Mathematics

College of Science and Engineering

2013

On a Class of Bilinear Pseudodifferential Operators

Árpád Bényi
Western Washington University, arpad.benyi@wwu.edu

Tadahiro Oh
Princeton University

Research Article

On a Class of Bilinear Pseudodifferential Operators

Árpád Bényi[1] and Tadahiro Oh[2]

[1] *Department of Mathematics, Western Washington University, 516 High Street, Bellingham, WA 98225, USA*
[2] *Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA*

Correspondence should be addressed to Árpád Bényi; arpad.benyi@wwu.edu

We provide a direct proof for the boundedness of pseudodifferential operators with symbols in the bilinear Hörmander class $BS_{1,\delta}^0$, $0 \le \delta < 1$. The proof uses a reduction to bilinear elementary symbols and Littlewood-Paley theory.

1. Introduction: Main Results and Examples

Coifman and Meyer's ideas on multilinear operators and their applications in partial differential equations (PDEs) have had a great impact in the future developments and growth witnessed in the topic of multilinear singular integrals. One of their classical results [1, Proposition 2, p. 154] is about the $L^p \times L^q \to L^r$ boundedness of a class of translation invariant bilinear operators (bilinear multiplier operators) given by

$$T_\sigma (f, g) (x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma (\xi, \eta) \, \widehat{f} (\xi) \, \widehat{g} (\eta) \, e^{ix \cdot (\xi + \eta)} d\xi \, d\eta. \quad (1)$$

We have the following.

Theorem A. *If $|\partial_\xi^\beta \partial_\eta^\gamma \sigma(\xi, \eta)| \le (1 + |\xi| + |\eta|)^{-|\beta| - |\gamma|}$ for all $\xi, \eta \in \mathbb{R}^n$ and all multi-indices β, γ, then T_σ has a bounded extension from $L^p \times L^q$ into L^r, for all $1 < p, q < \infty$ such that $1/p + 1/q = 1/r$.*

In fact, Coifman and Meyer's approach yields Theorem A only for $r > 1$. The optimal extension of their result to the range $r > 1/2$ (as implied in the theorem above) can be obtained using interpolation arguments and an end-point estimate $L^1 \times L^1$ into $L^{1/2,\infty}$ in the works of Grafakos and Torres [2] and Kenig and Stein [3].

Bilinear pseudodifferential operators are natural non-translation invariant generalizations of the translation invariant ones; they allow symbols to depend on the space variable

x as well. Let us then consider bilinear operators a priori defined from $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}'$ of the form

$$T_\sigma (f, g) (x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma (x, \xi, \eta) \, \widehat{f} (\xi) \, \widehat{g} (\eta) \, e^{ix \cdot (\xi + \eta)} \, d\xi \, d\eta. \quad (2)$$

Perhaps unsurprisingly, we impose then similar conditions on the derivatives of the symbol σ with the expectation that they would yield indeed bounded operators T_σ on appropriate spaces of functions. The estimates that we have in mind define the so-called bilinear Hörmander classes of symbols, denoted by $BS_{\rho,\delta}^m$. We say that $\sigma \in BS_{\rho,\delta}^m$ if

$$\left| \partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma (x, \xi, \eta) \right| \lesssim (1 + |\xi| + |\eta|)^{m + \delta |\alpha| - \rho(|\beta| + |\gamma|)} \quad (3)$$

for all $x, \xi, \eta \in \mathbb{R}^n$ and all multi-indices α, β, γ. Note that we need smoothness in x as in the linear Hörmander classes. As usual, the notation $a \lesssim b$ means that there exists a positive constant K (independent of a, b) such that $a \le Kb$.

With this terminology, we can restate Theorem A as follows.

If the x-independent symbol $\sigma(\xi, \eta)$ belongs to the class $BS_{1,0}^0$, then T_σ is bounded from $L^p \times L^q$ into L^r for all $1 < p, q < \infty$ such that $1/p + 1/q = 1/r$.

The condition of translation invariance (equivalently, the x-independence of the symbol) is superfluous. Moreover, the previous boundedness result can be shown to hold for the larger class of symbols $BS_{1,\delta}^0 \supseteq BS_{1,0}^0$, where $0 \le \delta < 1$. This is a known fact that is tightly connected to the bilinear Calderón-Zygmund theory developed by Grafakos and Torres in [2]

and the existence of a transposition symbolic calculus proved by Bényi et al. [4]. Let us briefly give an outline of how this follows. First, we note that the bilinear kernels associated to bilinear operators with symbols in $BS^0_{1,\delta}$, $0 \leq \delta < 1$, are bilinear Calderón-Zygmund operators in the sense of [2]. Second, we recall that [2, Corollary 1], which is an application of the bilinear $T(1)$ theorem therein, states the following.

Theorem B. *If T and its transposes, T^{*1} and T^{*2}, have symbols in $BS^0_{1,1}$, then they can be extended as bounded operators from $L^p \times L^q$ into L^r for $1 < p, q < \infty$ and $1/p + 1/q = 1/r$.*

Third, by [4, Theorem 2.1], we have the following.

Theorem C. *Assume that $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, and $\sigma \in BS^m_{\rho,\delta}$. Then, for $j = 1, 2$, $T^{*j}_\sigma = T_{\sigma^{*j}}$, where $\sigma^{*j} \in BS^m_{\rho,\delta}$.*

Finally, since $BS^0_{1,\delta} \subset BS^0_{1,1}$, we can directly combine Theorems B and C to recover the following optimal extension of the Coifman-Meyer result; note that now the symbol is allowed to depend on x while r is still allowed to be in the optimal interval $(1/2, \infty)$.

Theorem 1. *If σ is a symbol in $BS^0_{1,\delta}$, $0 \leq \delta < 1$, then T_σ has a bounded extension from $L^p \times L^q$ into L^r for all $1 < p, q < \infty$ such that $1/p + 1/q = 1/r$.*

Once we have the boundedness of the class $BS^0_{1,\delta}$ on products of Lebesgue spaces, a "reduction method" allows us to deduce also the boundedness of the class $BS^m_{1,\delta}$ on appropriate products of Sobolev spaces. Moreover, our estimates in this case come in the form of Leibniz-type rules; for more on these kinds of properties, see the work of Bernicot et al. [5]. In the particular case when the bilinear operator is just a differential operator, the Leibniz-type rules are referred to as Kato-Ponce's commutator estimates and are known to play a significant role in the study of the Euler and Navier-Stokes equations, see [6]; see also Kenig et al. [7] for further applications of commutators to nonlinear Schrödinger equations. Let $J^m = (I - \Delta)^{m/2}$ denote the linear Fourier multiplier operator with symbol $\langle \xi \rangle^m$, where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. By definition, we say that f belongs to the Sobolev space L^p_m if $J^m f \in L^p$. We have the following.

Theorem 2. *Let σ be a symbol in $BS^m_{1,\delta}$, $0 \leq \delta < 1$, $m \geq 0$, and let T_σ be its associated operator. Then there exist symbols σ_1 and σ_2 in $BS^0_{1,\delta}$ such that, for all $f, g \in \mathcal{S}$,*

$$T_\sigma (f, g) = T_{\sigma_1} (J^m f, g) + T_{\sigma_2} (f, J^m g). \tag{4}$$

In particular, then one has that T_σ has a bounded extension from $L^p_m \times L^q_m$ into L^r, provided that $1/p + 1/q = 1/r$, $1 < p, q < \infty$. Moreover,

$$\|T_\sigma (f, g)\|_{L^r} \lesssim \|f\|_{L^p_m} \|g\|_{L^q} + \|f\|_{L^p} \|g\|_{L^q_m}. \tag{5}$$

The proof of Theorem 2 follows a similar path as the one in the work of Bényi et al. [8, Theorem 2.7]. For the convenience of the reader, we sketch here the main steps in the argument. Let ϕ be a C^∞-function on $\mathbb{R}$ such that $0 \leq \phi \leq 1$, supp $\phi \subset [-2, 2]$, and $\phi(r) + \phi(1/r) = 1$ on $[0, \infty)$. Then (4) holds if we let

$$\sigma_1 (x, \xi, \eta) = \sigma (x, \xi, \eta) \phi \left(\frac{\langle \eta \rangle^2}{\langle \xi \rangle^2} \right) \langle \xi \rangle^{-m},$$

$$\sigma_2 (x, \xi, \eta) = \sigma (x, \xi, \eta) \phi \left(\frac{\langle \xi \rangle^2}{\langle \eta \rangle^2} \right) \langle \eta \rangle^{-m}. \tag{6}$$

Now, straightforward calculations that take into account the support condition on ϕ given that σ_1 and σ_2 belong to $BS^0_{1,\delta}$. The Leibniz-type estimate (5) follows now from Theorem 1 and (4).

It is also worthwhile to note that we can replace (5) with a more general Leibniz-type rule of the form

$$\|T_\sigma(f, g)\|_{L^r} \lesssim \|f\|_{L^{p_1}_m} \|g\|_{L^{q_1}} + \|g\|_{L^{p_2}_m} \|f\|_{L^{q_2}}, \tag{7}$$

where $1/p_1 + 1/q_1 = 1/p_2 + 1/q_2 = 1/r$, $1 < p_1, p_2, q_1, q_2 < \infty$.

One of the main reasons for the study of the Hörmander classes of bilinear pseudodifferential operators is the fact that the conditions imposed on the symbols arise naturally in PDEs. In particular, the bilinear Hörmader classes $BS^m_{\rho,\delta}$ model the product of two functions and their derivatives.

Example 3. Consider first a bilinear partial differential operator with variable coefficients

$$D_{k,\ell} (f, g) = \sum_{|\beta| \leq k} \sum_{|\gamma| \leq \ell} c_{\beta\gamma} (x) \frac{\partial^\beta f}{\partial x^\beta} \frac{\partial^\gamma g}{\partial x^\gamma}. \tag{8}$$

Note that $D_{k,\ell} = T_{\sigma_{k,\ell}}$, where the bilinear symbol is given by

$$\sigma_{k,\ell} (x, \xi, \eta) = (2\pi)^{-2n} \sum_{\beta, \gamma} c_{\beta\gamma} (x) (i\xi)^\beta (i\eta)^\gamma. \tag{9}$$

Assuming that the coefficients $c_{\beta\gamma}$ have bounded derivatives, it is easy to show that $\sigma_{k,\ell} \in BS^{k+\ell}_{1,0}$.

Example 4. The symbol in the previous example is almost equivalent to a multiplier of the form

$$\sigma_m (\xi, \eta) = \left(1 + |\xi|^2 + |\eta|^2 \right)^{m/2}. \tag{10}$$

Indeed, this symbol belongs to $BS^m_{1,0}$. We can also think of this symbol as the bilinear counterpart of the multiplier $\langle \xi \rangle^m$ that defines the linear operator J^m.

Example 5. With the notation in Example 4, the multipliers $\xi \sigma_{-1}(\xi, \eta)$ and $\eta \sigma_{-1}(\xi, \eta)$ belong to $BS^0_{1,0}$. In general, the multipliers $\sigma_{k+\ell}(\xi, \eta) = \xi^k \eta^\ell \sigma_{-1}(\xi, \eta)$ belong to $BS^{k+\ell}_{1,0}$.

Example 6. One of the recurrent techniques in PDE estimates is to truncate a given multiplier at the right scale. Consider now

$$\sigma (\xi, \eta) = \sigma_m (\xi, \eta) \sum_{a,b \in \mathbb{N}} c_{a,b} (x) \varphi \left(2^{-a} \xi \right) \chi \left(2^{-b} \eta \right), \tag{11}$$

where φ and χ are smooth "cutoff" functions supported in the annulus $\{1/2 \leq |\xi| \leq 2\}$, and the coefficients satisfy derivative estimates of the form

$$\left\| \partial_x^\alpha c_{a,b}(x) \right\|_{L^\infty} \lesssim 2^{\delta |\alpha| \max(a,b)}. \tag{12}$$

Elementary calculations show that $\sigma \in \mathrm{BS}_{1,\delta}^m$.

Remark 7. Theorems 1 and 2 lead to the natural question about the boundedness properties of other Hörmander classes of bilinear pseudodifferential operators. An interesting situation arises when we consider the bilinear Calderón-Vaillancourt class $\mathrm{BS}_{0,0}^0$. A result of Bényi and Torres [9] shows that, in this case, the $L^p \times L^q \rightarrow L^r$ boundedness fails. One can impose some additional conditions (besides being in $\mathrm{BS}_{0,0}^0$) on a symbol to guarantee that the corresponding bilinear pseudodifferential operator is $L^p \times L^q \rightarrow L^r$ bounded; see, for example, [9] and the recent work of Bernicot and Shrivastava [10]. However, there is a nice substitute for the Lebesgue space estimates. If we consider instead modulation spaces $M^{p,q}$ (see the excellent book by Gröchenig [11] for their definition and basic properties), we can show, for example, that if $\sigma \in \mathrm{BS}_{0,0}^0$ then $T_\sigma : L^2 \times L^2 \rightarrow M^{1,\infty}$ (which contains L^1). This and other more general boundedness results on modulation spaces for the class $\mathrm{BS}_{0,0}^0$ were obtained by Bényi et al. [12]. Then, this particular boundedness result with the reduction method employed in Theorem 2 allows us to also obtain the boundedness of the class $\mathrm{BS}_{0,0}^m$ from $L_m^2 \times L_m^2$ into $M^{1,\infty}$. Interestingly, we can also obtain the $L^p \times L^q \rightarrow L^r$ boundedness of the class $\mathrm{BS}_{0,0}^m$, but we have to require in this case the order m to depend on the Lebesgue exponents; see the work of Bényi et al. [13], also Miyachi and Tomita [14] for the optimality of the order m and the extension of the result in [13] below $r = 1$. The most general case of the classes $\mathrm{BS}_{\rho,\delta}^m$ is also given in [13].

In the remainder of the paper we will provide an alternate proof of Theorem 1 that does not use sophisticated tools such as the symbolic calculus. The proof is in the original spirit of the work of Coifman and Meyer that made use of the Littlewood-Paley theory. As such, we will only be concerned here with the boundedness into the target space L^r with $r > 1$. Of course, obtaining the full result for $r > 1/2$ is then possible because of the bilinear Calderón-Zygmund theory, which applies to our case. We will borrow some of the ideas from Bényi and Torres [15], which in turn go back to the nice exposition (in the linear case) by Journé [16], by making use of the so-called bilinear elementary symbols.

2. Proof of Theorem 1

We start with two lemmas that provide the anticipated decomposition of our symbol into bilinear elementary symbols. Since they are the immediate counterparts of [15, Lemma 1 and Lemma 2] to our class $\mathrm{BS}_{1,\delta}^0$, we will skip their proofs; see also [16, pp. 72–75] and [1, pp. 55–57]. The first reduction is as follows.

Lemma 8. *Fix a symbol σ in the class $\mathrm{BS}_{1,\delta}^0$, $0 \leq \delta < 1$, and an arbitrary large positive integer N. Then, for any $f, g \in \mathscr{S}$, $T_\sigma(f,g)$ can be written in the form*

$$T_\sigma(f,g) = \sum_{k,\ell \in \mathbb{Z}^n} d_{k\ell} T_{\sigma_{k\ell}}(f,g) + R(f,g), \tag{13}$$

where $\{d_{k\ell}\}$ is an absolutely convergent sequence of numbers,

$$\sigma_{k\ell}(x,\xi,\eta) = \sum_{j=0}^\infty \kappa_{jk\ell}(x) \psi_{k\ell}\left(2^{-j}\xi, 2^{-j}\eta\right), \tag{14}$$

with each $\psi_{k\ell}$ a C^∞-function supported on the set $\{1/3 \leq \max(|\xi|,|\eta|) \leq 1\}$,

$$\left| \partial_\xi^\beta \partial_\eta^\gamma \psi_{k\ell}(\xi,\eta) \right| \lesssim 1 \quad \forall |\beta|, |\gamma| \leq N,$$
$$\left| \partial^\alpha \kappa_{jk\ell}(x) \right| \lesssim 2^{j\delta|\alpha|} \quad \forall |\alpha| \geq 0, \tag{15}$$

and R is a bounded operator from $L^p \times L^q$ into L^r, for $1/p + 1/q = 1/r$, $1 < p, q, r < \infty$.

Now, if $\sigma_{k\ell}$ is any of the symbols in (14) and we knew a priori that $T_{\sigma_{k\ell}}$ are bounded from $L^p \times L^q$ into L^r with operator norms depending only on the implicit constants from (15), the fact that the sequence $\{d_{k\ell}\}$ is absolutely convergent immediately implies the $L^p \times L^q \rightarrow L^r$ boundedness of T_σ. Our first step has thus reduced the study of generic symbols in the class $\mathrm{BS}_{1,\delta}^0$ to symbols of the form

$$\sigma(x,\xi,\eta) = \sum_{j=0}^\infty m_j(x) \psi\left(2^{-j}\xi, 2^{-j}\eta\right), \tag{16}$$

where $\|\partial^\alpha m_j\|_{L^\infty} \lesssim 2^{j\delta|\alpha|}$, and ψ is supported in $\{1/3 \leq \max(|\xi|,|\eta|) \leq 1\}$.

Our second step is to further reduce the simpler looking symbol given in (16) to a sum of *bilinear elementary symbols*.

Lemma 9. *Let σ be as in (16). One can further reduce the study to symbols of the form*

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3, \tag{17}$$

where the elementary symbols σ_k, $k = 1, 2, 3$, are defined via

$$\sigma_k(x,\xi,\eta) = \sum_{j=0}^\infty m_j(x) \varphi_k\left(2^{-j}\xi\right) \chi_k\left(2^{-j}\eta\right), \tag{18}$$

with $\operatorname{supp} \varphi_1 \subseteq \{1/4 \leq |\xi| \leq 2\}$, $\operatorname{supp} \chi_1 \subseteq \{|\eta| \leq 1/8\}$, $\varphi_3 = \chi_1, \chi_3 = \varphi_1$, $\operatorname{supp} \varphi_2$, $\operatorname{supp} \chi_2 \subseteq \{1/20 \leq |\xi| \leq 2\}$, and $\|\partial^\alpha m_j\|_{L^\infty} \lesssim 2^{j\delta|\alpha|}$.

Therefore, we are now only faced with the question of boundedness for the two operators T_{σ_1} and T_{σ_2}, with σ_1 and σ_2 defined in Lemma 9; boundedness of T_{σ_3} follows from that of T_{σ_1} by symmetry. In the following, for $f, g \in \mathscr{S}$, we write

$$\widehat{f}_{jk}(\xi) = \varphi_k\left(2^{-j}\xi\right) \widehat{f}(\xi),$$
$$\widehat{g}_{jk}(\eta) = \chi_k\left(2^{-j}\eta\right) \widehat{g}(\eta). \tag{19}$$

In this case, for $k = 1, 2$, we can write

$$T_{\sigma_k}(f, g)(x) = \sum_{j=0}^{\infty} m_j(x) f_{jk}(x) g_{jk}(x). \tag{20}$$

Claim 1. T_{σ_2} is bounded from $L^p \times L^q$ into L^r.

Proof. By (20) and the Cauchy-Schwarz inequality, we can write

$$\left| T_{\sigma_2}(f, g) \right| \leq \sum_{j=0}^{\infty} |m_j| |f_{j2}| |g_{j2}|$$
$$\lesssim \left(\sum_{j=0}^{\infty} |f_{j2}|^2 \right)^{1/2} \left(\sum_{j=0}^{\infty} |g_{j2}|^2 \right)^{1/2}. \tag{21}$$

We used here the fact that the coefficients m_j are bounded, see Lemma 9. Using now Hölder's inequality, we have

$$\left\| T_{\sigma_2}(f, g) \right\|_{L^r} \lesssim \left\| \left(\sum_{j=0}^{\infty} |f_{j2}|^2 \right)^{1/2} \right\|_{L^p} \left\| \left(\sum_{j=0}^{\infty} |g_{j2}|^2 \right)^{1/2} \right\|_{L^q}. \tag{22}$$

Finally, the conditions on the supports of φ_2 and χ_2 allow us to make use of the Littlewood-Paley theory and conclude that

$$\left\| T_{\sigma_2}(f, g) \right\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^q}. \tag{23}$$

□

Obtaining the boundedness of the operator T_{σ_1} is a bit more delicate due to the support condition on χ_1, specifically having supp χ_1 contained in a disk rather than in an annulus. Nevertheless, we still claim the following.

Claim 2. T_{σ_1} is bounded from $L^p \times L^q$ into L^r.

Proof. Note first that we can still use the Littlewood-Paley theory on the f_{j1} part of the sum that defines $T_{\sigma_1}(f, g)(x) = \sum_{j=0}^{\infty} m_j(x) f_{j1}(x) g_{j1}(x)$. We have the following inequalities:

$$\left\| \left(\sum_{j=0}^{\infty} |f_{j2}|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p},$$
$$\left\| \sup_{j \geq 0} |g_{j1}| \right\|_{L^q} \lesssim \|g\|_{L^q}. \tag{24}$$

At this point, however, we must proceed more cautiously. Observe that

$$\text{supp } \widehat{f_{j1} g_{j1}} \subset \text{supp } \widehat{f_{j1}} + \text{supp } \widehat{g_{j1}} \subset \left\{ 2^{j-3} \leq |\xi| \leq 2^{j+3} \right\}. \tag{25}$$

Denoting then $h_{j1} := f_{j1} g_{j1}$, we now have $T_{\sigma_1}(f, g)(x) = \sum_{j=0}^{\infty} m_j h_{j1}$, where $\|\partial^\alpha m_j\|_{L^\infty} \lesssim 2^{j\delta|\alpha|}$, and h_{j1} satisfies the support condition (25). Assume for the moment that the following inequality holds:

$$\left\| \sum_{j=0}^{\infty} m_j h_{j1} \right\|_{L^r} \lesssim \left\| \left(\sum_{j=0}^{\infty} |h_{j1}|^2 \right)^{1/2} \right\|_{L^r}. \tag{26}$$

Then, the boundedness of the operator T_{σ_1} can be obtained as follows:

$$\left\| T_{\sigma_1}(f, g) \right\|_{L^r} = \left\| \sum_{j=0}^{\infty} m_j h_{j1} \right\|_{L^r} \lesssim \left\| \left(\sum_{j=0}^{\infty} |h_{j1}|^2 \right)^{1/2} \right\|_{L^r}$$
$$\lesssim \left\| \sup_{j \geq 0} |g_{j1}| \cdot \left(\sum_{j=0}^{\infty} |f_{j1}|^2 \right)^{1/2} \right\|_{L^r}$$
$$\lesssim \left\| \left(\sum_{j=0}^{\infty} |f_{j1}|^2 \right)^{1/2} \right\|_{L^p} \left\| \sup_{j \geq 0} |g_{j1}| \right\|_{L^q}$$
$$\lesssim \|f\|_{L^p} \|g\|_{L^q}. \tag{27}$$

□

The proof of Claim 2 assumed the estimate (26). Our next claim is that (26) is indeed true.

Claim 3. Assume that $\|\partial^\alpha m_j\|_{L^\infty} \lesssim 2^{j\delta|\alpha|}$ and supp $\widehat{h}_j \subset \{2^{j-3} \leq |\xi| \leq 2^{j+3}\}$. Then, for all $r > 1$, we have

$$\left\| \sum_{j=0}^{\infty} m_j h_j \right\|_{L^r} \lesssim \left\| \left(\sum_{j=0}^{\infty} |h_j|^2 \right)^{1/2} \right\|_{L^r}. \tag{28}$$

In our proof of this claim, we will make use of Journé's lemma [16, p. 69].

Lemma 10. *There exists a constant $C > 0$ such that, for all $j \geq 0$, $m_j = g_j + b_j$, where $\|g_j\|_{L^\infty} \leq C$, $\|b_j\|_{L^\infty} \leq C2^{(\delta-1)j}$, and supp $\widehat{h_j g_j} \subset \{2^j/72 \leq |\xi| \leq 9 \cdot 2^j\}$.*

Proof of Claim 3. First, we consider the case $r = 2$. With the notation in Lemma 10, it suffices to estimate $\|\sum_{j=0}^{\infty} b_j h_j\|_{L^2}$ and $\|\sum_{j=0}^{\infty} g_j h_j\|_{L^2}$.

The estimate on the "bad part" follows from the triangle and Cauchy-Schwarz inequalities and the control $\|b_j\|_{L^\infty} \lesssim 2^{(\delta-1)j}$; recall that $\delta < 1$:

$$\left\| \sum_{j=0}^{\infty} b_j h_j \right\|_{L^2} \leq \sum_{j=0}^{\infty} \|b_j\|_{L^\infty} \|h_j\|_{L^2}$$
$$\leq \left(\sum_{j=0}^{\infty} \|b_j\|_{L^\infty}^2 \right)^{1/2} \left(\sum_{j=0}^{\infty} \|h_j\|_{L^2}^2 \right)^{1/2}$$
$$\lesssim \left(\sum_{j=0}^{\infty} 2^{(2\delta-2)j} \right)^{1/2} \left(\sum_{j=0}^{\infty} \|h_j\|_{L^2}^2 \right)^{1/2}$$
$$\lesssim \left\| \left(\sum_{j=0}^{\infty} |h_j|^2 \right)^{1/2} \right\|_{L^2}. \tag{29}$$

Let us now look at the "good part". We start by noticing that, given the two summation indices $j, k \geq 0$, we have

$$\operatorname{supp} \widehat{g_j h_j} \subset \left\{ \frac{2^j}{72} \leq |\xi| \leq 9 \cdot 2^j \right\},$$

$$\operatorname{supp} \widehat{g_k h_k} \subset \left\{ \frac{2^k}{72} \leq |\xi| \leq 9 \cdot 2^k \right\}. \tag{30}$$

Thus, for $|j - k| \geq 11$, we have $\operatorname{supp} \widehat{g_j h_j} \cap \operatorname{supp} \widehat{g_k h_k} = \emptyset$. In view of this orthogonality, Plancherel's theorem with the estimate $\|g_j\|_{L^\infty} \lesssim 1$ gives

$$\left\| \sum_{j=0}^{\infty} g_j h_j \right\|_{L^2} \leq \sum_{i=0}^{11} \left\| \sum_{k=0}^{\infty} g_{i+11k} h_{i+11k} \right\|_{L^2} \lesssim \left(\sum_{j=0}^{\infty} \|g_j h_j\|_{L^2}^2 \right)^{1/2}$$

$$\leq \left(\sum_{j=0}^{\infty} \|g_j\|_{L^\infty}^2 \|h_j\|_{L^2}^2 \right)^{1/2} \lesssim \left\| \left(\sum_{j=0}^{\infty} |h_j|^2 \right)^{1/2} \right\|_{L^2}. \tag{31}$$

This completes the proof of the case $r = 2$. In the general case $r > 1$, we again seek the control of the "bad" and "good" parts. The estimate on the "bad" part follows virtually the same as in the case $r = 2$:

$$\left\| \sum_{j=0}^{\infty} b_j h_j \right\|_{L^r} \leq \left\| \left(\sum_{j=0}^{\infty} |b_j|^2 \right)^{1/2} \left(\sum_{j=0}^{\infty} |h_j|^2 \right)^{1/2} \right\|_{L^r}$$

$$\leq \left\| \left(\sum_{j=0}^{\infty} |b_j|^2 \right)^{1/2} \right\|_{L^\infty} \left\| \left(\sum_{j=0}^{\infty} |h_j|^2 \right)^{1/2} \right\|_{L^r} \tag{32}$$

$$\lesssim \left\| \left(\sum_{j=0}^{\infty} |h_j|^2 \right)^{1/2} \right\|_{L^r},$$

where we used Minkowski's integral inequality in the last step.

For the "good" part, we can think of $g_k h_k$ as being dyadic blocks in the Littlewood-Paley decomposition of the sum $S_i := \sum_{k \equiv i \pmod{11}} g_k h_k$. Thus, it will be enough to control uniformly (in the L^r norm) the sums S_i, $0 \leq i \leq 11$, in order to obtain the same bound on $\|\sum_{j \geq 0} g_j h_j\|_{L^r}$. The control on S_i however follows from the uniform estimate on the g_k's and an immediate application of Littlewood-Paley theory. $\square$

Acknowledgments

References

[1] R. R. Coifman and Y. Meyer, *Au Delà des Opérateurs Pseudo-Différentiels*, vol. 57 of *Astérisque*, 2nd edition, 1978.

[2] L. Grafakos and R. H. Torres, "Multilinear Calderón-Zygmund theory," *Advances in Mathematics*, vol. 165, no. 1, pp. 124–164, 2002.

[3] C. E. Kenig and E. M. Stein, "Multilinear estimates and fractional integration," *Mathematical Research Letters*, vol. 6, no. 1, pp. 1–15, 1999.

[4] Á. Bényi, D. Maldonado, V. Naibo, and R. H. Torres, "On the Hörmander classes of bilinear pseudodifferential operators," *Integral Equations and Operator Theory*, vol. 67, no. 3, pp. 341–364, 2010.

[5] F. Bernicot, D. Maldonado, K. Moen, and V. Naibo, "Bilinear Sobolev-Poincaré inequalities and Leibniz-type rules," *Journal of Geometric Analysis*. In press, http://arxiv.org/pdf/1104.3942.pdf.

[6] T. Kato and G. Ponce, "Commutator estimates and the Euler and Navier-Stokes equations," *Communications on Pure and Applied Mathematics*, vol. 41, no. 7, pp. 891–907, 1988.

[7] C. E. Kenig, G. Ponce, and L. Vega, "On unique continuation for nonlinear Schrödinger equations," *Communications on Pure and Applied Mathematics*, vol. 56, no. 9, pp. 1247–1262, 2003.

[8] Á. Bényi, A. R. Nahmod, and R. H. Torres, "Sobolev space estimates and symbolic calculus for bilinear pseudodifferential operators," *Journal of Geometric Analysis*, vol. 16, no. 3, pp. 431–453, 2006.

[9] Á. Bényi and R. H. Torres, "Almost orthogonality and a class of bounded bilinear pseudodifferential operators," *Mathematical Research Letters*, vol. 11, no. 1, pp. 1–11, 2004.

[10] F. Bernicot and S. Shrivastava, "Boundedness of smooth bilinear square functions and applications to some bilinear pseudo-differential operators," *Indiana University Mathematics Journal*, vol. 60, no. 1, pp. 233–268, 2011.

[11] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, Mass, USA, 2001.

[12] Á. Bényi, K. Gröchenig, C. Heil, and K. Okoudjou, "Modulation spaces and a class of bounded multilinear pseudodifferential operators," *Journal of Operator Theory*, vol. 54, pp. 301–313, 2005.

[13] Á. Bényi, F. Bernicot, D. Maldonado, V. Naibo, and R. H. Torres, "On the Hörmander classes of bilinear pseudodifferential operators, II," *Indiana University Mathematics Journal*. In press, http://arxiv.org/pdf/1112.0486.pdf .

[14] A. Miyachi and N. Tomita, "Calderón-Vaillancourt type theorem for bilinear pseudo-differential operators," *Indiana University Mathematics Journal*. In press.

[15] Á. Bényi and R. H. Torres, "Symbolic calculus and the transposes of bilinear pseudodifferential operators," *Communications in Partial Differential Equations*, vol. 28, no. 5-6, pp. 1161–1181, 2003.

[16] J.-L. Journé, *Calderón-Zygmund Operators, Pseudo-Differential Operators and the Cauchy Integral of Calderón*, vol. 994 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1983.