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The following two sequences are commonly used to define the number e:

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} \quad \text{and} \quad P_n = \left(1 + \frac{1}{n}\right)^n, \quad n \in \mathbb{N} \ \text{(positive integers)}.$$

In this note, we give a direct proof that $\{S_n\}$ and $\{P_n\}$ converge to the same limit. The main tool in our proof is the squeeze theorem, which is probably the easiest to prove among the limit theorems. However, to use it, we need to establish a relevant squeeze, which is the main result of this note.

Surprisingly, many elementary mathematical analysis textbooks do not include a proof that $\{S_n\}$ and $\{P_n\}$ converge to the same limit. The proofs in the classical book [**4**, Thm. 3.31] and in more recent [**1**, Prop. 3.3.1] and [**3**, App. 2] all use a limit theorem that they do not prove.

For completeness, we give the standard proof that $\{S_n\}$ is bounded above by 3. Clearly, $S_1 = 2 < 3$ and as $1/k! \le 1/\big((k-1)k\big)$ for all $k > 1$, we have

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} < 2 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) = 3 - \frac{1}{n}.$$

Thus, $S_n < 3$ for all $n \in \mathbb{N}$. Since $\{S_n\}$ is increasing, it converges by the monotone convergence theorem.

The squeeze. The squeeze that we mentioned earlier is

$$S_n - \frac{3}{2n} \le P_n \le S_n \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

Applying the squeeze theorem to (1) shows that $\{P_n\}$ converges to the same limit as $\{S_n\}$, namely e.

Proof. Since (1) is true for $n \le 2$, consider $n > 2$. Our proof of (1) is a succession of four steps, each suggesting the next one.

1. The binomial theorem yields an expanded expression for P_n:

$$P_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{1}{n^k} = 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!}\,\frac{n!}{(n-k)!\,n^k}. \tag{2}$$

2. For $k \in \{2, \ldots, n\}$, we rewrite the coefficient with $1/k!$ in (2) as the product of $k-1$ factors:

$$\frac{n!}{(n-k)!\,n^k} = \frac{(n-1)\cdots(n-k+1)}{n^{k-1}} = \left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right). \tag{3}$$

3. An upper bound for the product in (3) is clearly 1, so we look for its lower bound next. We proceed recursively. At each step, in some sense, we turn a product into a smaller sum.

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For $k = 2$ the product in (3) has only one term and clearly $(1 - 1/n) \geq 1 - 1/n$. For $k = 3$, we expand the product and drop a positive term:

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) = 1 - \frac{1+2}{n} + \frac{1 \cdot 2}{n} > 1 - \frac{1+2}{n}.$$

For $k = 4$, we multiply both sides above by $\left(1 - \frac{3}{n}\right)$, expand the product on the right, and drop a positive term:

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right) > \left(1 - \frac{1+2}{n}\right)\left(1 - \frac{3}{n}\right) > 1 - \frac{1+2+3}{n}.$$

Repeating this process a total of $k - 1$ times and, at the end, using the familiar formula $1 + 2 + \cdots + (k-1) = (k-1)k/2$ (whose history is given in [**2**]) yields

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) > 1 - \frac{1 + \cdots + (k-1)}{n} = 1 - \frac{(k-1)k}{2n}.$$

In conclusion, for the product in (3) we have

$$1 - \frac{(k-1)k}{2n} < \frac{n!}{n^k(n-k)!} < 1 \quad \text{for all } k \in \{2, \ldots, n\}. \tag{4}$$

4. The inequalities in (4) are applied to the right-most expression in (2) to establish the inequalities for P_n:

$$1 + 1 + \sum_{k=2}^{n} \frac{1}{k!}\left(1 - \frac{(k-1)k}{2n}\right) < P_n < 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \cdot 1 = S_n. \tag{5}$$

A simplification of the left-hand side of (5) leads to

$$\sum_{k=0}^{n} \frac{1}{k!} - \sum_{k=2}^{n} \frac{1}{k!}\frac{(k-1)k}{2n} = S_n - \frac{1}{2n}\sum_{k=2}^{n} \frac{1}{(k-2)!} = S_n - \frac{1}{2n}S_{n-2}.$$

Further, since $S_{n-2} < 3$, we have

$$S_n - \frac{1}{2n}S_{n-2} > S_n - \frac{3}{2n}.$$

Consequently, the left-hand side of (5) is greater than $S_n - 3/(2n)$ and the squeeze is established. ■

References

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