

4-2009

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## Recommended Citation

Binding, Paul and Ćurgus, Branko, "Riesz Bases of Root Vectors of Indefinite Sturm-Liouville Problems with Eigenparameter Dependent Boundary Conditions. II" (2009). *Mathematics*. 65.

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# Riesz bases of root vectors of indefinite Sturm-Liouville problems with eigenparameter dependent boundary conditions. II

Paul Binding and Branko Ćurgus

**Abstract.** We consider a regular indefinite Sturm-Liouville problem with two self-adjoint boundary conditions affinely dependent on the eigenparameter. We give sufficient conditions under which the root vectors of this Sturm-Liouville problem can be selected to form a Riesz basis of a corresponding weighted Hilbert space.

**Mathematics Subject Classification (2000).** Primary: 34B05, 47B50. Secondary: 34B09, 34B25, 47B25.

**Keywords.** Indefinite Sturm-Liouville problem, Riesz basis, Eigenvalue dependent boundary conditions, Krein space, definitizable operator.

## 1. Introduction

Consider the following eigenvalue problem

$$\begin{aligned} -f''(x) &= \lambda (\operatorname{sgn} x) f(x), & x \in [-1, 1], \\ f'(1) &= \lambda f(-1), \\ -f'(-1) &= \lambda f(1). \end{aligned}$$

Lengthy but straightforward calculations show the following: there exist an infinite number of real, simple, nonzero eigenvalues which accumulate only at  $-\infty$  and  $+\infty$ ; the number 0 is also a simple eigenvalue. Details can be found at the second author's web-site. It is natural to consider this problem in the Hilbert space  $L_2(-1, 1) \oplus \mathbb{C}^2$ . To our knowledge the following related question, which presents interesting mathematical challenges, has not been addressed. Is it possible to select eigenvectors of the given eigenvalue problem to form a Riesz basis of the above Hilbert space? In this article we answer such questions for a wide class

of indefinite Sturm-Liouville problems with  $\lambda$ -dependent boundary conditions. In particular, our Theorem 5.2 applies to the above simple example.

We consider a regular indefinite Sturm-Liouville eigenvalue problem of the form

$$-(pf')' + qf = \lambda rf \quad \text{on} \quad [-1, 1]. \quad (1.1)$$

We assume throughout that the coefficients  $1/p, q, r$  in (1.1) are real and integrable over  $[-1, 1]$ ,  $p(x) > 0$ , and  $xr(x) > 0$  for almost all  $x \in [-1, 1]$ . We impose the following eigenparameter dependent boundary conditions on equation (1.1):

$$\mathbf{M}\mathbf{b}(f) = \lambda \mathbf{N}\mathbf{b}(f), \quad (1.2)$$

where  $\mathbf{M}$  and  $\mathbf{N}$  are  $2 \times 4$  matrices and the boundary mapping  $\mathbf{b}$  is defined for all  $f$  in the domain of (1.1) by

$$\mathbf{b}(f) = [f(-1) \quad f(1) \quad (pf')(-1) \quad (pf')(1)]^T.$$

For our opening example

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

We remark that more general boundary conditions have been studied by many authors, recently for example in [3] and [4], but expansion theorems were not considered. Expansion theorems for polynomial boundary conditions and more general operators, but with weight  $r = 1$ , were given in [11] and [20].

In this article we study the problem (1.1), (1.2) in an operator theoretic setting established in [5]. Under Condition 2.1 below, a definitizable self-adjoint operator  $A$  in the Krein space  $L_{2,r}(-1, 1) \oplus \mathbb{C}_{\Delta}^2$  (actually  $A$  is quasi-uniformly positive as defined in [10]) is associated with the eigenvalue problem (1.1), (1.2). Here  $\Delta$  is a  $2 \times 2$  nonsingular Hermitean matrix which is determined by  $\mathbf{M}$  and  $\mathbf{N}$ ; see Section 2 for details. We remark that the topology of this Krein space is that of the corresponding Hilbert space  $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}^2$ . Here, and in the rest of the paper, we abbreviate  $L_{2,r}(-1, 1)$  to  $L_{2,r}$  and  $L_{2,|r|}(-1, 1)$  to  $L_{2,|r|}$ . For more details about Krein spaces and their operators see the standard reference [14] and [1] for recent developments.

Our main goal in this paper is to provide sufficient conditions on the coefficients in (1.1), (1.2) under which there is a Riesz basis of the above Hilbert space consisting of the union of bases for all the root subspaces of the above operator  $A$ . This will be referred to for the remainder of this section as the *Riesz basis property of  $A$* . We remark that the Riesz basis property of  $A$  is equivalent, modulo a finite dimensional subspace, to similarity of  $A$  to a self-adjoint operator in a Hilbert space. The latter similarity has been the subject of several recent papers (see for example [15] and [16]) involving Sturm-Liouville expressions on  $\mathbb{R}$  without boundary conditions.

Existence of Riesz bases and expansion theorems with a stronger topology, but in a smaller space corresponding to the form domain of the operator  $A$  (which in our case is a Pontryagin space), have been considered by many authors; see

[5, 21] and the references there. The results in [5] turned out to be independent of the number and the nature of the boundary conditions and the coefficients  $p$  and  $r$ . In contrast, the Riesz basis property depends nontrivially on the problem data even for the case when the boundary conditions are  $\lambda$ -independent (corresponding to  $\mathbf{N} = \mathbf{0}$  in our notation).

Sufficient conditions on  $r$  (near the turning point 0) for the Riesz basis property when  $\mathbf{N} = \mathbf{0}$  can be found in [2, 9, 12, 13, 18, 19], for example. That some condition is necessary, even in the case  $p = 1$ , was shown by Volkmer [22] who proved the existence of an odd  $r$  for which the Dirichlet problem (1.1) does not have this property. Recently Parfenov [17] gave a necessary and sufficient condition on an odd weight function  $r$ , near its turning point 0, for the Dirichlet problem (1.1) to have the Riesz basis property. In [6] we constructed an odd  $r$  for which the Dirichlet problem (1.1) has the Riesz basis property but the anti-periodic problem does not. This example shows that an additional condition on  $r$  near the boundary of  $[-1, 1]$  (which in some cases behaves as a second turning point, in addition to 0, for (1.1)) is needed for the general case of (1.2). Such conditions are given in [9] for  $\lambda$ -independent boundary conditions and in [7] for exactly one  $\lambda$ -dependent boundary condition (i.e., when  $\mathbf{N}$  has rank 1).

In this paper we consider the more difficult case of two  $\lambda$ -dependent boundary conditions. The method we use has its origins in the work of Beals [2]. Subsequently it was developed in [8] into a criterion (given below as Theorem 2.2) equivalent to the Riesz basis property of  $A$ . This criterion involves a positive homeomorphism  $W$  of the Krein space  $L_{2,r} \oplus \mathbb{C}_{\Delta}^2$  with the form domain of  $A$  as an invariant subspace. The explicit description of the form domain of  $A$  (given in Section 2) depends entirely on the number  $k \in \{0, 1, 2\}$  of boundary conditions which do not include derivatives in the  $\lambda$ -terms. We call such boundary conditions *essential*. Note that this differs from the usual terminology for  $\lambda$ -independent conditions. For example, in our terminology  $y'(1) = \lambda y(1)$  is an essential boundary condition.

The direct sum structure of the Krein space  $L_{2,r} \oplus \mathbb{C}_{\Delta}^2$  naturally leads us to consider the homeomorphism  $W$  as a block operator matrix, the top left entry  $W_{11}$  being an operator on  $L_{2,r}$ . Since it is clear from Section 2 that the functional components of the vectors in the form domain of  $A$  are (absolutely) continuous, we see that  $W_{11}$  induces a boundary matrix  $\mathbf{B}$  satisfying

$$\mathbf{B} \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix} = \begin{bmatrix} (W_{11}f)(-1) \\ (W_{11}f)(1) \end{bmatrix}.$$

An important hurdle, with analogues in several of above references, is to solve the inverse problem of finding a suitable  $W_{11}$  for a given matrix  $B$ . For example, in [7] (see also Section 3 below) such operators  $W_{11}$  were constructed with special diagonal  $\mathbf{B}$  under one-sided Beals type conditions at  $-1$  or  $1$ . In Section 4 we use conditions at  $-1$ , at  $1$ , and a condition connecting  $-1$  and  $1$  to produce  $W_{11}$  with an arbitrary prescribed boundary matrix  $\mathbf{B}$ .

In Sections 5 and 6 we complete the construction of  $W$ , thus establishing our sufficient conditions for the Riesz basis property. When there are no essential

boundary conditions ( $k = 0$ ), it turns out that the one-sided Beals type condition at 0 suffices; see Theorem 5.1. In other cases, however, we need conditions near the boundary of  $[-1, 1]$ . Conditions at 0, and at  $-1$  or  $1$ , are sufficient if  $k = 2$  and  $\Delta$  is definite. If  $\Delta$  is indefinite, then we also need the condition linking  $-1$  and  $1$ . In these cases it suffices to construct  $W$  as a block diagonal matrix. This is carried out in Theorem 5.2.

The most difficult case is  $k = 1$  which we tackle in Section 6. In this case we need not only off-diagonal blocks for  $W$ , but also a perturbation  $K$  of  $W_{11}$ , where  $K$  is an integral operator whose construction is rather delicate. Our final result Theorem 6.1 is as follows. If only one boundary point  $-1$  or  $1$  appears with  $\lambda$  in the essential boundary condition, then a Beals type condition at that point and at 0 are sufficient. Otherwise we need conditions at both boundary points and at 0, as well as the condition linking  $-1$  and  $1$ .

To conclude this introduction we remark that our conditions simplify drastically if  $p$  is even and  $r$  is odd, a case which has been studied by several authors [6, 17, 22]. In fact all the conditions that we impose on the boundary are then equivalent; see Example 4.3 and Corollary 6.5.

## 2. Operators associated with the eigenvalue problem

The maximal operator  $S_{\max}$  in  $L_{2,r}$  associated with (1.1) is defined by

$$S_{\max} : f \mapsto \ell(f) := \frac{1}{r}(-(pf)') + qf, \quad f \in \mathcal{D}(S_{\max}),$$

where

$$\mathcal{D}(S_{\max}) = \mathcal{D}_{\max} = \{f \in L_{2,r} : f, pf' \in AC[0, 1], \ell(f) \in L_{2,r}\}.$$

We define the boundary mapping  $\mathbf{b}$  by

$$\mathbf{b}(f) = [f(-1) \quad f(1) \quad (pf')(-1) \quad (pf')(1)]^T, \quad f \in \mathcal{D}(S_{\max}).$$

and the concomitant matrix  $\mathbf{Q}$  corresponding to  $\mathbf{b}$  by

$$\mathbf{Q} = i \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

The significance of  $\mathbf{Q}$  is captured by the following identity

$$\int_{-1}^1 (S_{\max} f \bar{g} - f S_{\max} \bar{g}) r = i \mathbf{b}(g)^* \mathbf{Q} \mathbf{b}(f), \quad f, g \in \mathcal{D}_{\max}.$$

We note that  $\mathbf{Q} = \mathbf{Q}^{-1}$ .

Throughout, we shall impose the following nondegeneracy and self-adjointness condition on the boundary data.

**Condition 2.1.** The boundary matrices  $\mathbf{M}$  and  $\mathbf{N}$  in (1.2) satisfy the following:

- (1) the  $4 \times 4$  matrix  $\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix}$  is nonsingular,  
 (2)  $\mathbf{MQM}^* = \mathbf{NQN}^* = 0$ ,  
 (3) the  $2 \times 2$  matrix  $i\mathbf{MQ}^{-1}\mathbf{N}^*$  is self-adjoint and invertible and we define

$$\Delta := -i(\mathbf{MQ}^{-1}\mathbf{N}^*)^{-1}.$$

Clearly the boundary value problem (1.1),(1.2) will not change if row reduction is applied to the coefficient matrix

$$[\mathbf{M} \ \mathbf{N}]. \quad (2.1)$$

In what follows we will assume that the matrix in (2.1) is row reduced to row echelon form (starting the reduction at the bottom right corner). In particular the matrix  $\mathbf{N}$  has the form

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_e & 0 \\ \mathbf{N}_1 & \mathbf{N}_n \end{bmatrix}.$$

The matrix  $0$  in the formula for  $\mathbf{N}$  is  $k \times 2$  with  $k \in \{0, 1, 2\}$ . The  $k \times 2$  matrix  $\mathbf{N}_e$  and the  $(2 - k) \times 2$  matrix  $\mathbf{N}_n$  are of maximal ranks.

There are three possible cases for  $\mathbf{N}$  in (2.1):

- (a)  $\mathbf{N}_n$  is a  $2 \times 2$  identity matrix (so  $k = 0$ ),  
 (b)  $\mathbf{N}_e$  and  $\mathbf{N}_n$  are nonsingular  $1 \times 2$  (row) matrices (so  $k = 1$ ),  
 (c)  $\mathbf{N}_e$  is a  $2 \times 2$  identity matrix (so  $k = 2$ ).

In case (a), both boundary conditions in (1.2) are *non-essential*, that is both rows on the right hand side of (1.2) contain derivatives. In case (b), the boundary condition corresponding to the first row in (1.2) is *essential*, that is no derivatives appear in this row on the right hand side; the second boundary condition in (1.2) is non-essential. In case (c), both boundary conditions in (1.2) are essential. Evidently  $k$  is the number of essential boundary conditions.

Next we define a Krein space operator associated with the problem (1.1),(1.2). We consider the linear space  $L_{2,r} \oplus \mathbb{C}_\Delta^2$ , equipped with the inner product

$$\left[ \begin{pmatrix} f \\ \mathbf{u} \end{pmatrix}, \begin{pmatrix} g \\ \mathbf{v} \end{pmatrix} \right] := \int_{-1}^1 f\bar{g}r + \mathbf{v}^* \Delta \mathbf{u}, \quad f, g \in L_{2,r}, \mathbf{u}, \mathbf{v} \in \mathbb{C}^2.$$

Then  $(L_{2,r} \oplus \mathbb{C}_\Delta^2, [\cdot, \cdot])$  is a Krein space. A fundamental symmetry on this Krein space is given by

$$J := \begin{bmatrix} J_0 & 0 \\ 0 & \text{sgn}(\Delta) \end{bmatrix},$$

where  $2 \times 2$  matrix  $\text{sgn}(\Delta)$  and  $J_0 : L_{2,r} \rightarrow L_{2,r}$  are defined by

$$\text{sgn}(\Delta) = |\Delta|^{-1} \Delta \quad \text{and} \quad (J_0 f)(t) := f(t) \text{sgn}(r(t)), \quad t \in [-1, 1].$$

Then  $\langle \cdot, \cdot \rangle := [J \cdot, \cdot]$  is a positive definite inner product which turns  $L_{2,r} \oplus \mathbb{C}_\Delta^2$  into a Hilbert space  $(L_{2,|r|} \oplus \mathbb{C}_{|\Delta}^2, \langle \cdot, \cdot \rangle)$ . The topology of  $L_{2,r} \oplus \mathbb{C}_\Delta^2$  is defined to

be that of  $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}^2$ , and a *Riesz basis* of  $L_{2,r} \oplus \mathbb{C}_{\Delta}^2$  is defined as a homeomorphic image of an orthonormal basis of  $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}^2$ .

We define the operator  $A$  in the Krein space  $L_{2,r} \oplus \mathbb{C}_{\Delta}^2$  on the domain

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} f \\ \mathbf{Nb}(f) \end{bmatrix} \in \mathcal{K} : f \in \mathcal{D}(S_{\max}) \right\}$$

by

$$A \begin{bmatrix} f \\ \mathbf{Nb}(f) \end{bmatrix} := \begin{bmatrix} S_{\max} f \\ \mathbf{Mb}(f) \end{bmatrix}, \quad f \in \mathcal{D}(A).$$

Using [5, Theorems 3.3 and 4.1] we see that this operator is definitizable with discrete spectrum in the Krein space  $L_{2,r} \oplus \mathbb{C}_{\Delta}^2$ . As in [7, Theorem 2.2], we then obtain the following, which is our basic tool.

**Theorem 2.2.** *Let  $\mathcal{F}(A)$  denote the form domain of  $A$ . Then there exists a Riesz basis of  $L_{2,r} \oplus \mathbb{C}_{\Delta}^2$  which consists of root vectors of  $A$  if and only if there exists a bounded, boundedly invertible, positive operator  $W$  in  $L_{2,r} \oplus \mathbb{C}_{\Delta}^2$  such that*

$$W \mathcal{F}(A) \subset \mathcal{F}(A).$$

In order to apply this result, we need to characterize the form domain  $\mathcal{F}(A)$ . To this end, let  $\mathcal{F}_{\max}$  be the set of all functions  $f$  in  $L_{2,r}$  which are absolutely continuous on  $[-1, 1]$  and such that  $\int_{-1}^1 p |f'|^2 < +\infty$ .

By [5, Theorem 4.2], there are three possible cases for the form domain  $\mathcal{F}(A)$  of  $A$ , corresponding to cases (a), (b) and (c) above.

(a) If  $\mathbf{N}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , then

$$\mathcal{F}(A) = \left\{ \begin{bmatrix} f \\ \mathbf{v} \end{bmatrix} \in \begin{matrix} L_{2,r} \\ \oplus \\ \mathbb{C}_{\Delta}^2 \end{matrix} : f \in \mathcal{F}_{\max}, \mathbf{v} \in \mathbb{C}^2 \right\}. \quad (2.2)$$

(b) If  $\mathbf{N}_e = [u \ v]$  with  $u, v \in \mathbb{C}$  and  $|u|^2 + |v|^2 \neq 0$ , then

$$\mathcal{F}(A) = \left\{ \begin{bmatrix} f \\ uf(-1) + vf(1) \\ z \end{bmatrix} \in \begin{matrix} L_{2,r} \\ \oplus \\ \mathbb{C}_{\Delta}^2 \end{matrix} : f \in \mathcal{F}_{\max}, z \in \mathbb{C} \right\}.$$

(c) If  $\mathbf{N}_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , then

$$\mathcal{F}(A) = \left\{ \begin{bmatrix} f \\ f(-1) \\ f(1) \end{bmatrix} \in \begin{matrix} L_{2,r} \\ \oplus \\ \mathbb{C}_{\Delta}^2 \end{matrix} : f \in \mathcal{F}_{\max} \right\}. \quad (2.3)$$

To construct an operator  $W$  as in Theorem 2.2 we need to impose conditions (to be given in the next two sections) on the coefficients  $p$  and  $r$  in (1.1). In all cases we need Condition 3.5 in a neighborhood of 0, and in some cases we need one of two Conditions, 3.7 or 3.8, on  $r$  in neighborhoods of  $-1$  or  $1$ . These will be discussed in Section 3. In some cases we also need Condition 4.1 connecting the boundary points  $-1$  and  $1$ . This is developed in Section 4.

### 3. Conditions at 0, $-1$ and $1$

In this section we recall the remaining concepts and results from [7, Sections 3, 4 and 5] which we need in this paper.

A closed interval of non-zero length is said to be a *left half-neighborhood* of its right endpoint and a *right half-neighborhood* of its left endpoint. Let  $\iota$  be a closed subinterval of  $[-1, 1]$ . By  $\mathcal{F}_{\max}(\iota)$  we denote the set of all functions  $f$  in  $L_{2,r}(\iota)$  which are absolutely continuous on  $\iota$  and such that  $\int_{\iota} p |f'|^2 < +\infty$ . With this notation we have  $\mathcal{F}_{\max} = \mathcal{F}_{\max}[-1, 1]$ .

**Definition 3.1.** Let  $p$  and  $r$  be the coefficients in (1.1). Let  $a, b \in [-1, 1]$  and let  $h_a$  and  $h_b$ , respectively, be half-neighborhoods of  $a$  and  $b$  which are contained in  $[-1, 1]$ . We say that the ordered pair  $(h_a, h_b)$  is *smoothly connected* if there exist

- (a) positive real numbers  $\epsilon$  and  $\tau$ ,
- (b) non-constant affine functions  $\alpha : [0, \epsilon] \rightarrow h_a$  and  $\beta : [0, \epsilon] \rightarrow h_b$ ,
- (c) non-negative real functions  $\rho$  and  $\varpi$  defined on  $[0, \epsilon]$

such that

- (i)  $\alpha(0) = a$  and  $\beta(0) = b$ ,
- (ii)  $p \circ \alpha$  and  $p \circ \beta$  are locally integrable on the interval  $(0, \epsilon]$ ,
- (iii)  $\rho \circ \alpha^{-1} \in \mathcal{F}_{\max}(\alpha([0, \epsilon]))$ ,
- (iv)  $1/\tau < \varpi < \tau$  a.e. on  $[0, \epsilon]$ ,
- (v)  $\rho(t) = \frac{|r(\beta(t))|}{|r(\alpha(t))|}$  and  $\varpi(t) = \frac{p(\beta(t))}{p(\alpha(t))}$  for  $t \in (0, \epsilon]$ .

The numbers  $\alpha', \beta'$  (the slopes of  $\alpha, \beta$ , respectively) and  $\rho(0)$  are called the *parameters* of the smooth connection.

A broad class of examples satisfying this definition can be given via the following one.

**Definition 3.2.** Let  $\nu$  and  $a$  be real numbers and let  $h_a$  be a half-neighborhood of  $a$ . Let  $g$  be a function defined on  $h_a$ . Then  $g$  is *of order  $\nu$  on  $h_a$*  if there exists  $g_1 \in C^1(h_a)$  such that

$$g(x) = |x - a|^\nu g_1(x) \quad \text{and} \quad g_1(x) \neq 0, \quad x \in h_a.$$

(The absolute value is missing in the corresponding definition in [7]).



**Example 3.3.** Let  $a, b \in \{-1, 0, 1\}$ . Let  $h_a$  and  $h_b$  be half-neighborhoods of  $a$  and  $b$ , respectively, and contained in  $[-1, 1]$ . For simplicity assume that  $p = 1$ . If  $r$  in (1.1) has order  $\nu$  ( $> -1$  to ensure integrability) on both half-neighborhoods  $h_a$  and  $h_b$  then as noted in [7] the half-neighborhoods  $h_a$  and  $h_b$  are smoothly connected. Moreover the parameters of the smooth connection are nonzero numbers. We remark that that  $p$  can be much more general – see [7, Example 3.4].

**Theorem 3.4.** Let  $\iota$  and  $j$  be closed intervals,  $\iota, j \in \{[-1, 0], [0, 1]\}$ . Let  $a$  be an endpoint of  $\iota$  and let  $b$  be an endpoint of  $j$ . Denote by  $a_1$  and  $b_1$ , respectively, the remaining endpoints. Assume that the half-neighborhoods  $\iota$  of  $a$  and  $j$  of  $b$  are smoothly connected with parameters  $\alpha', \beta'$  and  $\rho(0)$ . Then there exists an operator

$$S : L_{2,|r|}(\iota) \rightarrow L_{2,|r|}(j)$$

such that the following hold:

- (S-1)  $S \in \mathcal{L}(L_{2,|r|}(\iota), L_{2,|r|}(j))$ ,  $S^* \in \mathcal{L}(L_{2,|r|}(j), L_{2,|r|}(\iota))$ ;
- (S-2)  $(Sf)(x) = 0$ ,  $|x - b_1| \leq \frac{1}{2}$  for all  $f \in L_{2,|r|}(\iota)$  and  $(S^*g)(x) = 0$ ,  $|x - a_1| \leq \frac{1}{2}$  for all  $g \in L_{2,|r|}(j)$ ;
- (S-3)  $S\mathcal{F}_{\max}(\iota) \subset \mathcal{F}_{\max}(j)$ ,  $S^*\mathcal{F}_{\max}(j) \subset \mathcal{F}_{\max}(\iota)$ ;
- (S-4) For all  $f \in \mathcal{F}_{\max}(\iota)$  and all  $g \in \mathcal{F}_{\max}(j)$  we have

$$\lim_{\substack{y \rightarrow b \\ y \in j}} (Sf)(y) = |\alpha'| \lim_{\substack{x \rightarrow a \\ x \in \iota}} f(x), \quad \lim_{\substack{x \rightarrow a \\ x \in \iota}} (S^*g)(x) = |\beta'| \rho(0) \lim_{\substack{y \rightarrow b \\ y \in j}} g(y).$$

This is [7, Theorem 3.6].

**Condition 3.5 (Condition at 0).** Let  $p$  and  $r$  be coefficients in (1.1). Denote by  $h_{0-}$  a generic left and by  $h_{0+}$  a generic right half-neighborhood of 0. We assume that at least one of the four ordered pairs of half-neighborhoods

$$(h_{0-}, h_{0-}), \quad (h_{0-}, h_{0+}), \quad (h_{0+}, h_{0-}), \quad (h_{0+}, h_{0+}),$$

is smoothly connected with the connection parameters  $\alpha'_0, \beta'_0$  and  $\rho_0(0)$  such that  $|\alpha'_0| \neq |\beta'_0| \rho_0(0)$ .

We note from Example 3.3 that this condition is automatically satisfied if  $p = 1$  and  $r$  is of order  $\nu$  on some half-neighborhood of 0.

**Theorem 3.6.** Assume that the coefficients  $p$  and  $r$  satisfy Condition 3.5. Then there exists an operator

$$W_0 : L_{2,r} \rightarrow L_{2,r}$$

such that the following hold:

- (a)  $W_0$  is bounded on  $L_{2,|r|}$ ;
- (b)  $J_0 W_0 > I$ , in particular  $W_0^{-1}$  is bounded and  $W_0$  is positive on the Krein space  $L_{2,r}$ ;
- (c)  $(W_0 f)(x) = (J_0 f)(x)$ ,  $\frac{1}{2} \leq |x| \leq 1$ ,  $f \in L_{2,r}$ ;
- (d)  $W_0 \mathcal{F}_{\max} \subset \mathcal{F}_{\max}$ .

This is [7, Theorem 4.2].

**Condition 3.7 (Condition at  $-1$ ).** Let  $p$  and  $r$  be coefficients in (1.1). We assume that a right half neighborhood of  $-1$  is smoothly connected to a right half neighborhood of  $-1$  with the connection parameters  $\alpha'_{-1}, \beta'_{-1}$  and  $\rho_{-1}(0)$  such that  $|\alpha'_{-1}| \neq |\beta'_{-1}| \rho_{-1}(0)$ .

**Condition 3.8 (Condition at  $1$ ).** Let  $p$  and  $r$  be coefficients in (1.1). We assume that a left half-neighborhood of  $1$  is smoothly connected to a left half-neighborhood of  $1$  with the connection parameters  $\alpha'_{+1}, \beta'_{+1}$  and  $\rho_{+1}(0)$  such that  $|\alpha'_{+1}| \neq |\beta'_{+1}| \rho_{+1}(0)$ .

Again, we note from Example 3.3 that these conditions are automatically satisfied if  $p = 1$  and  $r$  is of order  $\nu_{-1}$  and  $\nu_{+1}$  on some half-neighborhood (in  $[-1, 1]$ ) of  $-1$  and  $1$ , respectively.

The following two propositions appear in [7] as Propositions 5.3 and 5.4, respectively.

**Proposition 3.9.** *Assume that the coefficients  $p$  and  $r$  satisfy Condition 3.7. Let  $b$  be an arbitrary complex number. Then there exists an operator*

$$W_{-1} : L_{2,r} \rightarrow L_{2,r}$$

such that the following hold:

- (a)  $W_{-1}$  is bounded on  $L_{2,|r|}$ ;
- (b)  $J_0 W_{-1} > I$ , in particular  $(W_{-1})^{-1}$  is bounded and  $W_{-1}$  is positive on the Krein space  $L_{2,r}$ ;
- (c)  $(W_{-1}f)(x) = (J_0f)(x)$ ,  $-\frac{1}{2} \leq x \leq 1$ ,  $f \in L_{2,r}$ ;
- (d)  $W_{-1}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}[-1, 0] \oplus \mathcal{F}_{\max}[0, 1]$ ;
- (e)  $(W_{-1}f)(-1) = bf(-1)$  for all  $f \in \mathcal{F}_{\max}$ .

**Proposition 3.10.** *Assume that the coefficients  $p$  and  $r$  satisfy Condition 3.8. Let  $b$  be an arbitrary complex number. Then there exists an operator*

$$W_{+1} : L_{2,r} \rightarrow L_{2,r}$$

such that the following hold:

- (a)  $W_{+1}$  is bounded on  $L_{2,|r|}$ ;
- (b)  $J_0 W_{+1} > I$ , in particular  $(W_{+1})^{-1}$  is bounded and  $W_{+1}$  is positive on the Krein space  $L_{2,r}$ ;
- (c)  $(W_{+1}f)(x) = (J_0f)(x)$ ,  $-1 \leq x \leq \frac{1}{2}$ ,  $f \in L_{2,r}$ ;
- (d)  $W_{+1}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}[-1, 0] \oplus \mathcal{F}_{\max}[0, 1]$ ;
- (e)  $(W_{+1}f)(1) = bf(1)$  for all  $f \in \mathcal{F}_{\max}$ .

#### 4. Mixed condition at $\pm 1$ and associated operator

In this section we establish analogues of the above results for a new condition involving both endpoints of the interval  $[-1, 1]$ .

**Condition 4.1 (Condition at  $-1, 1$ ).** Let  $p$  and  $r$  be the coefficients in (1.1). We assume that at least one of the following three conditions is satisfied.

- (A) There are two smooth connections each connecting a right half-neighborhood of  $-1$  to a left half-neighborhood of  $1$  with the connection parameters  $\alpha'_{mj}$ ,  $\beta'_{mj}$  and  $\rho_{mj}(0)$ ,  $j = 1, 2$ , such that

$$\begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\beta'_{m1}|\rho_{m1}(0) & |\beta'_{m2}|\rho_{m2}(0) \end{vmatrix} \neq 0. \quad (4.1)$$

- (B) There are two smooth connections each connecting a left half-neighborhood of  $1$  to a right half-neighborhood of  $-1$  with the connection parameters  $\alpha'_{mj}$ ,  $\beta'_{mj}$  and  $\rho_{mj}(0)$ ,  $j = 1, 2$ , such that (4.1) holds.
- (C) A right half-neighborhood of  $-1$  is smoothly connected to a left half-neighborhood of  $1$  with the connection parameters  $\alpha'_{m1}$ ,  $\beta'_{m1}$  and  $\rho_{m1}(0)$ , and a left half-neighborhood of  $1$  is smoothly connected to a right half-neighborhood of  $-1$  with the connection parameters  $\alpha'_{m2}$ ,  $\beta'_{m2}$  and  $\rho_{m2}(0)$ , such that

$$\begin{vmatrix} |\alpha'_{m1}| & |\beta'_{m2}|\rho_{m2}(0) \\ |\beta'_{m1}|\rho_{m1}(0) & |\alpha'_{m2}| \end{vmatrix} \neq 0.$$

**Example 4.2.** From Example 3.3 it follows that this condition is satisfied if  $p = 1$  and  $r$  has the same order  $\nu$  on a right half-neighborhood of  $-1$  and a left half-neighborhood of  $1$ .

**Example 4.3.** If  $p$  is an even function and  $r$  is odd, then it turns out that Conditions 3.7, 3.8 and 4.1 are equivalent. The first equivalence is clear. For the second, assume that Condition 3.8 is satisfied. Let  $\alpha_{+1}$  and  $\beta_{+1}$  be the corresponding affine functions from Definition 3.1 defined on  $[0, \epsilon]$ . Now define  $\alpha_{m1}(t) = \alpha_{+1}(t)$ ,  $\beta_{m1}(t) = -\beta_{+1}(t)$ ,  $t \in [0, \epsilon)$ , so  $\rho_{m1} = \rho_{+1}$ . Note that  $p$  is locally integrable on  $[\alpha_{+1}(\epsilon), 1)$  by Definition 3.1 (ii). Then define  $\alpha_{m2}(t) = 1 - t$ ,  $\beta_{m2}(t) = -1 + t$ ,  $t \in [0, 1 - \alpha_{+1}(\epsilon))$  and so  $\rho_{m2} = 1$ . Then Condition 4.1(B) is satisfied since (4.1) takes the form

$$\begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\beta'_{m1}|\rho_{m1}(0) & |\beta'_{m2}|\rho_{m2}(0) \end{vmatrix} = \begin{vmatrix} |\alpha'_{+1}| & 1 \\ |\beta'_{+1}|\rho_{+1}(0) & 1 \end{vmatrix}$$

which is nonzero by Condition 3.8. The proof of the converse is similar.

**Example 4.4.** We call a function  $g : [-1, 1] \rightarrow \mathbb{C}$  *nearly odd* (*nearly even*) if there exists a positive constant  $c \neq 1$  such that  $g(-x) = -c g(x)$  ( $g(-x) = c g(x)$ ) for almost all  $x \in (0, 1]$ . We note that if  $p$  is a nearly even function and  $r$  is nearly odd, both Conditions 3.5 and 4.1 are satisfied. Also, Conditions 3.7 and 3.8 are equivalent. The verification is straightforward.

**Example 4.5.** Let  $p = 1$  and  $r(x) = -1$  for  $x \in [-1, 0)$  and  $r(x) = 1 - x$  for  $x \in [0, 1]$ . It is not difficult to verify directly that these functions satisfy Conditions 3.5,

3.7 and 3.8, but not Condition 4.1. In addition notice that  $r$  is of order 0 in a right half-neighborhood of  $-1$  and of order 1 in a left half-neighborhood of 1.

The proof of the following theorem occupies the remainder of this section.

**Theorem 4.6.** *Assume that the coefficients  $p$  and  $r$  satisfy Conditions 3.7, 3.8 and 4.1. Let  $b_{jk}$ ,  $j, k = 1, 2$ , be arbitrary complex numbers. Then there exists an operator*

$$W_{s1} : L_{2,r} \rightarrow L_{2,r}$$

such that the following hold:

- (a)  $W_{s1}$  is bounded on the Hilbert space  $L_{2,|r|}$ ;
- (b)  $J_0 W_{s1} > I$ , in particular  $W_{s1}^{-1}$  is bounded and  $W_{s1}$  is positive on the Krein space  $L_{2,r}$ ;
- (c)  $(W_{s1}f)(x) = (J_0f)(x)$ ,  $-\frac{1}{2} \leq x \leq \frac{1}{2}$ ,  $f \in L_{2,r}$ ;
- (d)  $W_{s1}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}[-1, 0] \oplus \mathcal{F}_{\max}[0, 1]$ ;
- (e) 
$$\begin{bmatrix} (W_{s1}f)(-1) \\ (W_{s1}f)(1) \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix}.$$

*Proof.* We construct  $W_{s1}$  in the form

$$W_{s1} = J_0(X_{s1}^* X_{s1} + I),$$

where

$$X_{s1} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

is a block operator matrix corresponding to the decomposition

$$L_{2,|r|} = L_{2,|r|}(-1, 0) \oplus L_{2,|r|}(0, 1).$$

We split the proof into three parts. The off-diagonal and diagonal entries of  $X_{s1}$  are constructed in the first and second parts, respectively. In the third part we establish the stated properties of  $W_{s1}$ .

**1.** To construct the off-diagonal operators we treat each case (A), (B), (C) of Condition 4.1 separately.

**Case (A).** By Theorem 3.4 there exist operators

$$S_{mj} : L_{2,|r|}(-1, 0) \rightarrow L_{2,|r|}(0, 1), \quad j = 1, 2,$$

which satisfy (S-1)-(S-4) in Theorem 3.4 with  $\iota = [-1, 0]$ ,  $j = [0, 1]$ ,  $a = -1$  and  $b = 1$ . In particular, for  $f \in \mathcal{F}_{\max}[-1, 0]$  and  $j = 1, 2$ ,

$$(S_{mj}f)(1) = |\alpha'_{mj}| f(-1), \quad (S_{mj}^*f)(-1) = |\beta'_{mj}| \rho_{mj}(0) f(1).$$

To simplify the formulas we use the following notation

$$\Upsilon := \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\beta'_{m1}| \rho_{m1}(0) & |\beta'_{m2}| \rho_{m2}(0) \end{vmatrix}.$$

Define

$$X_{21} : L_{2,|r|}(-1, 0) \rightarrow L_{2,|r|}(0, 1),$$

by

$$X_{21} := b_{21} \Upsilon^{-1} \begin{vmatrix} S_{m1} & S_{m2} \\ |\beta'_{m1}| \rho_{m1}(0) & |\beta'_{m2}| \rho_{m2}(0) \end{vmatrix}.$$

Here and below we write such determinants as abbreviations for corresponding linear combinations of operators. For all  $f \in \mathcal{F}_{\max}[-1, 0]$  we have

$$(X_{21}f)(1) = b_{21} \Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| f(-1) & |\alpha'_{m2}| f(-1) \\ |\beta'_{m1}| \rho_{m1}(0) & |\beta'_{m2}| \rho_{m2}(0) \end{vmatrix} = b_{21} f(-1).$$

Also for all  $g \in \mathcal{F}_{\max}[0, 1]$  we have

$$(X_{21}^*g)(-1) = \bar{b}_{21} \Upsilon^{-1} \begin{vmatrix} |\beta'_{m1}| \rho_{m1}(0) g(1) & |\beta'_{m2}| \rho_{m2}(0) g(1) \\ |\beta'_{m1}| \rho_{m1}(0) & |\beta'_{m2}| \rho_{m2}(0) \end{vmatrix} = 0.$$

Now define the opposite off diagonal corner

$$X_{12} : L_{2,|r|}(0, 1) \rightarrow L_{2,|r|}(-1, 0),$$

by

$$X_{12} := -b_{12} \Upsilon^{-1} ( -|\alpha'_{m2}| S_{m1}^* + |\alpha'_{m1}| S_{m2}^* ) = -b_{12} \Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ S_{m1}^* & S_{m2}^* \end{vmatrix}.$$

Then for all  $f \in \mathcal{F}_{\max}[0, 1]$  we have

$$(X_{12}f)(-1) = -b_{12} \Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\beta'_{m1}| \rho_{m1}(0) f(1) & |\beta'_{m2}| \rho_{m2}(0) f(1) \end{vmatrix} = -b_{12} f(1).$$

Also

$$(X_{12}^*f)(1) = -\bar{b}_{12} \Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\alpha'_{m1}| f(-1) & |\alpha'_{m2}| f(-1) \end{vmatrix} = 0.$$

**Case (B).** By Theorem 3.4 there exist operators

$$S_{mj} : L_{2,|r|}(0, 1) \rightarrow L_{2,|r|}(-1, 0), \quad j = 1, 2,$$

which satisfy (S-1)-(S-4) in Theorem 3.4 with  $\iota = [0, 1]$ ,  $j = [-1, 0]$ ,  $a = 1$  and  $b = -1$ . In particular, for all  $f \in \mathcal{F}_{\max}[0, 1]$  and  $j = 1, 2$ ,

$$(S_{mj}f)(-1) = |\alpha'_{mj}| f(1), \quad (S_{mj}^*f)(1) = |\beta'_{mj}| \rho_{mj}(0) f(-1).$$

To simplify the formulas we continue to use the notation

$$\Upsilon := \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\beta'_{m1}| \rho_{m1}(0) & |\beta'_{m2}| \rho_{m2}(0) \end{vmatrix}.$$

Define

$$X_{12} : L_{2,|r|}(0, 1) \rightarrow L_{2,|r|}(-1, 0),$$

by

$$X_{12} = -b_{12} \Upsilon^{-1} \begin{vmatrix} S_{m1} & S_{m2} \\ |\beta'_{m1}| \rho_{m1}(0) & |\beta'_{m2}| \rho_{m2}(0) \end{vmatrix}.$$

Then for all  $f \in \mathcal{F}_{\max}[0, 1]$  we have

$$(X_{12}f)(-1) = -b_{12} \Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| f(1) & |\alpha'_{m2}| f(1) \\ |\beta'_{m1}| \rho_{m1}(0) & |\beta'_{m2}| \rho_{m2}(0) \end{vmatrix} = -b_{12} f(1)$$

and for all  $g \in \mathcal{F}_{\max}[-1, 0]$  we have

$$(X_{12}^*g)(1) = -b_{12} \Upsilon^{-1} \begin{vmatrix} |\beta'_{m1}| \rho_{m1}(0) g(-1) & |\beta'_{m2}| \rho_{m2}(0) g(-1) \\ |\beta'_{m1}| \rho_{m1}(0) & |\beta'_{m2}| \rho_{m2}(0) \end{vmatrix} = 0.$$

Now define the opposite off diagonal corner

$$X_{21} : L_{2,|r|}(-1, 0) \rightarrow L_{2,|r|}(0, 1),$$

by

$$X_{21} = b_{21} \Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ S_{m1}^* & S_{m2}^* \end{vmatrix}.$$

Then for all  $f \in \mathcal{F}_{\max}[-1, 0]$  we have

$$(X_{21}f)(1) = b_{21} \Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\beta'_{m1}| \rho_{m1}(0) f(-1) & |\beta'_{m2}| \rho_{m2}(0) f(-1) \end{vmatrix} = b_{21} f(-1)$$

and for all  $g \in \mathcal{F}_{\max}[0, 1]$  we have

$$(X_{21}^*g)(-1) = \bar{b}_{21} \Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\alpha'_{m1}| g(1) & |\alpha'_{m2}| g(1) \end{vmatrix} = 0.$$

**Case (C).** By Theorem 3.4 there exists an operator

$$S_{m1} : L_{2,|r|}(-1, 0) \rightarrow L_{2,|r|}(0, 1)$$

with the properties listed in Case (A) of this proof and there exists an operator

$$S_{m2} : L_{2,|r|}(0, 1) \rightarrow L_{2,|r|}(-1, 0)$$

with the properties listed in Case (B).

To simplify the formulas in this part of the proof we use the notation

$$\Upsilon := \begin{vmatrix} |\alpha'_{m1}| & |\beta'_{m2}| \rho_{m2}(0) \\ |\beta'_{m1}| \rho_{m1}(0) & |\alpha'_{m2}| \end{vmatrix}.$$

Define

$$X_{12} : L_{2,|r|}(0, 1) \rightarrow L_{2,|r|}(-1, 0)$$

by

$$X_{12} = -b_{12} \Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\beta'_{m2}| \rho_{m2}(0) \\ S_{m1}^* & S_{m2} \end{vmatrix}.$$

Then for all  $f \in \mathcal{F}_{\max}[0, 1]$  we have

$$(X_{12}f)(-1) = -b_{12} \Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\beta'_{m2}| \rho_{m2}(0) \\ |\beta'_{m1}| \rho_{m1}(0) f(1) & |\alpha'_{m2}| f(1) \end{vmatrix} = -b_{12} f(1)$$

and for all  $g \in \mathcal{F}_{\max}[-1, 0]$  we have

$$(X_{12}^*g)(1) = -\bar{b}_{12} \Upsilon^{-1} \begin{vmatrix} s_{m1} & \theta_{m2}(0) \\ s_{m1}g(-1) & \theta_{m2}(0)g(-1) \end{vmatrix} = 0.$$

The other off diagonal operator

$$X_{21} : L_{2,|r|}(-1, 0) \rightarrow L_{2,|r|}(0, 1)$$

is defined as:

$$X_{21} = b_{21} \Upsilon^{-1} \begin{vmatrix} S_{m1} & S_{m2}^* \\ |\beta'_{m1}| \rho_{m1}(0) & |\alpha'_{m2}| \end{vmatrix}.$$

Then for all  $f \in \mathcal{F}_{\max}[-1, 0]$  we have

$$(X_{21}f)(1) = b_{21} \Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| f(-1) & |\beta'_{m2}| \rho_{m2}(0) f(-1) \\ |\beta'_{m1}| \rho_{m1}(0) & |\alpha'_{m2}| \end{vmatrix} = b_{21} f(-1)$$

and for all  $g \in \mathcal{F}_{\max}[0, 1]$  we have

$$(X_{21}^*g)(-1) = \bar{b}_{21} \Upsilon^{-1} \begin{vmatrix} |\beta'_{m1}| \rho_{m1}(0) g(1) & |\alpha'_{m2}| g(1) \\ |\beta'_{m1}| \rho_{m1}(0) & |\alpha'_{m2}| \end{vmatrix} = 0.$$

We conclude this part of the proof by summarizing that in each of the three cases above we have defined operators

$$X_{12} : L_{2,|r|}(0, 1) \rightarrow L_{2,|r|}(-1, 0) \quad \text{and} \quad X_{21} : L_{2,|r|}(-1, 0) \rightarrow L_{2,|r|}(0, 1)$$

such that

$$\begin{aligned} X_{12} \mathcal{F}_{\max}[0, 1] &\subset \mathcal{F}_{\max}[-1, 0], & X_{12}^* \mathcal{F}_{\max}[-1, 0] &\subset \mathcal{F}_{\max}[1, 0], \\ X_{21}^* \mathcal{F}_{\max}[0, 1] &\subset \mathcal{F}_{\max}[-1, 0], & X_{21} \mathcal{F}_{\max}[-1, 0] &\subset \mathcal{F}_{\max}[1, 0], \end{aligned}$$

and for all  $f \in \mathcal{F}_{\max}[0, 1]$  and  $g \in \mathcal{F}_{\max}[-1, 0]$  we have

$$\begin{aligned} (X_{12}f)(-1) &= -b_{12} f(1), & (X_{12}^*g)(1) &= 0, \\ (X_{21}^*f)(-1) &= 0, & (X_{21}g)(1) &= b_{21} f(-1). \end{aligned}$$

This completes the construction of the off-diagonal entries of  $X_{s1}$ .

**2.** To construct the diagonal entries we need two self-adjoint operators  $P_{1,-}$  and  $P_{1,+}$  defined as follows. Let  $\phi_1 : [-1, 1] \rightarrow [0, 1]$  be an even function with  $\phi_1 \in C^1[-1, 1]$  and such that

$$\phi_1(-1) = 1, \quad \phi_1(x) = 0 \quad \text{for } 0 \leq |x| \leq 1/2, \quad \phi_1(1) = 1.$$

We now define

$$P_{1,-} : L_{2,|r|}(-1, 0) \rightarrow L_{2,|r|}(-1, 0) \quad \text{and} \quad P_{1,+} : L_{2,|r|}(0, 1) \rightarrow L_{2,|r|}(0, 1)$$

by

$$(P_{1,-}f)(x) = f(x)\phi_1(x), \quad f \in L_{2,|r|}(-1, 0), \quad x \in [-1, 0], \quad (4.2)$$

and

$$(P_{1,+}f)(x) = f(x)\phi_1(x), \quad f \in L_{2,|r|}(0, 1), \quad x \in [0, 1]. \quad (4.3)$$

These operators enjoy the following properties:

$$(P_{1,-}f)(x) = 0, \quad f \in L_{2,|r|}(-1, 0), \quad -\frac{1}{2} \leq x \leq 0,$$

$$(P_{1,+}f)(x) = 0, \quad f \in L_{2,|r|}(0, 1), \quad 0 \leq x \leq \frac{1}{2},$$

$$P_{1,-}\mathcal{F}_{\max}[-1, 0] \subset \mathcal{F}_{\max}[-1, 0], \quad P_{1,+}\mathcal{F}_{\max}[0, 1] \subset \mathcal{F}_{\max}[0, 1],$$

and

$$(P_{1,-}f)(-1) = f(-1), \quad f \in \mathcal{F}_{\max}[-1, 0],$$

$$(P_{1,+}f)(1) = f(1), \quad f \in \mathcal{F}_{\max}[0, 1].$$

Now we use Condition 3.7 to construct the operator  $X_{11}$ . As in Proposition 3.9, Theorem 3.4 implies that there exists an operator  $S_{-1} : L_{2,|r|}(-1, 0) \rightarrow L_{2,|r|}(-1, 0)$  with the properties listed there. In particular for all  $f \in \mathcal{F}_{\max}[-1, 0]$  we have

$$(S_{-1}f)(-1) = |\alpha'_{-1}|f(-1), \quad (S_{-1}^*f)(-1) = |\beta'_{-1}|\rho_{-1}(0)f(-1).$$

Since  $|\alpha'_{-1}| \neq |\beta'_{-1}|\rho_{-1}(0)$  we can choose complex numbers  $\gamma_1$  and  $\gamma_2$  such that

$$\gamma_1|\alpha'_{-1}| + \gamma_2 = -b_{11} - 1, \quad \bar{\gamma}_1|\beta'_{-1}|\rho_{-1}(0) + \bar{\gamma}_2 = 1.$$

Let  $P_{1,-}$  be the operator defined in (4.2). Put

$$X_{11} = \gamma_1 S_{-1} + \gamma_2 P_{1,-}.$$

Then for all  $f \in \mathcal{F}_{\max}[-1, 0]$  we have

$$(X_{11}f)(-1) = (-b_{11} - 1)f(-1), \quad (X_{11}^*f)(-1) = f(-1).$$

Note also that

$$X_{11}\mathcal{F}_{\max}[-1, 0] \subset \mathcal{F}_{\max}[-1, 0] \quad \text{and} \quad X_{11}^*\mathcal{F}_{\max}[-1, 0] \subset \mathcal{F}_{\max}[-1, 0].$$

To construct  $X_{22}$  we use Condition 3.8. By Theorem 3.4 there exists a bounded operator

$$S_{+1} : L_{2,|r|}(0, 1) \rightarrow L_{2,|r|}(0, 1)$$



such that

$$S_{+1}\mathcal{F}_{\max}[0, 1] \subset \mathcal{F}_{\max}[0, 1], \quad S_{+1}^*\mathcal{F}_{\max}[0, 1] \subset \mathcal{F}_{\max}[0, 1],$$

and for all  $f \in \mathcal{F}_{\max}[0, 1]$ ,

$$(S_{+1}f)(1) = |\alpha'_{+1}|f(1), \quad (S_{+1}^*f)(1) = |\beta'_{+1}|\rho_{+1}(0)f(-1).$$

Since  $|\alpha'_{+1}| \neq |\beta'_{+1}|\rho_{+1}(0)$  we can choose complex numbers  $\delta_1$  and  $\delta_2$  such that

$$\delta_1|\alpha'_{+1}| + \delta_2 = -b_{11} - 1, \quad \bar{\delta}_1|\beta'_{+1}|\rho_{+1}(0) + \bar{\delta}_2 = 1.$$

Let  $P_{1,+}$  be the operator defined in (4.3). Put

$$X_{22} = \delta_1 S_{+1} + \delta_2 P_{1,+}.$$

Then for all  $f \in \mathcal{F}_{\max}[0, 1]$  we have

$$(X_{22}f)(1) = (b_{22} - 1)f(1) \quad \text{and} \quad (X_{22}^*f)(1) = f(1).$$

Note also that

$$X_{22}\mathcal{F}_{\max}[0, 1] \subset \mathcal{F}_{\max}[0, 1] \quad \text{and} \quad X_{22}^*\mathcal{F}_{\max}[0, 1] \subset \mathcal{F}_{\max}[0, 1].$$

**3.** Now we formally define  $W_{s_1} := J_0(X_{s_1}^*X_{s_1} + I)$  where

$$X_{s_1} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

To complete the proof, we verify the properties of  $W_{s_1}$  stated in the theorem. Indeed, (a) and (b) are immediate, and since  $(X_{ij}f)(x) = 0$  whenever  $-\frac{1}{2} \leq x \leq \frac{1}{2}$ , (c) follows. Moreover, each of the operators  $X_{ij}$  maps  $\mathcal{F}_{\max}[-1, 0]$  or  $\mathcal{F}_{\max}[0, 1]$  to  $\mathcal{F}_{\max}[-1, 0]$  or  $\mathcal{F}_{\max}[0, 1]$  according to its position in the matrix, so (d) holds.

Finally, we check the effect of the individual components at the boundary points  $-1$  and  $1$ . Evidently

$$X_{s_1}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}, \quad X_{s_1}^*\mathcal{F}_{\max} \subset \mathcal{F}_{\max}.$$

Moreover for  $f, g \in \mathcal{F}_{\max}$  we have

$$\begin{bmatrix} (X_{s_1}f)(-1) \\ (X_{s_1}f)(1) \end{bmatrix} = \begin{bmatrix} (X_{11}f)(-1) + (X_{12}f)(-1) \\ (X_{21}f)(1) + (X_{22}f)(1) \end{bmatrix} = \begin{bmatrix} (-b_{11} - 1)f(-1) - b_{12}f(1) \\ b_{21}f(-1) + (b_{22} - 1)f(1) \end{bmatrix}$$

and

$$\begin{bmatrix} (X_{s_1}^*g)(-1) \\ (X_{s_1}^*g)(1) \end{bmatrix} = \begin{bmatrix} (X_{11}^*g)(-1) + (X_{21}^*g)(-1) \\ (X_{12}^*g)(1) + (X_{22}^*g)(1) \end{bmatrix} = \begin{bmatrix} g(-1) + 0 \\ 0 + g(1) \end{bmatrix}.$$

Substituting  $g = X_{s_1}f \in \mathcal{F}_{\max}$ , we get

$$\begin{bmatrix} (X_{s_1}^*X_{s_1}f)(-1) \\ (X_{s_1}^*X_{s_1}f)(1) \end{bmatrix} = \begin{bmatrix} (-b_{11} - 1)f(-1) - b_{12}f(1) \\ b_{21}f(-1) + (b_{22} - 1)f(1) \end{bmatrix}.$$

With  $Y_{s_1} = X_{s_1}^* X_{s_1} + I$  we have

$$\begin{bmatrix} (Y_{s_1}f)(-1) \\ (Y_{s_1}f)(1) \end{bmatrix} = \begin{bmatrix} -b_{11}f(-1) - b_{12}f(1) \\ b_{21}f(-1) + b_{22}f(1) \end{bmatrix} = \begin{bmatrix} -b_{11} & -b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix},$$

which proves (e) since  $W_{s_1} = J_0 Y_{s_1}$ .  $\square$

**Remark 4.7.** Notice that the operators  $W_{-1}$  and  $W_{+1}$  from Propositions 3.9 and 3.10 satisfy

$$\begin{bmatrix} (W_{-1}f)(-1) \\ (W_{-1}f)(1) \end{bmatrix} = \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} (W_{+1}f)(-1) \\ (W_{+1}f)(1) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix},$$

respectively, with arbitrary  $b \in \mathbb{C}$ . A stronger conclusion is contained in Theorem 4.6 (e) under stronger assumptions.

## 5. Two essential or two non-essential boundary conditions

The first theorem of this section deals with the case of two non-essential boundary conditions.

**Theorem 5.1.** *Assume that the following two conditions are satisfied.*

(a)  $\mathbf{N}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .

(b) *The coefficients  $p$  and  $r$  satisfy Condition 3.5.*

*Then there is a basis for each root subspace of  $A$ , so that the union of all these bases is a Riesz basis of  $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}$ .*

*Proof.* By (2.2), the form domain of  $A$  is given as

$$\mathcal{F}(A) = \left\{ \begin{bmatrix} f \\ \mathbf{v} \end{bmatrix} \in \begin{array}{c} L_{2,r} \\ \oplus \\ \mathbb{C}_\Delta^2 \end{array} : f \in \mathcal{F}_{\max}, \mathbf{v} \in \mathbb{C}^2 \right\}.$$

Recalling  $W_0$  from Theorem 3.6, we easily see that the operator

$$W = \begin{bmatrix} W_0 & 0 \\ 0 & \Delta^{-1} \end{bmatrix} : \begin{array}{c} L_{2,r} \\ \oplus \\ \mathbb{C}_\Delta^2 \end{array} \rightarrow \begin{array}{c} L_{2,r} \\ \oplus \\ \mathbb{C}_\Delta^2 \end{array}.$$

is bounded, boundedly invertible and positive in the Krein space  $L_{2,r} \oplus \mathbb{C}_\Delta^2$ . A simple verification shows that  $W\mathcal{F}(A) \subset \mathcal{F}(A)$  so the theorem follows from Theorem 2.2.  $\square$

We now consider the case of two essential conditions.

**Theorem 5.2.** *Assume that the following three conditions are satisfied.*

(a)  $\mathbf{N}_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .

(b) *The coefficients  $p$  and  $r$  satisfy Condition 3.5.*

(c) *One of the following holds:*

- (i)  $\Delta > 0$  and the coefficients  $p$  and  $r$  satisfy Condition 3.7.
- (ii)  $\Delta < 0$  and the coefficients  $p$  and  $r$  satisfy Condition 3.8.
- (iii) the coefficients  $p$  and  $r$  satisfy Conditions 3.7, 3.8 and 4.1.

Then there is a basis for each root subspace of  $A$ , so that the union of all these bases is a Riesz basis of  $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}$ .

*Proof.* Define the following two Krein spaces:

$$\mathcal{K}_0 := L_{2,r}\left(-\frac{1}{2}, \frac{1}{2}\right), \quad \mathcal{K}_1 := L_{2,r}\left(-1, -\frac{1}{2}\right)[\dot{+}]L_{2,r}\left(\frac{1}{2}, 1\right).$$

Extending functions in  $\mathcal{K}_0$  and  $\mathcal{K}_1$  by zero, we consider the spaces  $\mathcal{K}_0$  and  $\mathcal{K}_1$  as subspaces of  $L_{2,r}$ . Then

$$L_{2,r} = \mathcal{K}_0[\dot{+}]\mathcal{K}_1.$$

As in the previous proof our goal is to construct  $W : L_{2,r} \oplus \mathbb{C}_\Delta^2 \rightarrow L_{2,r} \oplus \mathbb{C}_\Delta^2$ . The first step is to define  $W_{01} : L_{2,r} \rightarrow L_{2,r}$ . We proceed by considering each case in (c) separately.

(i) Let  $W_0$  be the operator constructed in Theorem 3.6 and let  $W_{-1}$  be the operator constructed in Proposition 3.9 with  $b = 1$ . Property (c) in Theorem 3.6 and Proposition 3.9 imply that  $\mathcal{K}_0$  and  $\mathcal{K}_1$  are invariant under  $W_0$  and  $W_{-1}$ . Since we chose  $b = 1$ , we have  $(W_{-1}f)(-1) = f(-1)$  and  $(W_{-1}f)(1) = f(1)$ . Define

$$W_{01} := W_0|_{\mathcal{K}_0}[\dot{+}]W_{-1}|_{\mathcal{K}_1}. \quad (5.1)$$

Since  $W_0$  and  $W_{-1}$  are bounded, boundedly invertible and positive in the Krein space  $L_{2,r}$ , so is the operator  $W_{01}$ . Also,  $W_{01}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}$  and

$$\begin{bmatrix} (W_{01}f)(-1) \\ (W_{01}f)(1) \end{bmatrix} = \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix}. \quad (5.2)$$

(ii) Instead of  $W_{-1}$  in (i), we use the operator  $W_{+1}$  constructed in Proposition 3.10 with  $b = -1$ . Redefining the operator  $W_{01}$  as

$$W_{01} := W_0|_{\mathcal{K}_0}[\dot{+}]W_{+1}|_{\mathcal{K}_1}. \quad (5.3)$$

we see that it is again bounded, boundedly invertible, and positive in the Krein space  $L_{2,r}$ ,  $W_{01}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}$  and (since we use  $b = -1$ )

$$\begin{bmatrix} (W_{01}f)(-1) \\ (W_{01}f)(1) \end{bmatrix} = - \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix}. \quad (5.4)$$

(iii) This time we replace  $W_{-1}$  from (i) by  $W_{s1}$  from Theorem 4.6, so we define the operator

$$W_{01} := W_0|_{\mathcal{K}_0}[\dot{+}]W_{s1}|_{\mathcal{K}_1}, \quad (5.5)$$

which is again bounded, boundedly invertible and positive in the Krein space  $L_{2,r}$ . Also,  $W_{01}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}$  and

$$\begin{bmatrix} (W_{s1}f)(-1) \\ (W_{s1}f)(1) \end{bmatrix} = \Delta^{-1} \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix}. \tag{5.6}$$

Finally we define  $W : L_{2,r} \oplus \mathbb{C}_{\Delta}^2 \rightarrow L_{2,r} \oplus \mathbb{C}_{\Delta}^2$  by

$$W = \begin{bmatrix} W_{01} & 0 \\ 0 & I \end{bmatrix} \tag{5.7}$$

in case (c)(i),

$$W = \begin{bmatrix} W_{01} & 0 \\ 0 & -I \end{bmatrix} \tag{5.8}$$

in case (c)(ii), and

$$W = \begin{bmatrix} W_{01} & 0 \\ 0 & \Delta^{-1} \end{bmatrix} \tag{5.9}$$

in case (c)(iii).

By (2.3), the form domain of  $A$  is

$$\mathcal{F}(A) = \left\{ \begin{bmatrix} f \\ f(-1) \\ f(1) \end{bmatrix} \in \begin{matrix} L_{2,r} \\ \oplus \\ \mathbb{C}_{\Delta}^2 \end{matrix} : f \in \mathcal{F}_{\max} \right\}.$$

A straightforward verification shows that in each case (5.7), (5.8), and (5.9),  $W$  is a bounded, boundedly invertible, positive operator in the Krein space  $L_{2,r} \oplus \mathbb{C}_{\Delta}^2$ . Moreover  $W\mathcal{F}(A) \subset \mathcal{F}(A)$  via (5.2), (5.4) or (5.6). Now the theorem follows from Theorem 2.2.  $\square$

**Example 5.3.** Consider the eigenvalue problem

$$\begin{aligned} -f'' &= \lambda r f \\ f'(1) &= \lambda f(-1) \\ -f'(-1) &= \lambda f(1), \end{aligned}$$

where  $r(x) = \operatorname{sgn} x, x \in [-1, 1]$ , as in our example in the Introduction. Then clearly  $\mathbf{N}_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , giving (a) in Theorem 5.2 and (b) follows from the note

after Condition 3.5. Moreover, an easy computation gives  $\Delta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , which is indefinite. Condition (c) now follows from Examples 4.2 and 4.3, so Theorem 5.2 applies.

On the other hand, if instead we take  $r$  as in Example 4.5, then as we have seen, Condition 4.1 fails and hence so does (c)(iii) in Theorem 5.2. Therefore Theorem 5.2 gives no conclusion about a Riesz basis for this amended case.

## 6. One essential and one non-essential boundary condition

The main result of this section is the following theorem. Its proof will occupy the most of the section and then we will proceed to some examples.

**Theorem 6.1.** *Assume that the following three conditions are satisfied.*

- (a)  $\mathbf{N} = \begin{bmatrix} u & v & 0 & 0 \\ * & * & * & 1 \end{bmatrix}$  or  $\mathbf{N} = \begin{bmatrix} u & v & 0 & 0 \\ * & * & 1 & 0 \end{bmatrix}$ , where  $|u|^2 + |v|^2 > 0$  and the asterisks stand for arbitrary complex numbers.
- (b) The coefficients  $p$  and  $r$  satisfy Condition 3.5.
- (c) One of the following holds.
- (i)  $u = 1, v = 0$  and the coefficients  $p$  and  $r$  satisfy Condition 3.7.
  - (ii)  $u = 0, v = 1$  and the coefficients  $p$  and  $r$  satisfy Condition 3.8.
  - (iii)  $uv \neq 0$  and the coefficients  $p$  and  $r$  satisfy Conditions 3.7, 3.8 and 4.1.

Then there is a basis for each root subspace of  $A$ , so that the union of all these bases is a Riesz basis of  $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}$ .

*Proof.* It follows from (a) that the form domain of  $A$  is

$$\mathcal{F}(A) = \left\{ \begin{bmatrix} f \\ uf(-1) + vf(1) \\ z \end{bmatrix} \in \begin{matrix} L_{2,r} \\ \oplus \\ \mathbb{C}_{\Delta}^2 \end{matrix} : f \in \mathcal{F}_{\max}, z \in \mathbb{C} \right\}. \quad (6.1)$$

It is no restriction if we scale the first boundary condition so that

$$|u|^2 + |v|^2 = 1. \quad (6.2)$$

As in the previous proofs we shall construct  $W : L_{2,r} \oplus \mathbb{C}_{\Delta}^2 \rightarrow L_{2,r} \oplus \mathbb{C}_{\Delta}^2$  in blocks. We divide the proof into three parts and two lemmas.

1. First we define a bounded operator  $W_{01} : L_{2,r} \rightarrow L_{2,r}$  such that

$$J_0 W_{01} > I, \quad (6.3)$$

$$W_{01} \mathcal{F}_{\max} \subset \mathcal{F}_{\max}, \quad (6.4)$$

$$u(W_{01}f)(-1) + v(W_{01}f)(1) = 0, \quad f \in \mathcal{F}_{\max}. \quad (6.5)$$

We distinguish the three cases in (c) above.

(i) As in the proof of Theorem 5.2(i), we define  $W_{01}$  by (5.1), but now using  $b = 0$  instead of  $b = 1$ . Then  $W_{01}$  is a bounded operator in the Krein space  $L_{2,r}$ , and it satisfies (6.4) and  $(W_{01}f)(-1) = 0$ ,  $(W_{01}f)(1) = f(1)$  and hence (6.5). Inequality (6.3) follows from (5.1), Theorem 3.6(b) and Proposition 3.9(b).

(ii) This time we define  $W_{01}$  by (5.3), but now using  $b = 0$  instead of  $b = -1$ . Then  $W_{01}$  is a bounded operator in the Krein space  $L_{2,r}$ , it satisfies (6.4) and  $(W_{01}f)(-1) = -f(-1)$ ,  $(W_{01}f)(1) = 0$  and hence (6.5). In this case inequality (6.3) follows from (5.3), Theorem 3.6(b) and Proposition 3.10(b).

(iii) We now define  $W_{01}$  as in the proof of Theorem 5.2(iii), but instead of using  $\Delta^{-1}$  in 5.6 we use the zero  $2 \times 2$  matrix  $0$ . Then  $W_{01}$  is a bounded operator in the

Krein space  $L_{2,r}$ , it satisfies (6.4) and  $(W_{01}f)(-1) = 0$ ,  $(W_{01}f)(1) = 0$  and hence (6.5). Inequality (6.3) follows from (5.5), Theorem 3.6 (b) and Theorem 4.6 (b).

**2.** Next we define an integral operator  $K$  which will be a perturbation of  $W_{01}$ .

**2.1.** We start by writing the inverse of the matrix  $\Delta$  in the form

$$\Delta^{-1} = \begin{bmatrix} \eta_{11} & \eta_{12} \\ \bar{\eta}_{12} & \eta_{22} \end{bmatrix},$$

and setting  $\eta := \max\{|\eta_{11}|, |\eta_{12}|\} > 0$ , with  $\delta_2 \geq \delta_1 > 0$  as the eigenvalues of  $|\Delta|$ . We also define three positive constants

$$\alpha := \frac{\delta_2}{1 + 2\|r\|_1 \delta_2 \eta^2},$$

$$c := \frac{\alpha}{2\delta_2} \sqrt{\frac{\delta_1}{2}}, \tag{6.6}$$

$$\kappa := \frac{2\delta_2 \eta^2 \|r\|_1}{1 + 2\|r\|_1 \delta_2 \eta^2} = 2\alpha \eta^2 \|r\|_1. \tag{6.7}$$

Notice that

$$1 - \kappa = \frac{1}{1 + 2\|r\|_1 \delta_2 \eta^2} = \frac{\alpha}{\delta_2}. \tag{6.8}$$

**2.2.** Since  $r$  is integrable over  $[-1, 1]$ , we there exists  $\gamma \in [0, 1]$  such that

$$-\int_{-1}^{-\gamma} r + \int_{\gamma}^1 r \leq \left(\frac{c}{\alpha\eta}\right)^2. \tag{6.9}$$

Noting that  $p^{-1/2} \in L_2(0, 1) \subset L_1(0, 1)$  we can define

$$\phi(x) = \int_0^x p^{-1/2} \chi_{[\gamma, 1]}, \quad x \in [0, 1].$$

Extending  $\phi$  as an even function over  $[-1, 1]$  we see that  $\phi \in \mathcal{F}_{\max}$ . Since  $\phi(1)$  is a positive real number, we define  $\psi = \phi/\phi(1)$ . Clearly  $\psi : [-1, 1] \rightarrow [0, 1]$  is an even function in  $\mathcal{F}_{\max}$  such that

$$\psi(-1) = 1, \quad \psi(0) = 0, \quad \psi(1) = 1, \tag{6.10}$$

and, by (6.9),

$$\|\psi\|_{2,|r|} \leq \frac{c}{\alpha\eta}. \tag{6.11}$$

**2.3.** Define

$$\psi_j(x) = \begin{cases} \alpha \eta_{1j} \bar{u} \psi(x), & x \in [-1, 0), \\ \alpha \eta_{1j} \bar{v} \psi(x), & x \in [0, 1]. \end{cases} \tag{6.12}$$

Since  $\psi \in \mathcal{F}_{\max}$  and  $\psi(0) = 0$ , the functions  $\psi_1$  and  $\psi_2$  belong to  $\mathcal{F}_{\max}$ . Set

$$\omega(x) := \eta_{11} \overline{\psi_1(x)} + \eta_{12} \overline{\psi_2(x)}, \quad x \in [-1, 1],$$

and define  $k : [-1, 1] \times [-1, 1] \rightarrow \mathbb{C}$  by

$$k(x, t) = \begin{cases} \overline{u\omega(x)} & \text{if } t \leq -|x|, \\ \overline{v\omega(t)} & \text{if } x > |t|, \\ v\omega(x) & \text{if } t \geq |x|, \\ \overline{u\omega(t)} & \text{if } x < -|t|. \end{cases} \quad (6.13)$$

By the definitions of  $\psi_1, \psi_2$  and  $\omega$ , since  $\psi$  is a nonnegative even function, for all  $x \in [0, 1]$  we have

$$\overline{u\omega(-x)}, \overline{v\omega(x)} \in \mathbb{R}, \quad \text{and} \quad \overline{v\omega(-x)} = u\omega(x). \quad (6.14)$$

Since  $\omega$  is continuous, it follows from (6.14) and (6.13) that  $k$  is a continuous function. Moreover, by (6.2) and (6.12),

$$|\omega(t)| < \eta\eta\alpha + \eta\eta\alpha = 2\eta^2\alpha.$$

Therefore (6.7) shows that

$$|k(x, t)| \leq 2\eta^2\alpha = \frac{\kappa}{\|r\|_1}. \quad (6.15)$$

The first of our two lemmas is as follows.

**Lemma 6.2.** *Let  $K : L_{2,r} \rightarrow L_{2,r}$  be the integral operator defined by*

$$(Kf)(x) := \int_{-1}^1 k(x, t) f(t) r(t) dt, \quad f \in L_{2,r}.$$

Then

- (I) *The operator  $K$  is bounded and self-adjoint on  $L_{2,r}$  and  $\|K\|_{2,|r|} \leq \kappa$ .*
- (II) *The range of  $K$  is contained in  $\mathcal{F}_{\max}$ .*

*Proof.* (I) We first note that for  $f$  in  $L_{2,r}$  the function  $fr$  is integrable on  $(-1, 1)$ . In fact

$$\begin{aligned} \int_{-1}^1 |fr| &= \int_{-1}^1 |r|^{1/2} (|f||r|^{1/2}) \\ &\leq \left( \int_{-1}^1 |r| \right)^{1/2} \left( \int_{-1}^1 |f|^2 |r| \right)^{1/2} = \|r\|_1^{1/2} \|f\|_{2,r}. \end{aligned} \quad (6.16)$$

For  $f \in L_{2,|r|}$  we calculate

$$\begin{aligned} \|Kf\|_{2,|r|}^2 &\leq \int_{-1}^1 \int_{-1}^1 |k(x, t)| |f(t)| |r(t)| dt \int_{-1}^1 |k(x, s)| |f(s)| |r(s)| ds |r(x)| dx \\ &\leq \frac{\kappa^2}{\|r\|_1^2} \int_{-1}^1 \left( \int_{-1}^1 |f| |r| \right)^2 |r(x)| dx \leq \kappa \|f\|_{2,|r|}^2, \end{aligned}$$

by virtue of (6.15) and (6.16). Thus  $\|K\|_{2,|r|} \leq \kappa$ , so  $K$  is bounded, and self-adjointness follows from (6.13) since  $k(x, t) = \overline{k(t, x)}$ ,  $x, t \in [-1, 1]$ .

(II) Let  $f \in L_{2,r}$ . By definition, for  $-1 \leq x < 0$ ,

$$(Kf)(x) = u\overline{\omega(x)} \int_{-1}^x (fr)(t)dt + \bar{u} \int_x^{-x} (\omega fr)(t)dt + v\overline{\omega(x)} \int_{-x}^1 (fr)(t)dt$$

and, for  $0 < x \leq 1$ ,

$$(Kf)(x) = u\overline{\omega(x)} \int_{-1}^{-x} (fr)(t)dt + \bar{v} \int_x^{-x} (\omega fr)(t)dt + v\overline{\omega(x)} \int_x^1 (fr)(t)dt.$$

The function  $fr$  is integrable on  $(-1, 1)$  by (6.16). Since  $\omega \in \mathcal{F}_{\max}$  the function  $\omega fr$  is also integrable on  $(-1, 1)$ . Moreover,

$$\lim_{x \uparrow 0} (Kf)(x) = \lim_{x \downarrow 0} (Kf)(x) = (Kf)(0) = 0.$$

Therefore for each  $f \in L_{2,|r|}$  the function  $Kf$  is absolutely continuous on  $[-1, 1]$ . For almost all  $x \in [-1, 0)$ , we have

$$\begin{aligned} (Kf)'(x) &= u\bar{\omega}'(x) \int_{-1}^x (fr)(t)dt + v\bar{\omega}'(x) \int_{-x}^1 (fr)(t)dt \\ &\quad + u\overline{\omega(x)} (fr)(x) - \bar{u}\omega(x) (fr)(x) - \bar{u}\omega(-x) (fr)(-x) + v\overline{\omega(x)} (fr)(-x), \end{aligned}$$

and, for almost all  $x \in (0, 1]$ ,

$$\begin{aligned} (Kf)'(x) &= u\bar{\omega}'(x) \int_{-1}^{-x} (fr)(t)dt + v\bar{\omega}'(x) \int_x^1 (fr)(t)dt \\ &\quad - u\overline{\omega(x)} (fr)(-x) + \bar{v}\omega(-x) (fr)(-x) + \bar{v}\omega(x) (fr)(x) - v\overline{\omega(x)} (fr)(x). \end{aligned}$$

By (6.14) the terms not involving integrals in the above two equations cancel in pairs. Thus  $Kf \in \mathcal{F}_{\max}$  for all  $f \in L_{2,|r|}$  since  $\bar{\omega} \in \mathcal{F}_{\max}$ . This completes the proof of the lemma.  $\square$

**3.** We create off-diagonal blocks for  $W$  by means of the operator  $Z : \mathbb{C}_{\Delta}^2 \rightarrow L_{2,r}$  which we define by

$$Z\mathbf{a} := a_1 \psi_1 + a_2 \psi_2, \quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{C}^2.$$

The adjoint  $Z^{[*]} : L_{2,r} \rightarrow \mathbb{C}_{\Delta}^2$  of  $Z$  is given by

$$Z^{[*]}f = \Delta^{-1} \begin{bmatrix} [f, \psi_1] \\ [f, \psi_2] \end{bmatrix}, \quad f \in L_{2,r}.$$

Equalities (6.2), (6.11) and (6.12) yield  $\|\psi_1\|_{2,|r|} \leq c$  and  $\|\psi_2\|_{2,|r|} \leq c$ . Therefore

$$\begin{aligned} \int_{-1}^1 |Z\mathbf{a}|^2 |r| &\leq 2(|a_1|^2 \|\psi_1\|_{2,|r|}^2 + |a_2|^2 \|\psi_2\|_{2,|r|}^2) \\ &\leq 2c^2 \mathbf{a}^* \mathbf{a} \leq 2 \frac{c^2}{\delta_1} \mathbf{a}^* |\Delta| \mathbf{a}. \end{aligned}$$



Consequently, by (6.6),

$$\|Z\| = \|Z^{[*]}\| \leq c\sqrt{\frac{2}{\delta_1}} = \frac{\alpha}{2\delta_2}. \quad (6.17)$$

The second lemma we need is as follows.

**Lemma 6.3.** *Let the operator  $W : L_{2,r} \oplus \mathbb{C}_\Delta^2 \rightarrow L_{2,r} \oplus \mathbb{C}_\Delta^2$  be defined by*

$$W := \begin{bmatrix} W_{01} + K & Z \\ Z^{[*]} & \alpha \Delta^{-1} \end{bmatrix}.$$

Then

- (I)  $W$  is bounded and uniformly positive on  $L_{2,r} \oplus \mathbb{C}_\Delta^2$ .
- (II)  $W\mathcal{F}(A) \subset \mathcal{F}(A)$ .

*Proof.* (I) The operator  $W$  is bounded since each of its components is bounded. To prove that  $W$  is uniformly positive, we shall show that the operator  $JW$  is uniformly positive in the Hilbert space  $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}^2$ . From Lemma 6.2,  $\|K\| = \|JK\| \leq \kappa$  and

$$\left\| \begin{bmatrix} 0 & Z \\ Z^{[*]} & 0 \end{bmatrix} \right\| = \left\| J \begin{bmatrix} 0 & Z \\ Z^{[*]} & 0 \end{bmatrix} \right\| \leq \frac{\alpha}{2\delta_2}$$

follows from (6.17). Thus

$$\begin{aligned} \left\langle JW \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix}, \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} J_0 W_{01} & 0 \\ 0 & \alpha |\Delta|^{-1} \end{bmatrix} \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix}, \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix} \right\rangle \\ &\quad + \left\langle \begin{bmatrix} J_0 K & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix}, \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix} \right\rangle + \left\langle J \begin{bmatrix} 0 & Z \\ Z^{[*]} & 0 \end{bmatrix} \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix}, \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix} \right\rangle \\ &= \langle J_0 W_{01} f, f \rangle + \alpha \mathbf{a}^* \mathbf{a} + \langle J_0 K f, f \rangle + \left\langle J \begin{bmatrix} 0 & Z \\ Z^{[*]} & 0 \end{bmatrix} \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix}, \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix} \right\rangle \\ &\geq \langle f, f \rangle + \frac{\alpha}{\delta_2} \mathbf{a}^* |\Delta| \mathbf{a} - \kappa \langle f, f \rangle - \frac{\alpha}{2\delta_2} (\langle f, f \rangle + \mathbf{a}^* |\Delta| \mathbf{a}) \\ &\geq \left(1 - \kappa - \frac{\alpha}{2\delta_2}\right) \langle f, f \rangle + \left(\frac{\alpha}{\delta_2} - \frac{\alpha}{2\delta_2}\right) \mathbf{a}^* |\Delta| \mathbf{a} \\ &= \left(\frac{\alpha}{\delta_2} - \frac{\alpha}{2\delta_2}\right) \langle f, f \rangle + \frac{\alpha}{2\delta_2} \mathbf{a}^* |\Delta| \mathbf{a} \quad (\text{by (6.8)}) \\ &= \frac{\alpha}{2\delta_2} (\langle f, f \rangle + \mathbf{a}^* |\Delta| \mathbf{a}) \\ &= \frac{\alpha}{2\delta_2} \left\langle \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix}, \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix} \right\rangle, \end{aligned}$$

as required.

(II) We start with the identity

$$u(Kf)(-1) + v(Kf)(1) = \eta_{11} [f, \psi_1] + \eta_{12} [f, \psi_2], \quad f \in L_{2,|r|}, \quad (6.18)$$

which follows from the calculation

$$\begin{aligned}
 &u(Kf)(-1) + v(Kf)(1) \\
 &= u \int_{-1}^1 k(-1, t) f(t) r(t) dt + v \int_{-1}^1 k(1, t) f(t) r(t) dt \\
 &= u \int_{-1}^1 \bar{u}(\eta_{11}\bar{\psi}_1(t) + \eta_{12}\bar{\psi}_2(t)) f(t) r(t) dt \\
 &\quad + v \int_{-1}^1 \bar{v}(\eta_{11}\bar{\psi}_1(t) + \eta_{12}\bar{\psi}_2(t)) f(t) r(t) dt \\
 &= |u|^2\eta_{11}[f, \psi_1] + |u|^2\eta_{12}[f, \psi_2] + |v|^2\eta_{11}[f, \psi_1] + |v|^2\eta_{12}[f, \psi_2] \\
 &= \eta_{11}[f, \psi_1] + \eta_{12}[f, \psi_2].
 \end{aligned}$$

By (6.1), the general element of  $\mathcal{F}(A)$  takes the form

$$\begin{bmatrix} f \\ uf(-1) + vf(1) \\ z \end{bmatrix}$$

where  $f \in \mathcal{F}_{\max}$  and  $z \in \mathbb{C}$ . Applying  $W$  to this vector we obtain

$$w := \begin{bmatrix} g \\ \eta_{11}[f, \psi_1] + \eta_{12}[f, \psi_2] + \alpha \eta_{11}(uf(-1) + vf(1)) + \alpha \eta_{12}z \\ * \end{bmatrix},$$

where

$$g := W_{01}f + Kf + (uf(-1) + vf(1))\psi_1 + z\psi_2 \in \mathcal{F}_{\max}$$

by (6.4) and Lemma 6.2. Thus to prove that  $w \in \mathcal{F}(A)$ , it is enough to show that

$$ug(-1) + vg(1) = \eta_{11}[f, \psi_1] + \eta_{12}[f, \psi_2] + \alpha \eta_{11}(uf(-1) + vf(1)) + \alpha \eta_{12}z. \tag{6.19}$$

To this end we calculate

$$\begin{aligned}
 ug(-1) &= u((W_{01}f)(-1) + (Kf)(-1) + (uf(-1) + vf(1))\psi_1(-1) + z\psi_2(-1)) \\
 &= u(W_{01}f)(-1) + u(Kf)(-1) + \alpha |u|^2\eta_{11}(uf(-1) + vf(1)) + \alpha |u|^2\eta_{12}z
 \end{aligned}$$

from (6.10) and (6.12). Similarly

$$\begin{aligned}
 vg(1) &= v((W_{01}f)(1) + (Kf)(1) + (uf(-1) + vf(1))\psi_1(1) + z\psi_2(1)) \\
 &= v(W_{01}f)(1) + v(Kf)(1) + \alpha |v|^2\eta_{11}(uf(-1) + vf(1)) + \alpha |v|^2\eta_{12}z.
 \end{aligned}$$

Adding and using (6.5), (6.18) and (6.2), we obtain (6.19). This completes the proof of the lemma.  $\square$

The theorem now follows from Theorem 2.2 and Lemma 6.3.  $\square$

We now specialize Theorems 5.1, 5.2 and 6.1 to some of our earlier examples. First we consider Example 3.3 (cf. Example 4.2).

**Corollary 6.4.** *Assume that  $p = 1$  and  $r$  is of order  $\nu_0 > -1$  on a half-neighborhood of 0, and of order  $\nu_1 > -1$  on both a right half-neighborhood of  $-1$  and a left half-neighborhood of 1. Then there is a basis for each root subspace of  $A$ , so that the union of all these bases is a Riesz basis of  $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}$ .*

Now we consider Examples 4.3 and 4.4.

**Corollary 6.5.** *Assume that  $p$  is even,  $r$  is odd and that Condition 3.5 holds. If  $k = 0$  or Condition 3.7 holds, then there is a basis for each root subspace of  $A$ , so that the union of all these bases is a Riesz basis of  $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}$ .*

As a simple illustration of this corollary we could consider the eigenvalue problem stated in Example 5.3 but with  $r$  odd and of order  $\nu_0$  at 0 and  $\nu_1$  at 1 (and hence of order  $\nu_1$  at  $-1$ , since  $r$  is odd).

**Corollary 6.6.** *Assume that  $p$  is nearly even and  $r$  is nearly odd. If  $k = 0$  or Condition 3.7 holds, then there is a basis for each root subspace of  $A$ , so that the union of all these bases is a Riesz basis of  $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}$ .*

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