|^2 := (x^2 - 1) + (y^2 - 1) = |g|^2 - \text{Var}(g).$$

When $n = 2$, $|g|^4 = (x^2 + y^2)^2 = x^4 + 2x^2y^2 + y^4$ is Wick-ordered into

$$\begin{aligned} : |g|^4 : &= (x^4 - 6x^2 + 3) + 2(x^2 - 1)(y^2 - 1) + (y^4 - 6y^2 + 3) \\ &= x^4 + 2x^2y^2 + y^4 - 8(x^2 + y^2) + 8 \\ &= |g|^4 - 4\text{Var}(g)|g|^2 + 2\text{Var}(g)^2, \end{aligned}$$

where we used $H_4(x) = x^4 - 6x^2 + 3$.

When $n = 3$, $|g|^6 = (x^2 + y^2)^3 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6$ is Wick-ordered into

$$\begin{aligned} : |g|^6 : &= (x^6 - 15x^4 + 45x^2 - 15) + 3(x^4 - 6x^2 + 3)(y^2 - 1) \\ &\quad + 3(x^2 - 1)(y^4 - 6y^2 + 3) + (y^6 - 15y^4 + 45y^2 - 15) \\ &= |g|^6 - 9\text{Var}(g)|g|^4 + 18\text{Var}(g)^2|g|^2 - 6\text{Var}(g)^3, \end{aligned}$$

where we used $H_6(x) = x^6 - 15x^4 + 45x^2 - 15$.

In general, we have $: |g|^{2n} : \in \mathcal{K}_{2n}$. Moreover, we have

$$: |g|^{2n} : = |g|^{2n} + \sum_{j=0}^{n-1} a_j |g|^{2j} = |g|^{2n} + \sum_{j=0}^{n-1} b_j : |g|^{2j} : . \tag{3.20}$$

This follows from the fact that $|g|^{2n}$, as a polynomial in x and y only with even powers, is orthogonal to any homogeneous chaos of odd order, and it is radial, i.e., it depends only on $|g|^2 = x^2 + y^2$. Note that $: |g|^{2n} :$ can also be obtained from the Gram-Schmidt process applied to $|g|^{2k}$, $k = 0, \dots, n$ with $\mu_2 = (2\pi)^{-1} \exp(-(x^2 + y^2)/2)dxdy$.

With these preliminaries, we are ready to return to the proof of the large deviation estimate (3.13) for $p > 2$. Given $p > 2$, choose k such that $p \le 2k$. As in (3.14), by Hölder inequality with $\frac{1}{r} = \frac{1}{p} - \frac{1}{2k}$, we have

$$\big\|2^{-\frac{j}{p}} |g_n|\big\|_{\ell^p_{|n| \sim 2^j}} \lesssim \big\|2^{-\frac{j}{2k}} |g_n|\big\|_{\ell^{2k}_{|n| \sim 2^j}}, \tag{3.21}$$

i.e., we have $\|u\|_{\widehat{b}^s_{p,\infty}(\mathbb{T})} \lesssim \|u\|_{\widehat{b}^{1-\frac{1}{2k}}_{2k,\infty}(\mathbb{T})}$ for $(s-1)p = -1$. Hence, it suffices to prove (3.13) for $p = 2k$ and $s = 1 - \frac{1}{2k}$. Let

$$F_j(\omega) = 2^{-j} \sum_{|n| \sim 2^j} |g_n(\omega)|^{2k}. \tag{3.22}$$

Then, we have

$$Pr(\sup_j |F_j| > K^{2k}) \le \sum_{j=0}^{\infty} Pr(|F_j| > K^{2k}).$$

Hence, it suffices to prove

$$\sum_{j=0}^{\infty} Pr(|F_j| > K^{2k}) < e^{-cK^2}. \tag{3.23}$$

By (3.20), write F_j as a linear combination of homogeneous chaoses of order 2ℓ, $\ell = 0, 1, \dots, k$, i.e., we have $F_j = \sum_{\ell=0}^{k} F_j^{(\ell)}$, where $F_j^{(\ell)}$ is the component of F_j projected onto $\mathcal{K}_{2\ell}$. Then, it suffices to prove

$$\sum_{j=0}^{\infty} Pr(|F_j^{(\ell)}| > \tfrac{1}{k+1} K^{2k}) < e^{-cK^2} \tag{3.24}$$

for each $\ell = 0, 1, \dots, k$. By choosing K sufficiently large, we see that (3.24) trivially holds for $\ell = 0$, since $F_j^{(0)}$ is a constant independent of j and thus the left-hand side of (3.24) is 0 for large K. For $\ell \geq 1$, it follows from Lemma 3.11 that, for $q \geq 2$,

$$\|F_j^{(\ell)}\|_{L^q(\Omega)} \leq C_\ell q^\ell \|F_j^{(\ell)}\|_{L^2(\Omega)} = C'_\ell 2^{-\frac{j}{2}} q^\ell \tag{3.25}$$

where the constants C_ℓ and C'_ℓ are independent of j.

Let us now recall the following

Lemma 3.12 (Lemma 4.5 in [50]). *Suppose that we have, for all $q \geq 2$,*

$$\|F(\omega)\|_{L^q(\Omega)} \leq C N^{-\alpha} q^{\frac{n}{2}}$$

for some $\alpha, N, C > 0$ and $n \in \mathbb{N}$. Then, there exist c and C' depending on C and n but independent *of α and N such that*

$$Pr(|F(\omega)| > \lambda) \leq C' e^{-cN^{\frac{2\alpha}{n}} \lambda^{\frac{2}{n}}}.$$

Thus, from (3.25) and Lemma 3.12 with $n = 2\ell$, $N = 2^j$, $\alpha = \frac{1}{2}$, and $\lambda = \frac{1}{k+1} K^{2k}$, we have

$$Pr(|F_j^{(\ell)}| > \tfrac{1}{k+1} K^{2k}) < e^{-c_k 2^{\frac{j}{2\ell}} K^{\frac{2k}{\ell}}} < e^{-c_k 2^{\frac{j}{2\ell}} K^2}.$$

This establishes (3.24), and hence (3.23) and (3.13).

Remark 3.13. With $(s-1)p = -1$, we have

$$\mathbb{E}[\|u_j\|^p_{\widehat{b}^s_{p,\infty}(\mathbb{T})}] \sim \mathbb{E}[X_j^{(p)}] = c_p,$$

where $u_j = \mathbb{P}_{|n| \sim 2^j} u$, and $X_j^{(p)}$ is defined in (2.5). Also, note that

$$\mathbb{E}[F_j] = \mathbb{E}[F_j^{(0)}] = F_j^{(0)} = c_p.$$

Hence, it follows from the above computation for $\ell = 1, \dots, k$ that

$$Pr\big(\big|\|u\|_{\widehat{b}^s_{p,\infty}(\mathbb{T})} - c_p^{\frac{1}{p}}\big| > K\big) < e^{-cK^2} \tag{3.26}$$

$$Pr\big(\big|\|\mathbb{P}_{|n| \geq 2^N} u\|_{\widehat{b}^s_{p,\infty}(\mathbb{T})} - c_p^{\frac{1}{p}}\big| > K\big) < e^{-c2^{\frac{N}{p}} K^2}. \tag{3.27}$$

In probability theory, large deviation estimates are commonly stated as in the estimates (3.26) and (3.27). However, in applications to partial differential equations, it is more common to encounter these estimate in the form (3.7); see [4, 5, 6, 7, 8, 10, 32, 34, 35, 37].

3.3. **Large deviation estimates for $B^{\frac{1}{2}}_{p,\infty}$.** Lastly, we briefly discuss the large deviation estimates on the Besov spaces $B^s_{p,q}$ with the endpoint regularity $s = \frac{1}{2}$, $p < \infty$, and $q = \infty$:

$$Pr(\|u\|_{B^{\frac{1}{2}}_{p,\infty}} > K) < e^{-cK^2} \tag{3.28}$$

for some $c = c(p)$ and all sufficiently large $K \geq K_p$. For the non-endpoint result, the reader is referred to [41].

For $p \leq 2$, (3.28) follows from (3.13) once we note that $\|u\|_{B^{\frac{1}{2}}_{p,\infty}} \leq \|u\|_{B^{\frac{1}{2}}_{2,\infty}} = \|u\|_{\widehat{b}^{\frac{1}{2}}_{2,\infty}}$. When $p > 2$, (3.23) does not follow from (3.13) anymore. But, as in the proof of (3.13), it suffices to consider only the case $p = 2k$, $k \geq 2$. The proof for a general even index p involves lots of unwieldy technicalities. In the following we will sketch the argument for $p = 4$.

When $p = 4$, we have

$$\|u\|^4_{B^{\frac{1}{2}}_{4,\infty}} = \sup_j \Big\|2^{-\frac{j}{2}} \sum_{|n|\sim 2^j} g_n e^{int}\Big\|^4_{L^4_t} = \sup_j \big(\mathrm{I}^{(2)}_j + \mathrm{II}^{(2)}_j + \mathrm{III}^{(2)}_j\big),$$

where $\mathrm{I}^{(2)}_j$, $\mathrm{II}^{(2)}_j$, and $\mathrm{III}^{(2)}_j$ are defined in (2.16). In the following, we treat them separately.

First, note that

$$\big\{\omega : \mathrm{I}^{(2)}_j(\omega) > 2K^4\big\} \subset \bigcup_{\ell=1}^{2} \Big\{\omega : 2^{-j} \sum_{|n_\ell|\sim 2^j} |g_{n_\ell}(\omega)|^2 > K^2\Big\}.$$

Then, from (3.17), we have

$$Pr\big(\sup_j \mathrm{I}^{(2)}_j(\omega) > K^4\big) < e^{-cK^2}. \tag{3.29}$$

Next, note that $\mathrm{III}^{(2)}_j(\omega) > K^4$ if and only if $F_j(\omega) > 2^j K^4$, where $F_j(\omega)$ is defined in (3.22) with $k = 2$. Hence, from (3.23), we have

$$Pr\big(\sup_j \mathrm{III}^{(2)}_j(\omega) > K^4\big) < e^{-cK^2}. \tag{3.30}$$

Lastly, by expanding the complex Gaussians into their real and imaginary parts, it is not difficult to see that $\mathrm{II}^{(2)}_j$ is a homogeneous Wiener chaos of order 4 since each term in the sum is a product of four independent real-valued Gaussian random variables. Then, it follows from Lemma 3.11 that, for $q \geq 2$,

$$\|\mathrm{II}^{(2)}_j\|_{L^q(\Omega)} \leq Cq^2 \|\mathrm{II}^{(2)}_j\|_{L^2(\Omega)} = C' 2^{-\frac{j}{2}} q^2$$

where the constants C and C' are independent of j. Thus, from Lemma 3.12, we have $Pr(|\mathrm{II}^{(2)}_j| > K^4) < e^{-c2^{\frac{j}{4}}K^2}$. This immediately implies

$$Pr\big(\sup_j \big|\mathrm{II}^{(2)}_j(\omega)\big| > K^4\big) < e^{-cK^2}. \tag{3.31}$$

The large deviation estimate (3.28) follows from (3.29), (3.30), and (3.31).

For a general even index p, one needs to repeat the above argument, using (2.19). The estimates on $\mathrm{I}^{(k)}_j$ and $\mathrm{II}^{(k)}_j$ follow easily as before. In particular, note that $\mathrm{II}^{(k)}_j$ is a homogeneous Wiener chaos of order $2k$. One can then estimate the error terms by a combination of the arguments presented above. However, the actual computation becomes lengthy, and thus we omit details.

Appendix A. Brownian motion and Fourier-Wiener series

We present here a proof of the Fourier-Wiener series representation (2.2) of the mean zero Brownian loop $u(t)$ on $[0, 2\pi)$.

Let $b(t)$ be the complex-valued Brownian motion on $\mathbb{R}_+$ and $\beta(t) = b(t) - tb(2\pi)/2\pi$ be the corresponding periodic Brownian loop on $[0, 2\pi)$. Then, for $t \in [0, 2\pi)$, we have

$$d\beta = db - \frac{b(2\pi)}{2\pi} dt. \tag{A.1}$$

For $t \in [0, 2\pi)$, we write them in terms of the Wiener integrals:

$$b(t) = \int_{\mathbb{R}_+} \chi_{[0,t)} db(t') \quad \text{and} \quad \beta(t) = \int_0^{2\pi} \chi_{[0,t)} db(t') - \frac{b(2\pi)}{2\pi} \int_0^{2\pi} \chi_{[0,t)} dt'.$$

Given a (deterministic) periodic function f on $[0, 2\pi)$, we write $f = f_0 + f_1$ where $f_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t)dt$ and $f_1 = f - f_0$. Note that $\int_0^{2\pi} f_1(t)dt = 0$.

Now, define a conjugate[2] linear operator $T : L^2([0, 2\pi]) \to L^2(\Omega)$ given by

$$T(f)(\omega) := \int_0^{2\pi} \overline{f}(t) d\beta(t; \omega).$$

Then, from (A.1), we have

$$\begin{aligned} T(f) &= \int_0^{2\pi} (\overline{f}_0 + \overline{f}_1)(t) db(t) - \frac{b(2\pi)}{2\pi} \int_0^{2\pi} (\overline{f}_0 + \overline{f}_1)(t) dt \\ &= \int_0^{2\pi} \overline{f}_1(t) db(t) + \overline{f}_0 \int_0^{2\pi} db(t) - b(2\pi)\overline{f}_0 = \int_0^{2\pi} \overline{f}_1(t) db(t), \end{aligned}$$

where the last equality holds almost surely in ω since $b(0) = 0$ a.s. Thus, we see that $T(f)$ is given by the Wiener integral of the mean zero part $\overline{f}_1$. Hence, $T(f)$ is a Gaussian random variable with mean 0 and variance $2\|f_1\|^2_{L^2([0,2\pi])}$.[3] Namely, we have

$$\|T(f)\|_{L^2(\Omega)} = \sqrt{2}\|f_1\|_{L^2([0,2\pi])}.$$

Moreover, T behaves like a (conjugate) unitary operator from $L^2([0, 2\pi])$ onto $L^2(\Omega)$, that is

$$\mathbb{E}\big[T(f)\overline{T(g)}\big] = 2\overline{\langle f_1, g_1 \rangle}_{L^2}. \tag{A.2}$$

On the one hand, we have $\beta(t) = T(\chi_{[0,t)})$. On the other hand, we have

$$\chi_{[0,\tau)}(t) = \sum_{n \in \mathbb{Z}} a_n \overline{e}_n(t),$$

where $e_n(t) = (2\pi)^{-\frac{1}{2}} e^{int}$ and $a_n = \frac{e^{in\tau} - 1}{\sqrt{2\pi} in}$. Thus, we have

$$\beta(t) = T(\chi_{[0,t)}) = \sum_{n \neq 0} a_n g_n = \sum_{n \neq 0} \frac{g_n}{\sqrt{2\pi} in} e^{int} - \sum_{n \neq 0} \frac{g_n}{\sqrt{2\pi} in}, \tag{A.3}$$

[2]Instead of a linear operator, we have a conjugate linear operator due to the complex-valued setting.

[3]The factor 2 appears since we are considering the complex-valued Brownian motion. See Kuo's book [29] for properties of the (real-valued) Wiener integrals.

where $g_n = T(\overline{e}_n)$, $n \neq 0$, is a Gaussian random variable with mean 0 and variance $2\|e_n\|_{L^2}^2 = 2$. (Note that $T(e_0) = 0$.) Moreover, they are mutually independent from (A.2) and orthogonality of $\{e_n\}$. Hence, the original Brownian motion $b(t)$ can be represented as

$$b(t) = g_0 t + \sum_{n\neq 0} \frac{g_n}{\sqrt{2\pi} in} e^{int} - \sum_{n\neq 0} \frac{g_n}{\sqrt{2\pi} in},$$

where $g_0 = b(2\pi)/2\pi$ is a Gaussian random variable with mean 0 and variance 2. Also, by writing $g_0 = \frac{1}{2\pi}\int_0^{2\pi} db(t)$ a.s. and $g_n = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-int} db(t)$ for $n \neq 0$, it follows that g_0 is independent from $\{g_n\}_{n\neq 0}$.

Lastly, by subtracting the (spatial) mean of $\beta(t)$ over $[0, 2\pi)$ from (A.3), we obtain the Fourier-Wiener series representation of the mean zero periodic Brownian loop $u(t)$:

$$u(t) = \sum_{n\neq 0} \frac{g_n}{\sqrt{2\pi} in} e^{int}, \tag{A.4}$$

which is (2.2) up to the constant factor $\sqrt{2\pi} i$.

Remark A.1. Recall the following definition [25]. A Gaussian field on a Hilbert space H is a (conjugate) linear isometry (up to a multiplicative constant) of H into some Gaussian space. In the previous discussion, we constructed such a conjugate linear isometry in a concrete manner.

Appendix B. Brownian motion on the real line

In Remark 2.3, we defined the local-in-time versions of the time-frequency functions spaces in the following way. Given an interval $I \subset \mathbb{R}$, we let $M_s^{p,q}(I)$ be the restriction of $M_s^{p,q}(\mathbb{R})$ onto I via

$$\|u\|_{M_s^{p,q}(I)} = \inf\big\{\|v\|_{M_s^{p,q}(\mathbb{R})} : v = u \text{ on } I\big\}. \tag{B.1}$$

We define the local-in-time versions of other function spaces in an analogous manner.

In the following, we will show that, given a bounded interval I, Theorem 2.1 holds for the Brownian motion $b(t)$ on $\mathbb{R}$ in $M_s^{p,q}(I)$, $W_s^{p,q}(I)$, and $\widehat{b}^s_{p,q}(I)$. Given the complex-valued Brownian motion $b(t)$ on $\mathbb{R}$, it is known that $\sqrt{2a}b(t/a)$ and $b(t+t_0) - b(t_0)$, $a \neq 0$ and $t_0 \in \mathbb{R}$, are also Brownian motions. Hence, it suffices to show that $b(t)$ is bounded or unbounded a.s. in $M_s^{p,q}$, $W_s^{p,q}$, and $\widehat{b}^s_{p,q}$ restricted to some fixed bounded interval I.

We start by investigating the Brownian motion on local modulation spaces. We will repeatedly use the following proposition, the proof of which is deferred to the end of this appendix.

Proposition B.1. *Let f be a function on $\mathbb{R}$, and ϕ be a smooth cutoff function supported on $[0, 2\pi)$. Then, for $1 \le p, q \le \infty$ and $s \in \mathbb{R}$, we have*
(a)

$$\|\phi f\|_{\mathcal{F}L^{s,q}(\mathbb{T})} \lesssim \|f\|_{M_s^{p,q}(\mathbb{R})}. \tag{B.2}$$

(b)

$$\|\phi f\|_{M_s^{p,q}(\mathbb{R})} \lesssim \|f\|_{\mathcal{F}L^{s,q}(\mathbb{T})}. \tag{B.3}$$

From [42], we have

$$\|f\|_{M_s^{p,q}(\mathbb{R})} \sim \|f\|_{W_s^{p,q}(\mathbb{R})}, \quad 1 \le p, q \le \infty, \tag{B.4}$$

for any function f supported on a bounded interval I, where the implicit constant depends on p, q, and $|I|$.

B.1. **Boundedness of Brownian motion.** Recall that we are free to select the fixed interval I. In this case, we take $I = [\frac{1}{2}\pi, \frac{3}{2}\pi)$. Let ϕ be a smooth cutoff function supported on $[\frac{1}{4}\pi, \frac{7}{4}\pi)$ such that $\phi(t) = 1$ on I. Also, let $\widetilde{\phi}$ be a smooth cutoff function supported on $[0, 2\pi)$ such that $\widetilde{\phi}(t) = 1$ on $\operatorname{supp}\phi$. Note that $\phi = \phi\widetilde{\phi}$. Then, by Proposition B.1 (b), we have

$$\begin{aligned}\|f\|_{M^{p,q}_s(I)} &\leq \|\phi f\|_{M^{p,q}_s(\mathbb{R})} = \|\phi\widetilde{\phi} f\|_{M^{p,q}_s(\mathbb{R})} \lesssim \|\widetilde{\phi} f\|_{\mathcal{F}L^{s,q}(\mathbb{T})} \\ &\leq \|f\|_{\mathcal{F}L^{s,q}(\mathbb{T})} \sim \|f\|_{M^{p,q}_s(\mathbb{T})}, \end{aligned} \tag{B.5}$$

where the first inequality follows from the definition of the localized space and the last inequality follows from the boundedness of the multiplication by a smooth function (supported on $[0, 2\pi)$.) By (B.4) and (B.5), we also have

$$\|f\|_{W^{p,q}_s(I)} \lesssim \|f\|_{\mathcal{F}L^{s,q}(\mathbb{T})} \sim \|f\|_{W^{p,q}_s(\mathbb{T})}. \tag{B.6}$$

Now, let $\beta(t)$ be the periodic part of $b(t)$ on I, i.e.

$$\beta(t) = b(t) - b(\tfrac{1}{2}\pi) + \frac{b(\frac{3}{2}\pi) - b(\frac{1}{2}\pi)}{\pi}(t - \tfrac{1}{2}\pi).$$

Then, we have

$$\|b\|_{M^{p,q}_s(I)} \leq \|\beta\|_{M^{p,q}_s(I)} + C_\omega(I), \tag{B.7}$$

where $C_\omega(I) < \infty$ a.s. Then, from Theorem 2.1 with (B.5) and (B.7), we have

$$\begin{aligned}\|b\|_{M^{p,q}_s(I)} &\leq \|\beta\|_{M^{p,q}_s(I)} + C_\omega(I) \lesssim \|\phi\beta\|_{M^{p,q}_s(\mathbb{R})} + C_\omega(I) \\ &\lesssim \|\beta\|_{M^{p,q}_s(\mathbb{T})} + C_\omega(I) < \infty, \quad \text{a.s.}\end{aligned}$$

for $(s-1)q < -1$ with $q < \infty$ (and $s < 1$ when $q = \infty$.) The same boundedness result holds in $W^{p,q}_s(I)$ under the same condition.

B.2. **Unboundedness of Brownian motion.** In the following, we establish unboundedness of the Brownian motion on $\mathbb{R}$ in the function spaces restricted to bounded intervals on $\mathbb{R}$. Due to the definition (B.1) of the function spaces restricted to an interval, it is more difficult to establish a lower bound on the localized norms. (In order to to establish an upper bound on the localized norms, it suffices to show an upper bound on a single representation - see (B.5), whereas we need to show that a uniform lower bound exists for *all* representations in establishing a lower bound on the localized norms.) The trick we use is the following. Given a periodic function of period T, we establish a lower bound on the norms restricted to the interval of length $2T$.

Take $I = [0, \pi)$ such that $2I = [0, 2\pi) = \mathbb{T}$. Let ϕ_1 be a smooth cutoff function supported on $[\frac{1}{2}\pi - \varepsilon, \frac{3}{2}\pi + \varepsilon)$ with $\phi(t) = 1$ on $[\frac{1}{2}\pi + \varepsilon, \frac{3}{2}\pi - \varepsilon)$ for some small $\varepsilon > 0$ such that $\phi_1 + \phi_2 \equiv 1$ on $\mathbb{T}$, where $\phi_2(t) := \phi_1(t - \pi)$ on the periodic domain $\mathbb{T}$. Also, let $\widetilde{\phi}_1$ be a smooth cutoff function supported on $\mathbb{T}$ such that $\widetilde{\phi}_1(t) = 1$ on $\operatorname{supp}\phi_1$. Note that $\phi_1 = \phi_1\widetilde{\phi}_1$.

Let f be a periodic function of period π. Then, by Proposition B.1 (a), we have

$$\|f\|_{M^{p,q}_s(\mathbb{T})} \sim \|f\|_{\mathcal{F}L^{s,q}(\mathbb{T})} = \|(\phi_1 + \phi_2) f\|_{\mathcal{F}L^{s,q}(\mathbb{T})} \leq \|\phi_1 f\|_{\mathcal{F}L^{s,q}(\mathbb{T})} + \|\phi_2 f\|_{\mathcal{F}L^{s,q}(\mathbb{T})}$$

which is, noting that $\phi_2(t) f(t) = \phi_1(t - \pi) f(t - \pi)$,

$$= 2\|\phi_1\widetilde{\phi}_1 f\|_{\mathcal{F}L^{s,q}(\mathbb{T})} \lesssim \|\widetilde{\phi}_1 f\|_{M^{p,q}_s(\mathbb{R})}.$$

Let g any function on $\mathbb{R}$ such that $g = f$ on $2I = [0, 2\pi)$. Then, we have $\widetilde{\phi}_1 f = \widetilde{\phi}_1 g$. By the boundedness of the multiplication by a smooth function, we obtain

$$\|f\|_{M^{p,q}_s(\mathbb{T})} \lesssim \|\widetilde{\phi}_1 g\|_{M^{p,q}_s(\mathbb{R})} \lesssim \|g\|_{M^{p,q}_s(\mathbb{R})},$$

for any extension g. Hence, we have

$$\|f\|_{M^{p,q}_s(\mathbb{T})} \lesssim \|f\|_{M^{p,q}_s(2I)}, \tag{B.8}$$

for a periodic function f of period π. By the translation invariance property of the norm, we obtain

$$\|f\|_{M^{p,q}_s(\mathbb{T})} \lesssim \|f\|_{M^{p,q}_s(2I)} \leq \|f\|_{M^{p,q}_s(I)} + \|f\|_{M^{p,q}_s(I+\pi)} \lesssim \|f\|_{M^{p,q}_s(I)}. \tag{B.9}$$

By (B.4) and (B.9), we also have

$$\|f\|_{W^{p,q}_s(\mathbb{T})} \lesssim \|f\|_{W^{p,q}_s(I)}, \tag{B.10}$$

for a periodic function f of period π.

Now, let $\beta(t)$ be the periodic part of $b(t)$ on $[0, \pi)$, i.e. $\beta(t) = b(t) - tb(\pi)/\pi$. Then, we have

$$\|b\|_{M^{p,q}_s(I)} \geq \|\beta\|_{M^{p,q}_s(I)} - C_\omega(I), \tag{B.11}$$

where $C_\omega(I) < \infty$ a.s. Then, from Theorem 2.1 with (B.9) and (B.11), we have

$$\|b\|_{M^{p,q}_s(I)} \geq \|\beta\|_{M^{p,q}_s(I)} - C_\omega(I) \gtrsim \|\beta\|_{M^{p,q}_s(\mathbb{T})} - C_\omega(I) = \infty, \quad \text{a.s.}$$

for $(s-1)q \geq -1$ with $q < \infty$ (and $s \geq 1$ when $q = \infty$.) The same unboundedness result holds in $W^{p,q}_s(I)$ under the same condition.

A similar argument with the following proposition shows the boundedness or unboundedness of the Brownian motion in $\widehat{b}^s_{p,q}(I)$ for any bounded interval I.

Proposition B.2. *Let f be a function on $\mathbb{R}$, and ϕ be a smooth cutoff function supported on $[0, 2\pi)$. Then, for $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$, we have*
(a)

$$\|\phi f\|_{\widehat{b}^s_{p,q}(\mathbb{T})} \lesssim \|f\|_{\widehat{b}^s_{p,q}(\mathbb{R})}. \tag{B.12}$$

(b)

$$\|\phi f\|_{\widehat{b}^s_{p,q}(\mathbb{R})} \lesssim \|f\|_{\widehat{b}^s_{p,q}(\mathbb{T})}. \tag{B.13}$$

We conclude this appendix and our paper by presenting the proofs of Propositions B.1 and B.2.

Proof of Proposition B.1. (a) First, recall the support function ψ in the definition of the modulation spaces, i.e. ψ is a smooth cutoff function supported on $[-1, 1]$ such that $\sum_{n\in\mathbb{Z}} \psi(\xi - n) \equiv 1$.

Now, let $\eta(\xi)$ be a smooth cutoff function supported on $[-2, 2]$ such that $\eta(\xi) = 1$ on $[-1, 1]$. Then, we have $\psi(\xi)\eta(\xi) = \psi(\xi)$ for $\xi \in \mathbb{R}$. For $n \in \mathbb{Z}$, we let T_n and M_n denote the translation and modulation operators respectively, that is, $T_n\eta(\xi) = \eta(\xi - n)$ and

$M_n\phi(y) = \phi(y)e^{iny}$. Then, we have

$$\begin{aligned}
\widehat{\phi f}(n) &= \int_0^{2\pi} \phi(y)f(y)e^{-iny}dy = \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \psi(D-m)f(y)\overline{M_n\phi(y)}dy \\
&= \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \psi(\xi-m)\widehat{f}(\xi)\overline{\widehat{M_n\phi}(\xi)}d\xi = \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \psi(\xi-m)\widehat{f}(\xi)\overline{T_m\eta(\xi)\widehat{M_n\phi}(\xi)}d\xi \\
&= \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \psi(D-m)f(y)\overline{\widehat{T_m\eta} * M_n\phi(y)}dy.
\end{aligned}$$

First, we prove (B.2) for $s \geq 0$. For $m \neq n$, repeated integration by parts gives

$$\begin{aligned}
|\widehat{T_m\eta} * M_n\phi(x)| &= \bigg| \int_{\mathbb{R}} \widehat{\eta}(x-y)e^{im(x-y)}\phi(y)e^{iny}dy \bigg| = \bigg| \int_{\mathbb{R}} \widehat{\eta}(x-y)\phi(y)e^{i(n-m)y}dy \bigg| \\
&\lesssim \frac{1}{\langle n-m\rangle^{s+2}} \bigg| \int_{\mathbb{R}} \frac{1}{\langle x-y\rangle^2}\widetilde{\phi}(y)dy \bigg|
\end{aligned}$$

for some $\widetilde{\phi} \in \mathcal{S}(\mathbb{R})$. Thus, we have

$$\|\widehat{T_m\eta} * M_n\phi\|_{L^{p'}(\mathbb{R})} \lesssim \frac{1}{\langle n-m\rangle^{s+2}}, \quad m \neq n, \tag{B.14}$$

where p' is the Hölder conjugate exponent of $p \in [1,\infty]$. By Hölder inequality and (B.14), we have

$$\begin{aligned}
|\widehat{\phi f}(n)| &= \sum_{m\in\mathbb{Z}} \|\psi(D-m)f\|_{L^p(\mathbb{R})} \|\widehat{T_m\eta} * M_n\phi(y)\|_{L^{p'}(\mathbb{R})} \\
&\lesssim \sum_{m\in\mathbb{Z}} \frac{1}{\langle n-m\rangle^{s+2}} \|\psi(D-m)f\|_{L^p(\mathbb{R})}.
\end{aligned} \tag{B.15}$$

By $\langle n\rangle^s \lesssim \langle n-m\rangle^s\langle m\rangle^s$ for $s \geq 0$, (B.15) and Young's inequality, we obtain

$$\begin{aligned}
\|\phi f\|_{\mathcal{F}L^{s,q}(\mathbb{T})} = \|\langle n\rangle^s \widehat{\phi f}(n)\|_{\ell^q(\mathbb{Z})} &\lesssim \bigg\| \sum_{m\in\mathbb{Z}} \frac{1}{\langle n-m\rangle^2} \langle m\rangle^s \|\psi(D-m)f\|_{L^p(\mathbb{R})} \bigg\|_{\ell^q_n(\mathbb{Z})} \\
&\lesssim \big\| \langle n\rangle^s \|\psi(D-n)f\|_{L^p(\mathbb{R})} \big\|_{\ell^q_n(\mathbb{Z})} \lesssim \|f\|_{M^{p,q}_s(\mathbb{R})}.
\end{aligned}$$

Next, assume $s < 0$. In this case, we have $\langle n\rangle^s\langle n-m\rangle^s \lesssim \langle m\rangle^s$. By repeating the previous argument, we have

$$\|\widehat{T_m\eta} * M_n\phi\|_{L^{p'}(\mathbb{R})} \lesssim \frac{1}{\langle n-m\rangle^{-s+2}}, \quad m \neq n.$$

Then, the rest follows as before.

(b) Recall that ϕ is a smooth cutoff function supported on $[0, 2\pi)$. Let $n \in \mathbb{Z}$. Then, we have

$$\begin{aligned}
\psi(D-n)(\phi f)(x) &= \int_{\mathbb{R}} \psi(\xi - n)\widehat{\phi f}(\xi) e^{i\xi x} d\xi = \int_{\mathbb{R}} \psi(\xi - n) \int_0^{2\pi} \phi(y) f(y) e^{-i\xi y} dy\, e^{i\xi x} d\xi \\
&= \int_0^{2\pi} \phi(y) f(y) \int_{\mathbb{R}} \psi(\xi - n) e^{i\xi(x-y)} d\xi dy \\
&= e^{inx} \int_0^{2\pi} f(y)\phi(y)\widehat{\psi}(x-y) e^{-iny} dy \\
&= e^{inx} \int_0^{2\pi} \sum_{m\in\mathbb{Z}} \widehat{f}(n-m) e^{i(n-m)y} \phi(y)\widehat{\psi}(x-y) e^{-iny} dy \\
&= e^{inx} \sum_{m\in\mathbb{Z}} \widehat{f}(n-m) F_m(x),
\end{aligned} \tag{B.16}$$

where $\widehat{f}(n-m)$ are the Fourier coefficients of f (as a function on $\mathbb{T}$) at $n-m$ and

$$F_m(x) = \int_{\mathbb{R}} \phi(y)\widehat{\psi}(x-y) e^{-imy} dy.$$

First, we prove (B.3) for $s \geq 0$. By Young's inequality, we have

$$\|F_0(x)\|_{L^p(\mathbb{R})} = \bigg\| \int_{\mathbb{R}} \phi(y)\widehat{\psi}(x-y) dy \bigg\|_{L^p_x(\mathbb{R})} \lesssim 1. \tag{B.17}$$

For $m \neq 0$, by repeatedly integrating by parts as before, we obtain

$$|F_m(x)| \lesssim \frac{1}{\langle m \rangle^{s+2}} \bigg| \int_{\mathbb{R}} \frac{1}{\langle x-y \rangle^2} \widetilde{\phi}(y) dy \bigg|$$

for some $\widetilde{\phi} \in \mathcal{S}(\mathbb{R})$. Hence, by Young's inequality, we have

$$\|F_m(x)\|_{L^p(\mathbb{R})} \lesssim \frac{1}{\langle m \rangle^{s+2}}. \tag{B.18}$$

From (B.16), (B.17), and (B.18), we have

$$\|\psi(D-n) f(x)\|_{L^p(\mathbb{R})} \lesssim \sum_{m\in\mathbb{Z}} |\widehat{f}(n-m)| \frac{1}{\langle m \rangle^{s+2}}. \tag{B.19}$$

By $\langle n \rangle^s \lesssim \langle n-m \rangle^s \langle m \rangle^s$ for $s \geq 0$, (B.19) and Young's inequality, we obtain

$$\begin{aligned}
\|\phi f\|_{M^{p,q}_s(\mathbb{R})} &= \big\| \langle n \rangle^s \|\psi(D-n) f(x)\|_{L^p(\mathbb{R})} \big\|_{\ell^q_n(\mathbb{Z})} \lesssim \bigg\| \sum_{m\in\mathbb{Z}} \langle n-m \rangle^s |\widehat{f}(n-m)| \frac{1}{\langle m \rangle^2} \bigg\|_{\ell^q_n(\mathbb{Z})} \\
&\lesssim \|\langle n \rangle^s \widehat{f}(n)\|_{\ell^q_n(\mathbb{Z})} = \|f\|_{\mathcal{F}L^{s,q}(\mathbb{T})}.
\end{aligned}$$

The modification for $s < 0$ follows as in Part (a). □

Proof of Proposition B.2. (a) Let $\{\varphi_j\}_{j=0}^\infty$ be the support function in the definition of $\widehat{b}^s_{p,q}(\mathbb{R})$ (and the Besov space), i.e. $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp} \varphi_0 \subset \{|\xi| \leq 2\}$ and $\operatorname{supp} \varphi \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$ with $\varphi_0(\xi) + \sum_{j=1}^\infty \varphi_j(\xi) \equiv 1$, where $\varphi_j(\xi) = \varphi(2^{-j}\xi)$.

Let $\eta_0(\xi)$ be a smooth cutoff function supported on $\{|\xi| \leq 3\}$ such that $\eta_0(\xi) = 1$ on $\{|\xi| \leq 2\}$. For $j \geq 1$, let $\eta_j(\xi) = \eta(2^{-j}\xi)$, where η is a smooth cutoff function supported on

$\{\frac{1}{4} \le |\xi| \le 3\}$ such that $\eta(\xi) = 1$ on $\{\frac{1}{2} \le |\xi| \le 2\}$. Then, we have $\varphi_j(\xi)\eta_j(\xi) = \varphi_j(\xi)$ for $\xi \in \mathbb{R}$ and $j \ge 0$. Also, recall the notation $M_n\phi(y) = \phi(y)e^{iny}$. Then, we have

$$
\begin{aligned}
|\widehat{\phi f}(n)| &= \bigg| \int_0^{2\pi} \phi(y)f(y)e^{-iny}dy \bigg| = \bigg| \sum_{j=0}^{\infty} \int_{\mathbb{R}} \phi(y)\varphi_j(D)f(y)e^{-iny}dy \bigg| \\
&= \bigg| \sum_{j=0}^{\infty} \int_{\mathbb{R}} \varphi_j(\xi)\widehat{f}(\xi)\overline{\eta_j(\xi)\widehat{M_n\phi}(\xi)}dy \bigg| \le \sum_{j=0}^{\infty} \|\varphi_j \widehat{f}\|_{L^p_\xi(\mathbb{R})} \|\eta_j \widehat{M_n\phi}\|_{L^{p'}_\xi(\mathbb{R})}.
\end{aligned}
\tag{B.20}
$$

Since $\widehat{\phi} \in \mathcal{S}(\mathbb{R})$, we have

$$
\|\eta_0 \widehat{M_n\phi}\|_{L^{p'}_\xi(\mathbb{R})} = \bigg(\int |\eta_0(\xi)\widehat{\phi}(\xi - n)|^{p'} d\xi \bigg)^{\frac{1}{p'}} \lesssim c_{0,n},
$$

and

$$
\begin{aligned}
\|\eta_j \widehat{M_n\phi}\|_{L^{p'}_\xi(\mathbb{R})} &= \bigg(\int |\eta(2^{-j}\xi)\widehat{\phi}(\xi - n)|^{p'} d\xi \bigg)^{\frac{1}{p'}} \\
&\lesssim c_{j,n} \bigg(\int_{|\xi| \sim 2^j} \frac{1}{\langle \xi - n \rangle^2} d\xi \bigg)^{\frac{1}{p'}} \lesssim c_{j,n}, \quad j \ge 1,
\end{aligned}
$$

where

$$
c_{j,n} = \begin{cases} 1, & |j-k| \le 2, \\ \langle 2^j - n \rangle^{-2-s}, & |j-k| \ge 3 \text{ and } s \ge 0, \\ \langle 2^j - n \rangle^{-2+s} & |j-k| \ge 3 \text{ and } s < 0, \end{cases} \qquad \text{with } |n| \sim 2^k.
\tag{B.21}
$$

From (B.20), we have

$$
\|\phi f\|_{\widehat{b}^s_{p,q}(\mathbb{T})} = \big\| \|\langle n \rangle^s \widehat{\phi f}(n)\|_{\ell^p_{|n| \sim 2^k}} \big\|_{\ell^q_k} \lesssim \bigg\| \bigg\| 2^{ks} \sum_{j=0}^{\infty} c_{j,n} \|\varphi_j \widehat{f}\|_{L^p_\xi(\mathbb{R})} \bigg\|_{\ell^p_{|n| \sim 2^k}} \bigg\|_{\ell^q_k}.
\tag{B.22}
$$

The contribution from $|j - k| \le 2$ is easily estimated by

$$
\lesssim \big\| 2^{ks} \|\varphi_k \widehat{f}\|_{L^p_\xi(\mathbb{R})} \big\|_{\ell^q_k} \sim \|f\|_{\widehat{b}^s_{p,q}(\mathbb{R})}.
$$

When $j \ge k + 3$, we have $c_{j,n} \sim 2^{-(2+s)j}$ for $|n| \sim 2^k$. Then, the contribution to (B.22) in this case is estimated by

$$
\begin{aligned}
&\bigg\| \bigg\| 2^{ks} \sum_{j \ge k+3} c_{j,n} \|\varphi_j \widehat{f}\|_{L^p_\xi(\mathbb{R})} \bigg\|_{\ell^p_{|n| \sim 2^k}} \bigg\|_{\ell^q_k} \\
&\qquad \lesssim \bigg\| \Big\| \big\| 2^{js} \|\varphi_j \widehat{f}\|_{L^p_\xi(\mathbb{R})} \big\|_{\ell^q_j} \Big(\sum_{j \ge k+3} 2^{(k-j)sq'} 2^{-q'(2+s)j} \Big)^{\frac{1}{q'}} \Big\|_{\ell^p_{|n| \sim 2^k}} \bigg\|_{\ell^q_k} \\
&\qquad \lesssim \|f\|_{\widehat{b}^s_{p,q}(\mathbb{R})} \bigg(\sum_k \sum_{|n| \sim 2^k} \sum_{j \ge k+3} 2^{(k-j)s} 2^{-(2+s)j} \bigg) \lesssim \|f\|_{\widehat{b}^s_{p,q}(\mathbb{R})},
\end{aligned}
$$

where we used Hölder inequality in j in the first inequality.

When $j \le k-3$, we have $c_{j,n} \sim 2^{-(2+s)k}$ for $|n| \sim 2^k$. Then, the contribution to (B.22) in this case is estimated by

$$\begin{aligned}
&\bigg\| \Big\| 2^{ks} \sum_{j \le k-3} c_{j,n} \|\varphi_j \widehat{f}\|_{L^p_\xi(\mathbb{R})} \Big\|_{\ell^p_{|n|\sim 2^k}} \bigg\|_{\ell^q_k} \\
&\qquad \lesssim \bigg\| \big\| 2^{js} \|\varphi_j \widehat{f}\|_{L^p_\xi(\mathbb{R})} \big\|_{\ell^q_j} \Big(\sum_{j \le k-3} 2^{(k-j)sq'} 2^{-q'(2+s)k} \Big)^{\frac{1}{q'}} \bigg\|_{\ell^p_{|n|\sim 2^k}} \bigg\|_{\ell^q_k} \\
&\qquad \lesssim \|f\|_{\widehat{b}^s_{p,q}(\mathbb{R})} \bigg(\sum_k \sum_{|n|\sim 2^k} \sum_{j \le k-3} 2^{(k-j)s} 2^{-(2+s)k} \bigg) \lesssim \|f\|_{\widehat{b}^s_{p,q}(\mathbb{R})}.
\end{aligned}$$

Using (B.21), a similar computation yields (B.4) for $s < 0$.

(b) Consider again ϕ to be a smooth cutoff function supported on $[0, 2\pi)$. Then, we have

$$\begin{aligned}
\widehat{\phi f}(\xi) &= \int_{\mathbb{R}} \phi(y) f(y) e^{-i\xi y} dy \\
&= \int_0^{2\pi} \phi(y) f(y) e^{-iny} dy + \int_0^{2\pi} f(y) \phi_{n,\xi}(y) e^{-iny} dy \\
&= \widehat{\phi f}(n) + \widehat{\phi_{n,\xi} f}(n), \qquad \text{(B.23)}
\end{aligned}$$

where $\phi_{n,\xi}(y) = \phi(y)(e^{-i(\xi-n)y} - 1)$.

First, we prove (B.13) for $s \ge 0$. For $n \in \mathbb{Z}$, let $I_n = [n-1, n+1]$. Then, for $\xi \in I_n$, we have

$$\begin{aligned}
|\widehat{\phi_{n,\xi}}(m)| &= \bigg| \int_0^{2\pi} \phi(y)(e^{-i(\xi-n)y} - 1) e^{-imy} dy \bigg| = \bigg| \int_{\mathbb{R}} \phi(y)(e^{-i(\xi-n)y} - 1) e^{-imy} dy \bigg| \\
&= |\widehat{\phi}(\xi - n + m) - \widehat{\phi}(m)| \le \sup_{\zeta \in [m-1, m+1]} |\partial_\zeta \widehat{\phi}(\zeta)| |\xi - n| \lesssim \langle m \rangle^{-2-s}. \qquad \text{(B.24)}
\end{aligned}$$

Also, we have $|\widehat{\phi}(m)| \lesssim \langle m \rangle^{-2-s}$.

From (B.23) and (B.24), we have

$$\begin{aligned}
\|\phi f\|_{\widehat{b}^s_{p,q}(\mathbb{R})} &= \big\| 2^{js} \|\varphi_j(\xi) \widehat{\phi f}(\xi)\|_{L^p_\xi(\mathbb{R})} \big\|_{\ell^q_j} \lesssim \bigg\| 2^{js} \Big(\sum_{|n|\sim 2^j} \int_{\xi \in I_n} |\widehat{\phi f}(\xi)|^p d\xi \Big)^{\frac{1}{p}} \bigg\|_{\ell^q_j} \\
&\lesssim \bigg\| 2^{js} \Big(\sum_{|n|\sim 2^j} \Big| \sum_m \widehat{f}(n-m) \langle m \rangle^{-2-s} \Big|^p \Big)^{\frac{1}{p}} \bigg\|_{\ell^q_j}. \qquad \text{(B.25)}
\end{aligned}$$

Note that, for $|n| \sim 2^j$, (i) $|n-m| \sim 2^j$ implies $|m| \lesssim 2^j$, (ii) $|n-m| \ll 2^j$ implies $|m| \sim 2^j$, and (iii) $|n-m| \gg 2^j$ implies $|m| \gg 2^j$. First, we consider the case (i). In this case, the contribution to (B.25) is estimated by

$$\lesssim \bigg\| 2^{js} \|\widehat{f}(n)\|_{\ell^p_{|n|\sim 2^j}} \Big(\sum_{|m| \lesssim 2^j} \langle m \rangle^{-2-s} \Big) \bigg\|_{\ell^q_j} \lesssim \|f\|_{\widehat{b}^s_{p,q}(\mathbb{T})}.$$

Next, we consider the case (ii). By Hölder inequality, the contribution to (B.25) is estimated by

$$\lesssim \bigg\| 2^{js} \|\widehat{f}(n)\|_{\ell^p_{|n|\ll 2^j}} \Big(\sum_{|m|\sim 2^j} \langle m \rangle^{-2-s} \Big) \bigg\|_{\ell^q_j} \lesssim \bigg\| j^{\frac{1}{q'}} 2^{-j} \big\| \|\widehat{f}(n)\|_{\ell^p_{|n|\sim 2^k}} \big\|_{\ell^q_k} \bigg\|_{\ell^q_j} \lesssim \|f\|_{\widehat{b}^s_{p,q}(\mathbb{T})}.$$

Lastly, we consider the case (iii). In this case, we have $|n-m| \sim |m|$. By Hölder inequality, the contribution to (B.25) is estimated by

$$\lesssim \Big\| 2^{js} \sum_{|m|\sim 2^k \gg 2^j} 2^{-(2+s)k} \|\widehat{f}(n)\|_{\ell^p_{|n|\sim 2^k}} \Big) \Big\|_{\ell^q_j}$$

$$\lesssim \big\| 2^{ks} \|\widehat{f}(n)\|_{\ell^p_{|n|\sim 2^k}} \big\|_{\ell^q_k} \Big\| \Big(\sum_{k\geq j} 2^{-(2+s)q'k} \Big)^{\frac{1}{q'}} \Big\|_{\ell^q_j} \lesssim \|f\|_{\widehat{b}^s_{p,q}(\mathbb{T})}.$$

When $s<0$, we can instead choose $|\widehat{\phi_{n,\xi}}(m)|, |\widehat{\phi}(m)| \lesssim \langle m\rangle^{-2+s}$ in (B.24). The modification is straightforward, and thus we omit the details. □

References

[1] A. Ayache and N. Tzvetkov, *L^p properties for Gaussian random series,* Trans. Amer. Math. Soc. 360 (2008), 4425–4439.

[2] Á. Bényi, K. Gröchenig, L. Rogers, and K. Okoudjou, *Unimodular Fourier multipliers for modulation spaces,* J. Func. Anal. 246 (2007), 366–384.

[3] Á. Bényi and K. Okoudjou, *Local well-posedness of nonlinear dispersive equations on modulation spaces,* Bull. London Math. Soc. 41 (2009), 549–558.

[4] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures,* Comm. Math. Phys. 166 (1994), 1–26.

[5] J. Bourgain, *On the Cauchy and invariant measure problem for the periodic Zakharov system,* Duke Math. J. 76 (1994), 175–202.

[6] J. Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrdinger equation,* Comm. Math. Phys. 176 (1996), no. 2, 421–445.

[7] J. Bourgain, *Invariant measures for the Gross-Pitaevskii equation,* J. Math. Pures Appl. (9) 76 (1997), no. 8, 649–702.

[8] N. Burq and N. Tzvetkov, *Invariant measure for a three dimensional nonlinear wave equation,* Int. Math. Res. Not. (2007), no. 22, Art. ID rnm108, 26pp.

[9] N. Burq and N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations. I. Local theory,* Invent. Math. 173 (2008), no. 3, 449–475.

[10] N. Burq and N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations. II. A global existence result,* Invent. Math. 173 (2008), no. 3, 477–496.

[11] Z. Ciesielski, *Modulus of smoothness of the Brownian motion in the L^p norm,* Proc. Conf. Approximation Theory (1991), Varna.

[12] Z. Ciesielski, *Orlicz spaces, spline systems, and Brownian motion*, Constr. Approx. 9 (1993), no. 2–3, 191–208.

[13] Z. Ciesielski, G. Kerkyacharian, and B. Roynette, *Quelques espaces fonctionnels associés à des processus gaussiens*, Studia Math. 107 (1993), no. 2, 171–204.

[14] E. Cordero and F. Nicola, *Strichartz estimates in Wiener amalgam spaces for the Schrödinger equation,* Math. Nachr. 28 (2008), 25–41.

[15] E. Cordero and F. Nicola, *Metaplectic representation on Wiener amalgam spaces and applications to the Schrödinger equation,* J. Func. Anal. 254 (2008), 506–534.

[16] I. Daubechies, *The wavelet transform, time-frequency localization and signal analysis,* IEEE Trans. Inform. Theory 36 (1990), 961–1005.

[17] H. G. Feichtinger and K. Gröchenig, *Banach spaces related to to integrable group representations and their atomic decompositions, I,* J. Func. Anal. 86 (1989), 307–340.

[18] H. G. Feichtinger and K. Gröchenig, *Banach spaces related to to integrable group representations and their atomic decompositions, II,* Monatsh. Math. 108 (1989), 129–148.

[19] M. X. Fernique, *Intégrabilité des Vecteurs Gaussiens,* Academie des Sciences, Paris, Comptes Rendus, 270, Séries A (1970), 1698–1699.

[20] Y. V. Galperin and K. Gröchenig, *Uncertainty principles as embeddings of modulation spaces,* J. Math. Anal. Appl. 274 (2002), 181–202.

[21] K. Gröchenig, *Foundations of Time-Frequency Analysis,* Birkhäuser, Boston, 2001.

[22] L. Gross, *Abstract Wiener spaces,* Proc. 5th Berkeley Sym. Math. Stat. Prob. 2 (1965), 31–42.
[23] A. Grossmann and J. Morlet, *Decomposition of Hardy functions into square integrable wavelets of constant shape,* SIAM J. Math. Anal. 15 (1984), 723–736.
[24] T.P. Hytönen and M.C. Veraar, *On Besov regularity of Brownian motions in infinite dimensions,* Probab. Math. Statist. 28 (2008), 143–162.
[25] S. Janson, *Gaussian Hilbert Spaces,* Cambridge Tracts in Mathematics, 129. Cambridge University Press, Cambridge, 1997. x+340 pp.
[26] J. P. Kahane, *Some Random Series of Functions,* Second edition. Cambridge Studies in Advanced Mathematics, 5. Cambridge University Press, Cambridge, 1985. xiv+305 pp.
[27] M. Kobayashi and M. Sugimoto, *The inclusion between Sobolev and modulation spaces,* J. Func. Anal., to appear.
[28] H. Kuo, *Gaussian Measures in Banach Spaces,* Lecture Notes in Math. 463, Springer-Verlag, New York, 1975.
[29] H. Kuo, *Introduction to Stochastic Integration,* Universitext. Springer, New York, 2006. xiv+278 pp.
[30] J. Lebowitz, H. Rose, E. Speer, *Statistical mechanics of the nonlinear Schrödinger equation,* J. Statist. Phys. 50 (1988), no. 3–4, 657–687.
[31] M. Ledoux and M. Talagrand, *Probability in Banach spaces. Isoperimetry and Processes,* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 23. Springer-Verlag, Berlin, 1991. xii+480 pp.
[32] A. Nahmod, T. Oh, L. Rey-Bellet, and G. Staffilani, *Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS*, J. Eur. Math. Soc., to appear.
[33] D. Nualart, *Stochastic calculus with respect to fractional Brownian motion,* Ann. Fac. Sci. Toulouse Math. 15 (2006), no. 1, 63–78.
[34] T. Oh, *Invariant Gibbs measures and a.s. global well-posedness for coupled KdV systems,* Diff. Integ. Eq. 22 (2009), no. 7–8, 637–668.
[35] T. Oh, *Invariance of the white noise for KdV*, Comm. Math. Phys. 292 (2009), no. 1, 217–236. Also, see *Erratum: "Invariance of the white noise for KdV"*, in preparation.
[36] T. Oh, *Periodic stochastic Korteweg-de Vries equation with the additive space-time white noise*, Anal. PDE. 2 (2009), no. 3, 281–304.
[37] T. Oh, *Invariance of the Gibbs Measure for the Schrödinger-Benjamin-Ono System,* SIAM J. Math. Anal. 41 (2009), no. 6, 2207–2225.
[38] K. A. Okoudjou, *Embeddings of some classical Banach spaces into modulation spaces,* Proc. Amer. Math. Soc. 132 (2004), 1639–1647.
[39] E. Paley, A. Zygmund, *On some series of functions (1) (2) (3)*, Proc. Camb. Philos. Soc. 26 (1930), 337–357, 458–474; 28 (1932), 190–205.
[40] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion,* Third edition. Grundlehren der Mathematischen Wissenschaften, 293. Springer-Verlag, Berlin, 1999. xiv+602 pp.
[41] B. Roynette, *Mouvement brownien et espaces de Besov,* Stochastics Stochastics Rep. 43 (1993), 221–260.
[42] M. Ruzhansky, M. Sugimoto, J. Toft, and N. Tomita, *Changes of variables in modulation and Wiener amalgam spaces,* Math. Nachr., to appear.
[43] L. Saulis and A. Nakas, *Asymptotic expansions for large deviations when the Cramér condition is violated* (Russian. Lithuanian, English summary), Litovsk. Mat. Sb. 13 (1973), no. 1, 199–219.
[44] S. Sheffield, *Gaussian free fields for mathematicians,* Probab. Theory Related Fields 139 (2007), no. 3–4, 521–541.
[45] R. L. Schilling, *Function spaces as path spaces of Feller processes,* Math. Nachr. 217 (2000), 147–174.
[46] A. N. Shiryaev, *Probability,* Translated from the first (1980) Russian edition by R. P. Boas. Second edition. Graduate Texts in Mathematics, 95. Springer-Verlag, New York, 1996. xvi+623 pp.
[47] M. Sugimoto and N. Tomita, *The dilation property of modulation spaces and their inclusion relation with Besov spaces,* J. Func. Anal. 248 (2007), 79–106.
[48] L. Tartar, *An Introduction to Sobolev Spaces and Interpolation Spaces,* Lecture Notes of the Unione Matematica Italiana, 3. Springer, Berlin; UMI, Bologna, 2007. xxvi+218 pp.
[49] J. Toft, *Convolution and embeddings for weighted modulation spaces,* in: "Advances in pseudo-differential operators," Oper. Theory Adv. Appl. 155, Birkhäuser, Basel 2004, 165–186.
[50] N. Tzvetkov, *Construction of a Gibbs measure associated to the periodic Benjamin-Ono equation*, Probab. Theory and Relat. Fields DOI 10.1007/s00440-008-0197-z.

[51] M. Veraar, *Regularity of Gaussian white noise on the d-dimensional torus,* arXiv:1010.6219v2 [math.PR].
[52] B. Wang, L. Zhao, and B. Guo, *Isometric decomposition operators, function spaces $E^{\lambda}_{p,q}$ and applications to nonlinear evolution equations,* J. Func. Anal. 233 (2006), 1–39.
[53] B. Wang and C. Huang, *Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations,* J. Diff. Equations 239 (2007), 213–250.
[54] B. Wang and H. Hudzik, *The global Cauchy problem for the NLS and NLKG with small rough data,* J. Diff. Equations 232 (2007), 36–73.
[55] B. Wang, L. Han, and C. Huang, *Global well-posedness and scatering for the derivative nonolinear Schrödinger equation with small rough data,* Ann. I.H. Poincaré 26 (2009), 2253–2281.
[56] P. Zhidkov, *Korteweg-de Vries and Nonlinear Schrödinger Equations: Qualitative Theory,* Lecture Notes in Math. 1756, Springer-Verlag, 2001.

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