Making Sense with Math: an Introduction to Math for People in College

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Introduction

In which you will learn why I wrote this book, and for whom.

Is this book for you?

You might be wondering if this is the right book for you.

If you are a bit uncertain about any of the mathematical statements in the box on the next page, or if you feel the least anxious about any of them, I’m so glad you picked this book up: it is for you!

When you have completed this book, those mathematical statements will seem like common-sense statements.

The choices I made while writing this book are based on the following two assumptions about you:

1. You are committed to your education and want to learn and accomplish educational and professional goals
2. You are worried about the mathematics required for you to achieve those goals.

If you are worried about the mathematics required for you to achieve your goals, but you are not committed to your education, I suggest you take a break from learning math, and come back when you are excited and motivated. It’s ok to have other priorities and I don’t take it personally!
MATHEMATICAL STATEMENTS

\[
\begin{align*}
\frac{\text{pos}}{\text{pos}} &= \text{pos} & \frac{\text{neg}}{\text{pos}} &= \text{neg} & \text{pos} \cdot \text{pos} &= \text{pos} \\
\frac{\text{pos}}{\text{neg}} &= \text{neg} & \text{neg} \cdot \text{pos} &= \text{neg} & \text{neg} \cdot \text{neg} &= \text{pos} \\
\frac{\text{neg}}{\text{neg}} &= \text{pos} & \text{pos} \cdot \text{neg} &= \text{neg} & A^+ - A &= 0 \\
A - (-C) &= A + C & A - C &= A + (-C) & \frac{a}{b} + \frac{c}{d} &= \frac{ad + cb}{bd} \\
&\quad \text{(if } b \neq 0 \text{ and } d \neq 0) & a(bc) &= b(ac) = c(ab) \\
\frac{a}{b} &= a \cdot \frac{1}{b} & \frac{a}{b} \cdot \frac{c}{d} &= \frac{bd}{ad} & a(b + c) &= ab + ac \\
&\quad \text{(if } b \neq 0 \text{ and } d \neq 0) \\
\frac{a}{b} = \frac{ad}{bc} & a + b &= a + b & a - b &= \frac{1}{a^b} \\
&\quad \text{(if } b \neq 0 \text{ and } d \neq 0 \text{ and } c \neq 0) & \frac{a^b}{a^c} = a^{b-c} & \frac{a^b}{a^c} = a^{b-c} \\
\sqrt[n]{x} &= x^{\frac{1}{n}} & a^0 &= 1 & (a^b)^c &= a^{bc} \\
\frac{1}{a^b} & (a^b)(a^c) = a^{b+c} & (a^b) + (a^c) &= a^{b-c} \\
&\quad \text{Division by zero doesn’t make sense}
\end{align*}
\]
**Why should you trust me, and this book, to help you learn math?**

I’ve been teaching math to college students since 1999. I love it, and every year I learn more from my students about how to teach and about the amazingly diverse ways people can think and reason. Over the years, I’ve used many textbooks, websites, and curricula, so, what’s the need for this book?

Most math books for college students start out reviewing “rules” in an introductory chapter. The review usually goes like this: here are the “rules”, here are some examples of using those “rules” and here are 10 to 100 exercises where you will practice using those “rules” and then you’ll be tested on them. I have spent almost 2 decades studying how adults learn arithmetic and math, and I have observed the absurdity of expecting a person to learn math by showing them rules and then requiring that they practice those rules.

The problem with that approach, even if it seems familiar and comfortable to you, is that people learn, in part, by connecting new ideas and perspectives to what they already understand, and correcting any previous misunderstandings. This process takes time and effort. Memorizing rules to quickly retrieve them won’t be useful to you when you are trying to apply mathematics to an unfamiliar situation.

This book will help you grow the habits of mind that will allow you to make problems easier through the use of mathematics. These habits of mind include:

- attending to your mistakes.
- connecting new ideas and approaches with prior ideas
- using those connections to refine and revise your prior understanding and build new knowledge for yourself
- being precise
- approaching problems you do not know how to solve with, if not enthusiasm, at least confidence.
- Solving problems by transforming difficult-to-understand statements into easier-to-understand ones.
**My Inspirations**

Many books and articles about how people learn math (or anything) have influenced the structure and content of this book, as well as my personal experience teaching and getting to know hundreds of students. But most significantly, the hard work, good humor, and amazing minds of my students are my inspiration.

**Structure of Read this Before you Take College Math**

*Read This Before you Take College Math* is a workbook, and you should write all over it. As you go through the chapters, the mathematical ideas build on each other, and each page incorporates prior ideas and concepts. Please read and do the work sequentially, and do not jump around topic to topic; the book tells a coherent story that lays a foundation. Expect yourself to do each and every exercise, in order, and do not continue to the next exercise without understanding the previous one. If you don’t understand the previous one, read the text a little more slowly. Talking to others about what you are reading is one of the best ways to solidify understanding.

It will take several weeks, if you think and read and work on it for a few hours every day, to learn the material in *Read This Before You Take College Math*. I use it as a primary text for a quarter-long (10 week) course.

This book will encourage you to build the habit of self-reflection and productive practice that you will need to succeed in your mathematical education. If you don’t do all of the writing, you should not expect that your efforts will help you achieve your larger goals. Your goal is to get in the mental arena, and work to significantly change your mathematical thinking.
What is Mathematics to You?

Read this book with a pencil in your hand. If you don’t have one, now is the time to get up and grab a pencil, because I’m going to ask you to write some of your thoughts down now, before you read any farther. Don’t skip this part. Really.

- What do you believe mathematics is? (There is no wrong answer! This is just for you to put your thoughts into words.)
- What do you believe mathematics is good for?
- What are some emotions that come up for you when you think about taking a math class?
- Do you believe you have the ability to do mathematics?
- Do you believe your ability to do mathematics can change over time?
- Why do you think you should learn this material? And, if your answer to this question is “so I can pass and get a degree”, answer this question instead: Why is learning this math required for your degree?

Let’s look at each of those questions.

What do you believe mathematics is? And, what do you believe mathematics is good for?

Did you write that math is working with numbers or formulas? Perhaps you wrote that mathematics is hard, or easy, or silly or stupid, stressful, torture, or “not for you”. Whatever you wrote, it is valid and reflects your experiences and thinking. I ask only that you be willing and stay open-minded to consider new ways of thinking about mathematics.

What do mathematicians say? If you wrote that mathematics is logical, quantitative problem-solving, you agree pretty well with mathematicians. If you wrote that mathematics allows us to simplify relationships between and among quantities to allow deeper understanding of essential characteristics of those relationships, you agree pretty well with mathematicians. If you wrote that mathematics allows us to compare seemingly unconnected situations to make problem-solving easier, then you agree with mathematicians. If you wrote that understanding mathematics takes creativity and perseverance, then you agree with mathematicians. Mathematicians would also agree that math includes the practice of solving problems, some concrete and related to a real-life problems, and some abstract and perhaps not (yet) applicable to a particular real-life problem people are trying to solve.

What are some emotions that come up for you when you think about taking a math class?

What do you believe about yourself and mathematics? Are you “just not good at math”? Do you “just hate math”? Those of us who become mathematicians do so for a variety of reasons, but most mathematicians would agree that they got support and were given encouragement along the way by someone who inspired them. Perhaps mathematicians liked the company of other mathematicians. And, for reasons that are mysterious, mathematicians enjoy doing mathematics, in the same way that some
people enjoy mountain climbing or dancing till dawn. *No one is born with the ability to do algebra or understand fraction notation, and every one of us struggles with new concepts and ideas.*

**Do you believe you have the ability to do mathematics? And do you believe your ability to do mathematics can change over time?**

An amazing thing happens when we learn: we physically change our bodies, and these changes are measurable in our brains! (please read the excellent books *Mindset* by Carol Dwek if you are interested in the psychology of learning, and Zull, J. E. *The Art of Changing the Brain* if you are interested in teaching). The fact that learning new concepts and ideas physically alters our brains has enormous implications for learning. When any human being is faced with learning new concepts and skills that are difficult and challenging they will occasionally fail. If we believe we can change the way our brains work through self-reflection and productive practice, then failure is an opportunity to recognize there is something we “don’t know” and set about “getting to know it”. If we believe we cannot change the way our brains work, then failure means we “aren’t good at this thing” and perhaps we “just aren’t that type of person” or “we’d better try something else”.

I am not recommending that we should all go around seeking failure and doing activities that bring us no pleasure in order to “grow our brains”. I am simply suggesting that if you have learned that you are not “a math person” it would be enormously helpful if you could change your belief to “I am not a math person *yet*”.

And, while you are thinking about how to become a “math person”, keep in mind there are many ways to be a good mathematician and solve problems. When I or anyone shows you “how to do” a problem they are only showing you how they look at the problem and why they look at it that way. You may see a different way of solving the same problem or looking at the problem. If you can logically justify your solution, examine the pros and cons of your solution, then you are doing mathematics and have become a “math person”. Growing your ability to problem-solve, learning to explain your logical thinking and noticing when your own understanding is flawed, and learning to understand the logical thinking of others and notice when their thinking is not quite logical, are the fundamentals of any mathematics you may learn.

When you are learning, keep an open mind, but, like the bumper sticker says, not so open that what you already know falls out. Build new understanding by connecting ideas to what you already know, and if you find that what you thought you knew turns out not to be true, be willing to change your mind and gently say goodbye to your old ideas. But don’t ignore them... make connections with what you know, what you’ve been taught before, while being open to new approaches and ideas.

And, lastly-but very importantly-consider your emotional reaction to mathematics. You will be able to learn more easily if you can be eager to learn new concepts. Accept any bad experiences with math you may have had, be gentle with yourself, and get ready to move on.
Why do you think you should learn this material? And, if your answer to this question is “so I can pass and get a degree”, answer this question instead: Why is learning this math required for your degree?

This is a difficult one for me to answer, as I don’t know what your educational, career, or vocational goals are. Any field in science or engineering clearly requires the ability to think mathematically. But what if you aspire to be a poet? Mathematical thinking might not help you there (but it certainly wouldn’t hurt). You most certainly want to be healthy, and you probably want to buy things, and if you are lucky you live in a place where you can vote. Determining which health insurance policy is right for you, figuring out how much money you would need to drive and maintain a vehicle or feed your horses, deciding what tax policies affect your community—all of these require comfort with numbers. Every degree requires that students learn in-depth facts and ways of thinking related to that subject, but every degree also requires that students have a basic knowledge to make good decisions and be able to communicate. Learning mathematical thinking is part of every human beings birthright. You deserve to be able to think in powerful ways, and mathematical thinking is very powerful.

Ok, if this book is for you, here we go!
Chapter 1: What is a Number?

In which you will learn:

- It is useful to visualize some numbers as points on number lines
- It is useful to visualize some numbers as arrows on number lines
- It is useful to conceptualize some numbers as repetitions or pieces of something.
- Numbers can be positive or negative
- It is often useful to use letters to represent numbers

Chapter 1

What the heck is a number?

Of course, you use numbers all the time. Our paychecks, our bills, how much we weigh, how tall we are, what percent fat our milk is, how likely it is that we graduate college in 4 years, how fast we run ... these are all described with numbers. So, whether you can define a number or not, you probably are good with numbers, at least some types of numbers. Wouldn’t it be fabulous if you saw numbers in a way that will help you experience a lot of “ah–ha!” moments in your math classes, while building your confidence and capacity to use numbers to answer complicated questions?

There are several ways people can learn: we can have new experiences, and, remembering and reflecting on those experiences, we can say, “I’ve learned something”. “I learned that I enjoy eating chocolate” is something you can learn from the experience of eating chocolate. “I learned that fire burns my hand” is something you learn when you trip and fall into the fire. How we learn to understand mathematical concepts is a little bit different: we learn new concepts by fitting them into existing concepts that we already have. Unfortunately, many of you have had experiences with math that are more like falling into a fire and getting burned than building new understanding based on previous thoughts and ideas. For those of you who have had those “burning math experiences”, please accept my sincerest and most heartfelt apology.

Back to my initial question: What is a number? Clearly, if you are reading this, you already have some concept of a number. Stop for a moment and think about what, to you, a number means. What does the number 3 bring to mind? What about the number –3? What about 0.1?
What are your ideas of what the meaning of a number is?

There are many useful ways to think about the kinds of numbers you’ll need to understand to be successful in college math. In some of your math classes, you might learn names and distinctions among several types of numbers: counting numbers, integers, real numbers, rational numbers, imaginary numbers, etc... I will not focus on the names of numbers or categories of numbers, and leave that for you to learn in your future math classes if needed. Being able to name a thing does not indicate that one understands it, and, somewhat paradoxically, sometimes naming gets in the way of understanding.

My promise to you, as you read this book and work through the problems and exercises, is that you will build your understanding of numbers so that when and if you are required to distinguish an integer from an irrational number the distinction will be obvious and easy to learn. You will have built a conceptual framework of mathematics into which you can fit new concepts, and upon which you can build new understanding. By working through this book, you will build your problem–solving abilities, and you will practice thinking flexibly and creatively, but you won't learn many definitions or mathematical jargon.

To keep it simple and relevant I’m introducing some ways to conceptualize numbers. These might be new to you! I’d like you to wrap your head around all of them (of course there are many, many more; these are just common, and I think they are useful):

1. Point on a number line

   - A number could be represented by a point on a number line. You could, for example, represent 5 apples as a point on number line.

   - If the number line you are looking at goes right to left and left to right (horizontally, like the horizon), positive numbers are to the right of zero, and negative numbers are to the left of zero (this is just what is called “convention”, meaning it’s what was decided a long time ago; it doesn’t always have to be that way).

   - If the number line goes up and down (like an old–fashioned analog thermometer), any number above 0 is a positive number, and any number below zero is a negative number.
2. An arrow on a number line

A number could be represented by an arrow with a certain length on a number line. A number that represents a change from one thing to another thing is often written as an arrow. For example, if you started with $15 and ended up with $20, the amount of money you gained could be shown as an arrow with a length of 5.

- If the number line you are looking at is horizontal (like the horizon), an arrow pointing to the right is a positive number, and its length is its value. No matter where the arrow starts or ends with respect to zero, if it’s pointing to the right, it refers to a positive number. If it’s pointing to the left, it’s a negative number. So, if you start at 20 and go to 15, the arrow from 20 to 15 represents the number –5 (pronounced negative five).

- If the number line you are looking at is vertical (up and down), an arrow pointing up is a positive number, and its length is its value. No matter where the arrow is with respect to zero, if it’s pointing up, it refers to a positive number. If it’s pointing down, it’s a negative number.

3. Conceptualizing numbers as repetitions or pieces.

The first two ways of conceptualizing numbers were visual on number lines. But, as you know when you pull out your wallet and pull out 5 ten-dollar bills, a number could mean “how many repeats” of some other number, or “some part of, or piece of” some other number. In that example, “5” refers to the number of times you repeat the 10-dollar bill.

If you are dealing with the type of number that is a “repetition,” or a “piece” then this number is usually being multiplied to or multiplied by another number (more on this on the chapter on multiplication and division). If you have 5 ten-dollar bills, you could say “I have 5 multiplied to 10 dollars”, or “I have 10 dollars, multiplied by 5”
• For example, if I said I had 5 threes’, I could draw 5 arrows pointing right, each with length “three”, connected beginning–to–end. In this case the number 5 refers to the number of repetitions of arrows. If I said I had 4 negative sevens’, I could draw 4 left–pointing arrows, all connected, each with length 7. The result would be one big left–pointing arrow with length 28. This idea will be explained more in the chapter on multiplying, and I don’t recommend skipping ahead to that chapter. Be patient, you’ll get there.

These are three ways to think about numbers: points on a number line, arrows on a number line, or repetitions of arrows. Of course, you have seen other representations, such as, for example:

• A common way to represent numbers is, simply with dots or tally marks. 4 dots could represent the number 4, and 10 tally marks could represent the number 10.

• A popular representation for numbers, especially fractions, is with pictures that look like slices of pizza.

You can use any of these visualizations, and others, to help you work with numbers, but, in this book, I focus on the two visualizations using number lines (points on a number line or arrows on a number line) and the concept of “number of repetitions” or “pieces”. You will find that as you learn, you create your own unique perspective and understanding of mathematics based on your own, unique, creative thinking.

How do you decide which representation or concept will help you answer the particular problem facing you? That’s what this book is about! These ideas will lay the foundation to think creatively about mathematics; how you most readily think about a particular number in a particular situation might be different from how I think about the same situation. How you think of a particular number depends on what makes the most sense to you in a given situation. Sometimes it makes sense to think of arrows with a certain length and direction. Sometimes it makes more sense to think of a point on a number line. And sometimes, a number is a repetition of arrows (a repetition of numbers) or a piece of a certain arrow.

Note that “the point on a number line representing 3”, “an arrow pointing to the right with a length of 3”, “the concept of 3 repetitions”, or a representation of 3 using three dots, are all mathematically equal. How you visualize or conceptualize numbers does not affect if they are considered equal or not.

Even though these concepts and visualizations of "three–ness" might be different, they are all equal to 3.
Negative numbers

It’s important to understand that the sign of the number is part of the number itself. The number negative 5 is one number, and includes the symbol for a negative (−) and the digit 5.

If a number is represented as a point on a number line, its “opposite” is the point on the number line on the other side of the 0 and the same distance from zero. The "opposite" of 5 is negative 5, and the "opposite" of negative 3 is 3. The "opposite of a negative is a positive". The opposite of −6 is 6, and the opposite of 4 is −4.

Ok … sit up a little taller and lean in, because it’s about to get a little more complicated… are you ready?

Sometimes, we use the word "negative" to mean the same thing as "opposite". This can cause confusion. Previously we said that a negative number is to the left of the zero when on the horizontal number line. Now, we are seeing that the word negative can also mean the opposite of any number. And, when stated this way, the negative of a number could be a positive number. Aaargh! "negative −7" would therefore be 7, because it's "the opposite of negative 7".

The “opposite”, or “negative” of a left–pointing arrow with length 7 is a right–pointing arrow with length 7 (meaning the opposite of −7 is 7). The “opposite” or “negative” of 3 repetitions is −3 repetitions. What exactly the phrase “negative repetitions” means might not be clear now, but we will explore the idea of “negative repetitions” in the chapter on multiplication.

We use the “−“ symbol to show “opposite”; it’s the same as a subtraction symbol, or the “minus sign” as in “7 − 5 = 2” and this is both convenient and confusing. **Subtraction is something you do to numbers and the negative sign is part of a number, but we use the same symbols for “subtraction”, “negative” and “opposite”**.

For example, 10 minus 5 is written 10 − 5. This represents something you do with the two numbers 10 and 5. The number “negative 5” is written as −5. This is just a number… you are not doing anything to it. And, to write “the opposite of negative 5", you’d write − − 5, which is the same as 5!

We use the same symbol for:

1. the act of subtraction, as in a minus sign
2. to show a negative number, as in −5,
3. to show the opposite of a number, like −x, or −(−5)

For now, keep in mind that −5 means the number “negative 5”. And, the opposite of the number −5 could be written −(−5), and means the same as the number 5.

**Connecting to previous ideas.**

Now, go back and look at what you wrote for ideas of what a number meant to you… can you connect your ideas to the 3 representations I gave you?
Are you thinking to yourself, “what were those three representations?” If you forgot, GREAT! Forgetting something, and then remembering is a great way to learn. Can you recall the three representations of numbers that I introduced without looking back?

1. Can you recall them here, now, without looking?

   a. How is representing a number as a point on a number line, an arrow, or a repetition of arrows connected to the ideas you wrote about what a number is? How are they different?

Using letters instead of numbers:

If you have used letters to represent numbers, take a moment to think about, and use this space to write down what ideas you remember about that practice. What does it mean to you? Write some examples you might remember of mathematical tasks with letters that represent numbers. Even if you don’t remember very well, write what you can. If you’ve never had any algebra: keep this space blank.

Why use letters instead of numbers?

There are three ways that people use letters in mathematics that you will find useful:

1. To describe ideas about numbers in general:

   For example, if you subtract 1 from any number, you’ll get the same result as if you had added negative 1 to that number. You can express this statement like this:

   $$x - 1 = x + (-1)$$

   In that equation, $x$ stands for any number. It does not represent some unknown number; it would not make sense to “solve for $x$. ” In this example, $x$ just means “any number in general.” Another example of using a letter or letters to stand for numbers in general would be this equation:

   $$(-x)(-y) = (xy)$$

   What this means is that the opposite of $x$ times the opposite of $y$ is the same as $x$ times $y$. $x$ and $y$ stand for any numbers in general. This is not a statement about any kind of relationship between two variables, and you could not solve for one variable or the other. It is simply a statement about numbers in general.

2. To represent some specific unknown number or numbers:

   Another way we use letters in mathematics is to represent a specific unknown number or numbers. For example, suppose I am able to save $15 every week, and I wonder how long it will take for me to have about $230. This equation might help me answer this question:

   $$15x = 230$$

   $x$ stands for the number of weeks that I save 15. It’s the number of times I have to repeat 15 before I get to 230. This equation can be solved, it has one solution, and $x$ stands for the number that makes this equation true. It doesn’t
stand for numbers in general, it stands for a very specific number (15 \( \frac{1}{3} \), in case you were wondering. So, you’d have to wait 16 weeks to get to at least $230). When letters are used in mathematics in this way, we often are trying to solve an equation. **Solving an equation simply means finding all the values for the letters (the unknown numbers) that would make the equation true.**

3. **To describe relationships between or among quantities:**

Another way that mathematicians use letters in equations is to show **relationships among quantities**. Consider the same example, above, in which I save $15 every week. In two weeks, I’ll have saved $26, in 3 weeks I’ll have saved $39 dollars, etc…I wonder… how is this savings described mathematically? This situation is described by this equation:

\[ 13x = T \]

\( x \) in this equation stands for the number of weeks that I put $13 into my piggy bank, and \( T \) is the total that is in the bank. All of the pairs of numbers that make this equation true are solutions to this equation. The equation describes a relationship between two quantities.

Here’s another example: suppose you are in Austin, Texas, and you want to call your friend in Anchorage, Alaska, but you don’t want to wake her up. What time is it in Anchorage, Alaska? There are 3 hours’ different between the time zone in Texas and the main Alaska time zone. An equation that describes this relationship is

\[ T - A = 3 \]

\( T \) stands for the time in Austin, Texas; \( A \) stands for the time in Anchorage, Alaska. This equation describes the relationship between the two times, and is always true, not matter what time it is.

I think you are ready for some exercises! Enough reading…

**Exercises:**

First, take a look back and review what you wrote in your notes above, and then do the following exercises. The solutions are at the back of the book, so please check yourself as you go and examine your mistakes. It’s important that you work slowly and thoughtfully. This is NOT a timed test. The most important favor you can do yourself is to celebrate any errors you make. If you don’t make any errors, you are probably not learning. So, celebrate your errors, examine them, understand what went wrong. Never ever, ever ignore them or hope they will go away on their own. When you make an error, think “hooray… now I have a mystery to solve”.

For each of the following, identify the number that is being represented or described. Feel free to use a number line to help you, but write your answer as a positive or negative number:

2. An arrow with a length of 3 pointing to the left.

3. The number represented by this arrow:
4. The number of times this unknown negative number is repeated:

5. The point on a number line that is halfway between negative 4 and positive 2.

6. An arrow that starts at 8 and goes to 10.

7. An arrow that starts at 10 and ends at 8.

8. An arrow that starts at –3 and goes to –1.

9. An arrow that starts at –1 and goes to –3.

10. The number of times the arrows are repeated:

11. The number of times the arrows are repeated:

12. The point on a number line that is halfway between –2 and 0.

13. The point on a number line that is halfway between –9 and –7.

14. The point on a number line that is halfway between positive 5 and negative 5.

15. The opposite of the number –3.

16. The opposite of the number 3.

17. If x represents –8, what is the opposite of x? Put another way, if x = –8, what is –x?

18. If x = 7, what is the opposite of negative x?

19. If x = –5, what is negative x?

20. If negative x = 5, what is x?

21. If negative x = 3, what is x?

22. If the opposite of x = 8, what is x?

23. If the opposite of x = –3, what is x?

24. If x is -10, is “negative x” a positive or a negative number?
In the following exercises, decide whether

I: the equation describes something true about all numbers, with perhaps some exceptions. If there are exceptions, what are they?

III: the equation describes a relationship between quantities where the value of one depends on the value of another.

III. it is possible to make a complete list of all of the solutions. If so, make the list.

25. \( x + y = y + x \)
26. \( x - 1 = x + (-1) \)
27. \( x - 1 = 10 \)
28. \( 2x = 10 \)
29. \( 2x = y \)
30. \( 2x = x + x \)
31. \( 2x + y = 2x + 3 \)
32. \( x^2 = (x)(x) \)
33. \( x^2 = 9 \)
34. \( x^2 = y \)
35. \( x + x + x = 2x + x \)
36. \( x + x = 4 \)
37. \( x + x = 2 \)
38. \( x - y = x + (-y) \)
39. \( x + 2 = y \)
40. \( x = y \)
41. \( x = 2 \)
42. \( x = 1 \)
43. \( x = x \)
Make note of what mistakes you made in the previous exercises, or when you had to look up the answer. DON’T discount any mistakes as “silly”: confront them and change your thinking!

- How will you need to change the way you think about things to avoid making that mistake again?

- How will you need to change the way you think about numbers, so you don’t have to look up the answers you looked up?

You might try asking friends, relatives and classmates how they think about questions you missed. You might have to read more slowly and do the exercises more slowly to examine your thinking. Good luck!

{if you easily breezed through this, that’s ok! But, I hope, even if you weren’t challenged, you changed you mind about something. If you were challenged, Hooray! Learning is everybody’s birthright.}

I hope you learned:

- It is useful to visualize some numbers as points on number lines
- It is useful to visualize some numbers as arrows on number lines
- It is useful to conceptualize some numbers as repetitions or pieces of something.
- Numbers can be positive or negative
- It is often useful to use letters to represent numbers
Chapter 2: Addition

In which you will learn:

- Adding means combining quantities
- You can simplify some combinations, but not others
- Whether you are combining positive or negative numbers, you can generalize any addition with this equation: \( x_1 + \Delta x = x_2 \). The signs of the numbers \( x_1, \Delta x, \) and \( x_2 \) are included in their symbolic representations.
- If \( \Delta x \) is a negative number, \( x_2 \) is less than \( x_1 \).

Adding

So, you’ve learned that I am going to represent and think about numbers in one of three ways: points on a number line, arrows with direction and length, or repetitions or pieces of arrows. But what really matters about numbers is what we DO with them.

Here, write down whatever you remember about addition, and write different examples of addition. Do you remember how to add large numbers and “carry” digits (how do you add 473 and 649, for example?) Do you remember how to add fractions? If you don’t remember, don’t stress: make a note (something like “I know I used to be able to add fractions, but I don’t remember how now”). Be clear in what you write, because you will come back later to look at your work through new eyes after you read this chapter.
Addition is **combining two or more numbers**. If I have 3 bags of gold, and you have 5 bags of gold, and we combine our bags, we will have 8 bags of gold. Altogether, we’ll have 8 bags of gold. So far, so good, yes?

Suppose I have 3 bags of gold, and you have 5 used toothbrushes. Can I combine my bags of gold and your toothbrushes? YES! Of course, why not? What I would get is 3 bags of gold and 5 toothbrushes jumbled together in a pile. Can I combine the “3” and the “5”, NO! The 3 and the 5 refer to different things: bags of gold in one case and used toothbrushes in the other.

You can simplify the combination of numbers when they refer to the same thing. Mathematicians sometimes say “you can combine like terms”. (A “like term” is what mathematicians call a “type of thing”, perhaps because they wouldn’t feel professional using the word “thing”). You can add 3 bananas to 6 bananas and get 9 bananas. You could keep saying “6 plus 3” bananas, but it’s simpler to say 9 instead of “6 plus 3”. You can add 3 bananas to 6 apples by putting them together in your grocery bag, but, you can’t simplify what you have combined. 3 bananas + 6 apples can’t be simplified.\(^1\)

Bottom line: you can simplify combinations if you are combining “like terms”, and “like terms” means “same things”. So, if I am adding 3 pennies to 10 pennies, I could more simply describe what I have by saying I have 13 pennies. If I am adding 3 dirty socks and 8 hedgehogs, I can do it by putting them into a sack, but I’d still have 3 dirty socks and 8 hedgehogs, and that won’t simplify at all.

**Exercises with adding “like terms”**

If possible, simplify the following combinations so your result is 1 number and 1 noun. You might have to change how you describe some of the objects. For example, a dollar could be described as 10 dimes or 4 quarters. The answers are at the end of the chapter, but you learn by thinking and even struggling. Jumping to the answer won’t help you change your brain!

1. 2 dollars and 5 dollars
2. 2 dollars and 3 dimes
3. 2 dollars and 10 dimes
4. 2 dollars and 16 dimes
5. 2 pencils and 7 chairs
6. 2 dollars 12 dimes and 120 pennies
7. 3 hundred plus 9 hundred
8. 23 hundred plus 8 hundred
9. 2 thousand plus 3 hundred plus 8 hundred
10. Do “23 hundred” and “2 thousand 3 hundred” refer to the same number?

\(^1\) Although you could make an excellent fruit smoothie.
11. Do “2 tens plus 8 ones” and “28” refer to the same quantity?
12. Do “3 tens plus 8 tens” and “11 tens” refer to the same quantity?
13. What is a more common way to refer to the quantity “11 tens”?
14. 2 tens plus 6 tens
15. 2 tens plus 4 ones
16. 2 x’s plus 3 x’s (x refers to some quantity that you don’t know or don’t feel like specifying)
17. 2 x’s plus 3 y’s (x and y refer to some quantities that you don’t know, or don’t feel like specifying)

**Visualizing addition with number lines, adding ΔX (“Delta X”)**

Suppose I have 5 beans. I could represent this number of beans by drawing a number line and noting the number 5.

Now, I add 3 beans to the 5 beans. These 3 beans represent the “change” that I’ve added to my original amount of beans. Since I am combining beans and beans, I can combine the numbers. Here’s a way to represent this on a number line:

Addition of two quantities can be represented visually with a point on a number line showing the initial amount (your starting point), and an arrow showing what you are adding. The arrow is the “change” or “what you added”. Where the arrow ends up is the result of the combination.

Mathematics is often about generalization; this means writing ideas down that apply to numbers in general, not just specific numbers about beans. I could start with any arbitrary amount of anything and I could represent it with a letter or other type of symbol. I can represent it with a letter even if I do know what the actual amount is.

For example, say I’ve decided to represent the amount of something that I start with as $x_1$. This is pronounced “X sub one” or “X one”. The little subscript “1” is to remind
me that it’s my starting point. I could also have called it the $x_A$, or $x_0$ pronounced “X sub A” or “X zero”. I could label it “S” for “start”. It doesn’t matter so much how you label it, as long as you know it represents the amount of something you start with, and can be visualized as a point on a number line. Note that $x_1$ is a different number than $x_2$ or $x_3$.

In mathematics and science it is often common to call the amount something has changed “delta”, as in the Greek letter $\Delta$. I know I promised that I wouldn’t use many definitions or jargon, but this one is so important and useful I am going to insist upon using it. Bottom line: what you add to an initial amount is called “delta”.

All of this is leading to this generalized, symbolic, representation of addition. This equation shows the relationship among 3 numbers:

$$x_1 + \Delta x = x_2$$

The initial amount, meaning what I start with, is called $x_1$, and what I am adding to it is called $\Delta x$, (pronounced “delta X”) and if they are combined, their combination is the same as the number called $x_2$. $\Delta x$, $x_1$, and $x_2$ all represent numbers, and the symbol “=” simply means “the same as” or even more simply “is” (the = symbol does not mean “the answer is”, because “the answer” depends on what question you are asking).

The important aspect of this visualization of addition is that you start with a number represented by a point on the number line and end up with a number represented by another point on a number line. Note that in the example shown in the picture, $\Delta x$ is positive, so the arrow that represents $\Delta x$ points to the right. $\Delta x$ is what is added, and is represented by an arrow. $\Delta x$ is the difference between where I ended up and where I started on the number line.

Now, remember those 3 beans combined with the 5 beans? These 3 beans could represent the “change” that I’ve added to my original amount of beans. Since I am combining beans and beans, I can combine the numbers. Here’s a way to represent this on a number line, using $x_1$, $\Delta x$, and $x_2$. Here is a picture of $5 + 3 = 8$:

**Adding a Negative Change**

Remember: a number is negative if it’s a left-pointing arrow.

So, the visualization of $x_1 + \Delta x = x_2$ could look like this, if $\Delta x$ happens to be a negative number!
\( \Delta x \), when represented with an arrow, always goes from \( x_1 \) to \( x_2 \), and means “the difference between \( x_2 \) and \( x_1 \)” The difference between two numbers can be negative or positive; the difference means the quantity added to one number to get to another number.

**The difference between \( x_2 \) and \( x_1 \) is \( x_2 - x_1 \)**

This can be visualized as an arrow going from \( x_1 \) to \( x_2 \)

**CAUTION:** Students often look at this picture:

And think it must represent a subtraction, because you end up at a smaller number than you started with. However, this equation \( x_1 + \Delta x = x_2 \) has no subtraction sign or negative sign! *The fact that any of these numbers are negative is included in the symbols and is not written out explicitly.*

Did you get that? **Whether or not numbers are positive or negative is INCLUDED in the symbol that represents them.**

\( x_1 \) can represent 5
\( x_1 \) can represent \(-5\)
\( \Delta x \) can represent 200
\( \Delta x \) can represent \(-2\)

Try to answer this question “I added an unknown number to 5. Is my result bigger or smaller than 5?”

Actually, you have to answer “I don’t know, because I don’t know if I added a positive or negative number to 5. If I added a negative number, the answer is less than 5. If I added a positive number, the answer is bigger than 5”.

How do you write “I added an unknown number to 5?” the unknown number can be represented by a symbol, \( \Delta x \), and that addition of \( \Delta x \) to 5 looks like this:

\[ 5 + \Delta x = x_2 \]

Remember, \( \Delta x \) **could be negative or positive.**

Enough reading. You should try some exercises that require you think think about these ideas:
18. My daughter has $-10 (she’s in debt), and my son has $50.
   a. Show on this number line how much money you would have to give my daughter so that she ends up with the same amount of money my son has.

   

   b. Write an addition equation that shows the relationship among the three numbers: -10, Δx, and 50
   c. What is the number Δx? Is Δx a positive or negative number?

19. My daughter has D amount of money, and my son has S amount of money.
   If D is less than S, you’d have to give my daughter a positive Δx amount of money for her to end up with what my son has. See the picture below.

   

   If D is more than S, you’d have to give her a debt (add a negative amount to the money she has) for her to end up with the same amount of money as my son. See the picture below.

   

   a. Write an equation that shows the relationship among the three numbers: D, Δx, and S.
   b. Do you understand that it is the same equation whether or not my daughter has more or less money than my son? Explain in your own words how that can be the same equation (it has something to do with the idea that the sign of a number is included in the variable). If you had any
negative symbol, subtraction symbol, minus sign, opposite sign anywhere in your equation, you were wrong!

c. If S is 50 dollars and D is -30 dollars, what is $\Delta x$? In other words, what do I give my daughter so she ends up with what my son has? Write the equation that gives you your answer.

d. If S is -10 dollars and D is -5 dollars, what is $\Delta x$? In other words, what do I give my daughter so she ends up with what my son has? Write the equation that gives you your answer.

write the specific addition problems that are represented visually in the following number lines. (even if you can represent what is written as a subtraction problem, write it as an addition problem.)

20. $x_1 + \Delta x = x_2$; write the specific addition statement that is represented visually in the following number line

![Number Line 1](image1)

21. $x_1 + \Delta x = x_2$; write the specific addition statement that is represented visually in the following number line

![Number Line 2](image2)

22. $x_1 + \Delta x = x_2$; write the specific addition statement that is represented visually in the following number line

![Number Line 3](image3)
23. \(x_1 + \Delta x = x_2\), write the specific addition statement that is represented visually in the following number line

(continued)

24. \(x_1 + \Delta x = x_2\); write the specific addition statement that is represented visually in the following number line

25. \(x_1 + \Delta x = x_2\); write the specific addition statement that is represented visually in the following number line

26. \(x_1 + \Delta x = x_2\); write the specific addition statement that is represented visually in the following number line

27. \(x_1 + \Delta x = x_2\); write the specific addition statement that is represented visually in the following number line
28. All of the exercises above were addition exercises. There was an initial value and it is combined with another value. However, some of the exercises included negative numbers. What was your reaction to the negative numbers? Did you try to re-think the problems as subtraction? Why? What do you remember about that from past classes? If your tendency is to change addition problems to subtraction problems, that’s ok, but if you did, it’s important that you understand the idea of addition as

\[ x_1 + \Delta x = x_2 \]

And that any of these three symbols might represent negative numbers.

Write your reaction to realizing that problems that you may automatically consider subtraction are actually addition problems using negative numbers.

Some common reactions are: “in my head, I change an addition to a subtraction so I can actually solve the equation”

**Why is it important to distinguish that addition with negative numbers is different than subtraction?**

Addition with negative numbers is different than subtraction, because remember that addition is combining two or more numbers. In all of our examples in this chapter, we have been combining numbers, we have not removed anything or figured out the difference between numbers.

The meaning of combining numbers if one or both of them are negative depends on the context.

If the “thing” we are talking about is money, then

- a negative number would be a debt.
- If I combine my debt with another debt, I get more in debt! \((x_1 \text{ is negative and } \Delta x \text{ is negative and } x_2 \text{ is negative})\)
- If I start out a little in debt and combine that with winning the lottery, I end up with a bunch of money. (I wish). \((x_1 \text{ is negative, } \Delta x \text{ is big big positive, and } x_2 \text{ is positive})\)

The "thing" could be elevation;

- I could be scuba diving and start at a certain depth and then go deeper. \((x_1 \text{ is negative and } \Delta x \text{ is negative and } x_2 \text{ is negative})\)

The "thing" could be temperature;

- it was -40F outside somewhere in Minnesota, and then it got colder: the new temperature is a combination of a negative with another negative number \((x_1 \text{ is negative or positive? } \Delta x? \ x_2? \text{ }^2\)

\[ \text{ }^2 (x_1 \text{ is negative and } \Delta x \text{ is negative and } x_2 \text{ is negative})\]
You will see in the section on subtraction, that, indeed, subtraction and addition of negative numbers are related. Be patient, please.

Here are more questions for you to answer... be sure to struggle and think before you check solutions, but do check the solutions. It’s not good if you practice incorrectly.

29. Suppose I am combining some assets (represented by positive numbers) and debts (represented by negative numbers) to figure out how much money I have in total. To look at this like a mathematician (and to look at it in a way that will help you in math class) I want to look at this combination through the mental lens of this structure: \( x_1 + \Delta x = x_2 \)

First, I think about the money I own, because I like to start out with happy thoughts. Suppose I have $200. Then, I combine that with my debt of -$300.

\[
x_1 = \$200 \\
\Delta x = -$300
\]

What is \( x_2 \)? Draw a number line representing this addition.

30. Next, I am going to go about this a different way.

First, I think about my debt, because it’s really on my mind: -$300. Then, I combine that with the money I have: $200.

\[
x_1 = -$300 \\
\Delta x = $200
\]

What is \( x_2 \)? Draw a number line.

31. Question: Do you end up with the same result if you add numbers in different order?

You might visualize 3+-2 differently than -2+3, but when you combine numbers, the result never depends on what number you call \( x_1 \) and what number you call \( \Delta x \). The end result, what we’re calling \( x_2 \) will be the same.

32. Question: Does it EVER matter in what order you add numbers? What about when the numbers are both negative or both positive?

33. If I add \( X \) (an unknown number) to 7, will the result be more or less than 7? (or, is it impossible to tell?)

34. If I add \( X \) (an unknown number) to 7, will the result be more or less than \( X \)? (or, is it impossible to tell?)

35. I add \( Y \) to -2 and I get the number \( N \). Is \( N \) less than or bigger than -2, or, is it impossible to tell?)

36. I add \( Y \) to -2 and I get the number \( N \). Is \( N \) less than or bigger than \( Y \), or, is it impossible to tell?)

37. How could you write “adding \( Y \) to -2 is \( N \)” as an equation? (choose all that are correct)

a. \(-2 + Y = N\)

b. \( Y + -2 = N\)
Look at this equation:

\[ x_1 + \Delta x = x_2 \]

At the beginning of this chapter, you wrote what you remembered about adding. Take a look back at what you wrote… how have your ideas changed?

If you wrote that you weren’t sure how to add fractions, you will have to wait till the chapters on fractions to get to that. For now, you could start pondering how adding fractions, for example 2 thirds plus 4 fifths, is similar to adding two quantities that are described with different terms, like pennies and dimes.

Use this space to summarize what you learned about addition, especially how you physically changed your mind by considering something in a new way. Congratulations! That’s hard work! Be sure to be kind to yourself after all this thinking. You need breaks, and keep a balance in your life, but your education is worth a lot of hard work!
I hope you learned:

- Adding means combining quantities
- You can simplify some combinations, but not others
- Whether you are combining positive or negative numbers, you can generalize any addition with this equation: \( x_1 + \Delta x = x_2 \). The signs of the numbers \( x_1, \Delta x, \) and \( x_2 \) are included in their symbolic representations.
- If \( \Delta x \) is a negative number, \( x_2 \) is less than \( x_1 \).

Now, after your hard work, you can wrap your head around these ideas, and use them! Celebrate your progress!
Chapter 3: What is an Equal Sign? Equivalent Equations in Addition and Subtraction

In which you will learn:

- Once you have created an equation that describes a situation, you might make useful discoveries and statements about that same situation if you “do the same thing to both sides of the equal sign.”
- Subtraction “undoes” addition
- Finding a solution to an equation means finding the values of any variables in that equation that make the equation a true statement.

*What is an Equal Sign?*

In the chapter “What is a Number?”, you learned that a number is equivalent to another number regardless of how you are visualizing it: arrow, point on number line, number of repetitions.

*Solutions to Equations*

An equation is a statement that one thing is the same as another thing. As an example, consider the equation $x + 3 = 10$, which is simply the statement that “if I combine $x$ and 3, I’ll get 10”.

If an equation has variables (letters that stand for unknowns), then finding the values of those variables that make the equation true is called “finding the solution to that equation”.

In the above example, the solution to $x + 3 = 10$ is $x = 7$, because if you replace $x$ with 7, you get $7 + 3 = 10$, which is a true statement.

Sometimes, you can find the solutions to equations simply by considering the equation and asking yourself what values of the variables would make the equation true, and sometimes, you get to do some algebra to solve equations. How exciting!

For example, consider this equation:

$$(x)(x) = 25$$

You might know how to find the solutions to this equation through various algebraic techniques, but it might also be faster to ask yourself “what are all the values of $x$ that could make that equation true?” In other words, you are finding all the numbers that make this statement true: “this number, if multiplied to itself, will give you the number 25”.

For that equation, there are two solutions: $x = -5$ and $x = 5$. You could get to that solution without doing any formal algebra.

*Keeping Your Balance in Math, or “doing same thing to both sides of the equal sign”*
Suppose you have an equation that you know accurately describes the relationship among some quantities. For example, consider the time in New York and the time in Los Angeles. You know that New York is 3 hours ahead of Los Angeles, so an equation that describes the relationship between the times in both cities is:

\[ L + 3 = N \]
\[ x_1 + \Delta x = x_2 \]

\( L \) is the time in Los Angeles \((x_1)\), \( N \) is the time in New York \((x_2)\), and \( \Delta x \) is 3. This equation describes this relationship no matter what time it is. I start with the time in LA, I add 3 to end up with the time in New York.

But if I started out asking you to write an equation describing the situation “NY is 3 hours ahead of LA”, some people would write:

\[ 3 = N - L \]
\[ \Delta x = x_2 - x_1 \]

The difference between the two cities’ times is 3 (shown by the symbol \( \Delta x \)).

Some people would write, when given the task of writing an equation that describes the relationship between the two cities’ time,

\[ L = N - 3 \]
\[ x_1 = x_2 - \Delta x \]

If I remove 3 from the time in New York, I’ll end up with the time in Los Angeles.

All of these equations describe the relationship between the time in LA and NY. These are equivalent equations, and the statements that follow each equation are equivalent statements:

- \( L + 3 = N, add 3 \text{ hours to LA time to get NY time} \)
- \( 3 = N - L, \text{the difference between NY and LA is 3 hours} \)
- \( L = N - 3, \text{subtract 3 hours from NY to get LA time} \)

How would you get from one of these equations to another, if you didn’t want to actually think too hard? This is the heart of algebra:

**If an equation correctly describes a situation, any math operation you do to both sides of the equal sign might result in another equation that could also correctly describe the situation.**

The word “might” is important because some mathematical operations, when done to both sides of an equal sign, will not give you an equivalent equation. You’ll think about some of those kinds of mathematical operations in later chapters, and some in later math classes. This is why you must always check your answer in the original equation

To answer questions with numbers, all you have to do is construct a true equation, then do the same thing to both sides of the equal sign until you have another true equation that is more helpful to you. Then, check to see if your answer makes sense. It might be that you did some math that introduced some solutions that don’t fit your original equation, or it might be that you accidentally tried to divide by zero.
Equivalent Equation Examples

Exercises
In the following examples, explain what mathematical operation is done to the first equation to get to the second equation. (you are NOT necessarily solving for \(x\), you’ll get to do that later…). Please check your answers as you go because it’s important you don’t re-enforce wrong ideas!

1. \(2 + 3 = 5\) \(2 = 5 - 3\)
2. \(x - 3 = 5\) \(x = 8\)
3. \(x - 3 = 5\) \(x - 8 = 0\)
4. \(3 - x = 5\) \(3 = 5 + x\)
5. \(x_1 + \Delta x = x_2\) \(x_1 = x_2 - \Delta x\)
   (these two equations show the relationship between addition and “removing what was added”)  
6. \(x_1 + \Delta x = x_2\) \(\Delta x = x_2 - x_1\)
   (these two equations show the relationship between addition and “finding the difference”)  

Check all of your answers. If you were unsure of your answer, or didn’t get the same answer as in the solutions, write what you needed help with here. Write what was confusing. It’s especially helpful to frame your thinking with a prompt like:

“I used to think …., but now I think ….”

Or,

“I didn’t understand that …., but now I understand why that works ….”

More on equivalent equations: isolating \(x\)
Much of algebra is devoted to “solving for \(x\)”, where \(x\) is a number you don’t know, but you know a true statement that involves \(x\)

If you know that, then you also know that any equation (as long as you don’t divide by zero) that you can get to by doing the same mathematical operation to both sides of the equal sign might also be true. If your goal is to figure out what number \(x\) is, then you need to think about what you can do to both sides of an equal sign so you get an equation that has \(x\) all by itself on one or the other side of the equal sign. This is called “isolating \(x\)”, or, simply “solving for \(x\)” or “solving the equation”.

Learning efficient ways to isolate \(x\), or solve for \(x\), takes a lot of practice, and some types of equations are easier to work with than others. In algebra and precalculus classes you will learn various techniques and procedures, but keep in mind they are often simply variations on doing the same thing to both sides of the equal sign, and as long as you are doing that, you won’t be doing anything wrong (but don’t divide by zero). You might not always be efficient unless you practice a lot, but as long as you
“do the same thing to both sides of the equal sign” you are doing ok\(^1\), and probably on the right track.

Exercises: Solve for \(x\) in the following equations. Some might take multiple steps. Keep “doing the same thing” to both sides of resulting equations until you have isolated \(x\).

7. \(x + 3 = 6\)
8. \(x - 3 = -5\)
9. \(x + 2 = -7\)
10. \(3 - x = 5\)
11. \(x_1 + \Delta x = x_2\) (solve for \(x_1\))
12. \(x_1 + \Delta x = x_2\) (solve for \(\Delta x\))

13. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The difference between Fred and Melinda’s money is $100
   b. If you add 100 to Fred’s money, you’ll have the same as Melinda’s money

14. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The difference between Fred and Melinda’s money is $100
   b. If you add 100 to Melinda’s money, you’ll have the same as Fred’s money

15. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The difference between Fred and Melinda’s money is $100
   b. If you subtract 100 from Melinda’s money, you’ll have the same as Fred’s money

16. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The difference between Fred and Melinda’s money is $100
   b. If you subtract 100 from Fred’s money, you’ll have the same as Melinda’s money

17. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The temperature in Fairbanks, Alaska is 15 degrees more than the temperature in Seattle, Washington
   b. Fairbank’s temperature is Seattle’s temperature minus 15 degrees

18. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The temperature in Fairbanks, Alaska is 15 degrees more than the temperature in Seattle, Washington
   b. Seattle’s temperature is Fairbanks’ temperature minus 15 degrees

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\(^1\) But don’t divide by zero, and always check if the solution you come up with is a solution to the original equation, you may have introduced some extra solutions, or overlooked some, as well.
19. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. Seattle’s temperature is Fairbanks’ temperature minus 15 degrees
   b. Fairbanks’ temperature is Seattle’s temperature plus 15 degrees

20. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The difference between Fairbanks’ temperature and Seattle’s temperature is 15
   b. Fairbanks’ temperature is Seattle’s temperature plus 15 degrees

21. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The difference between Fairbanks’ temperature and Seattle’s temperature is 15
   b. Seattle’s temperature is 15 degrees more than Fairbanks’

22. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. Fairbanks’ temperature is Seattle’s temperature plus 15 degrees
   b. Fairbanks’ temperature minus 15 gives you Seattle’s temperature.

Before you read any further, check your solutions with the answers (ideally, you should be checking as you go). It would be great if you made many errors, because that means you have some learning to do, and that you are getting your money’s worth out of the time you are putting into your learning. If you made no errors: good job, as well… you must be working carefully.

It is VERY important that you pay attention to every mistake. You might notice your mistakes right away and say to yourself “that’s a little mistake, I *get* it”. But, dismissing your mistakes casually is dangerous to your learning. Every mistake is important for you to learn from.

Here, write down every problem you did incorrectly, analyze your mistakes, and make a strategy for how to avoid each of those mistakes.

I hope you learned:

- Once you have created an equation that describes a situation, you might make useful discoveries and statements about that same situation if you “do the same thing to both sides of the equal sign.”
- Subtraction “undoes” addition
- Finding a solution to an equation means finding the values of any variables in that equation that make the equation a true statement.
Chapter 4: Subtraction is Sometimes Removal, Sometimes Finding the Difference

In which you will learn:

- that subtraction represents:
  - the removal of one quantity from another
  OR
  - finding the difference between two quantities
- how to make life easier by knowing when to change some addition problems to subtraction problems that are easier to simplify.
- how to make life easier by knowing when to change some subtraction problems to addition problems that are easier to simplify.
- that some questions that seem completely unrelated may be answered with the help of the same mathematical expressions

Subtracting

How do you think of subtraction? Someone stealing your stuff? That’s how I remember it being introduced to me: Roxane has 5 cookies, and Sarah took two of them, how many does Roxane have left? ¹

I would bet that many readers are used to simplifying expressions like $9 - (-3)$, and you might automatically simplify that to 12.

This chapter is all about helping you learn to explain what, precisely, $9 - (-3)$ could mean, and why it simplifies to 14.

It’s frustrating for some people to have to go back and think about the reasoning behind rules they have accepted long ago. Once you “get it”, though, it’s so satisfying! You don’t have to mindlessly follow rules anymore! Making sense of procedures is in your power.

Ok… enough motivation:

An important way to look at subtraction is the difference between two numbers: Roxane has 5 cookies, Sarah has 2 cookies, what’s the difference between the number of cookies Roxane has and the number of cookies Sarah has?

Here, take a moment to write everything you remember about what subtracting means. Think of different examples of subtracting. If you can, try to remember how to deal

¹ Perhaps its stories like this that start children thinking about math negatively (meaning not happily)!
with subtracting when you have negative numbers, or when you have fractions, or large or small numbers. Just brain-dump:

**Subtraction and Addition: Connecting these Concepts**

Remember representing addition this way?

\[ x_2 = x_1 + \Delta x \]

This could be visualized these two different ways, depending if \( \Delta x \) is positive or negative. This first number line, below, shows the addition of a negative number:

![Number Line Addition Example](image)

Start at a point \( x_1 \) and add \( \Delta x \) to it to arrive at \( x_2 \).

So, for example, if \( \Delta x \) is negative, then you might have an equation that looks like this: \( 5 + -2 = 3 \). This might describe the situation where I have 5 dollars plus a debt of 2 dollars (like an IOU note). This would look like the number line above.

**Subtraction is Removal, or Taking Away.**

Now, ask yourself, what does \( 5 - 2 \) mean?

This expression might describe the situation where I had 5 dollars and Bobbie stole 2 dollars from me, I’d end up with 3 dollars. That’s the view of subtraction as “taking away”.

What would this look like on a number line? I have $5, and that $5 contains $2 and some other amount. If I remove the $2, what do I end up with?

Here’s a picture of it:

![Number Line Subtraction Example](image)

If you think about \( 5 - 2 \) as “removing 2 from 5”, you end up at \( x_1 = 3 \).
STOP and notice the direction of the arrow. You are removing a positive number, so the arrow that represents the number 2 is pointing to the right. Essentially, $5 - 2$ is asking you “you added 2 to some number and ended up with 5. What was that original number?”

Here is a summary of the two concepts “Addition” and “Subtraction as Removal”:

**Addition** is combining numbers; you can think of one of the numbers as a point on a number line $x_1$, and the other number as an arrow on a number line, $\Delta x$. The result of this combination is the point on a number line where the arrow ends: $x_2$.

**Subtraction as removal** is figuring out what $x_1$ is, when all you know is $x_2$ (where the arrow ends), and the length and direction of the arrow $\Delta x$.

1. a. Show on this number line “remove 3 from 10”, $10 - 3$. Remember: 10 is $x_2$. 3 is the arrow that ends at 10, it is $\Delta x$. You are trying to figure out what $x_1$ is. $x_1$ is where the arrow begins.

   ![Number Line Diagram 1](image)

   b. Simplify this expression: $10 - 3$

2. Show on a number line “Combine 10 and $-3$”, and simplify the expression: $10 + (-3)$

   ![Number Line Diagram 2](image)
3. Show on this number line “remove 8 from 12”, $12 - 8$. Remember: 12 is the result of addition. 8 is the arrow that ends at 12, it is $\Delta x$. You are trying to figure out what $x_1$ is. $x_1$ is where the arrow begins.

4. Show on a number line the combination of 12 and $-8$, in other words simplify $12 + -8$

5. Show on this number line “remove 9 from 15”, $15 - 9$. Remember: 15 is the result of some addition, $x_2$. 9 is the arrow that ends at 15, it is $\Delta x$. You are trying to figure out what $x_1$ is. $x_1$ is where the arrow begins.

6. Show on this number line the combination of 15 and $-9$, in other words, show $15 + -9$ on this number line.

Simplify $15 + -9$: 

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7. Back up!
   a. Write down the equations for question 1 and question 2, what do you notice?
   b. Write down the equations for question 3 and question 4, what do you notice?
   c. Write down the equations for question 5 and question 6, what do you notice?
   d. Do you think that if you need to simplify a subtraction problem it might sometimes be easier to transform it into an addition expression? And, sometimes, if you have an addition problem, it might be sometimes easier to transform it into a subtraction problem that you know has the same answer?

8. Show on this number line “remove 9 from 6”, \( 6 - 9 \). Remember: 6 is the result of some addition, \( 6 = x_2 \). 9 is the arrow, \( 9 = \Delta x \). You are trying to figure out what \( x_1 \) is. \( x_1 \) is where the arrow begins. This one is a bit trickier than the previous exercises.

\[
\begin{array}{c|c|c}
   x_1 & \Delta x = 9 & x_2 = 6 \\
\end{array}
\]

9. Wait.. what? That previous question might seem kinda of weird. How can you remove a 9 from 6 if 9 is bigger than 6? Hmm… Suppose you have $6 because you combined $9 and a $3 debt; if you take away $9, you’ll end up with a $3 debt. Are you thinking “this approach is making something that should be simple (subtraction) more complicated that it needs to be!”? Memorizing rules might seem simple, but if you don’t make sense of those rules, they can’t help you solve problems outside of a narrow context of math class.
   a. Why do you think I might be asking you to think deeply about “taking away” rather than giving you a set of rules on how to subtract numbers?
   b. If your goal is to quickly answer many subtraction problems, then this approach is probably more time consuming than what you have been taught. But, if your goals include understanding how to use subtraction to make sense of numbers, and answering questions
involving “removing one thing from another thing”, this approach is what you need. In your life, no one will ever, outside of a math class, ask you to answer many subtraction problems. Being able to make sense of the idea of “removing something” is useful, do you agree?

10. Show on this number line: combine 6 and $-9$. The expression is $6 + -9$

![Number line diagram](image)
11. What do you end up with if you remove 10 from -6. Use a number line to show subtraction as “removing the ∆x”.

Hint: the expression that means “remove 10 from −6” is $-6 - 10$

12. What do you get if you combine 10 and $-6$? Show on the number line and write the expression that means “combine 10 with $-6$”:

13. What do you get if you remove 5 from $-5$? To make sense of this, imagine that you have a total of a debt of $5. You got there by having $5 and some other amount. What was that other amount that you had to combine with $5 to end up with a debt of $5$?

14. What do you get if you combine $-5$ and $-5$?
15. So far this chapter, I’ve been leading you down a garden path, and it’s time for you to STOP and look around and figure out where I’ve led you.

a. Write the expression that means “remove 3 from 10”, and then simplify that expression:
   Write the expression that means “add 10 and −3”, and then simplify

b. Write the expression that means “remove 8 from 12”, and then simplify that expression
   Write the expression that means “add 12 and −8”, and then simplify

c. Write the expression that means “remove 9 from 15”, and then simplify that expression
   Write the expression that means “add 15 and −9”, and then simplify

d. Write the expression that means “remove 9 from 6”, and then simplify that expression
   Write the expression that means “add 6 and −9”, and then simplify

e. Write the expression that means “remove 10 from −6” and then simplify that expression
   Write the expression that means “add −6 and −10”, and then simplify

f. Write the expression that means “remove 5 from −5”, and then simplify that expression
   Write the expression that means “add −5 and −5”, and then simplify

16. Do you think that, in general, “removing A from B”
   \[ A - B \]
   is mathematically equivalent to “adding A to the opposite of B”
   \[ A + (-B) \]
A very powerful idea in mathematics is that you can find expressions that are mathematically equivalent to each other. So, if you are trying to figure something out, and it is hard to wrap your brain around it, you can find an equivalent expression that is easier to wrap your brain around.

Isn’t that cool?
Subtraction as “finding the difference”; Subtraction as finding $\Delta x$

Subtraction can mean “removal”. It can also mean “finding the difference”. The word “difference” here is just about the same as how I might use it speaking to someone in a conversation. What’s the difference between $10$ and $7$? What would I have to add to $7$ to get to $10$? Those questions can be answered with subtraction.

“What is the difference between 7 and 4?” This is written mathematically: $7-4=3$

It’s strange, and you maybe have never thought of subtraction this way, but $7-4$ means two different things:

- $(7-4)$ means “Remove 4 from 7”

AND

- $(7-4)$ means “what do you need to add to 4 to get to 7?” or, “what is the difference between 7 and 4?”

Here are some examples to help you visualize the “finding the difference” meaning of subtraction:

10-7= 3 means “start at 7 and count up to 10”.

“What is the difference between 10 and 7?”

In this case, $7=x_1$, $10=x_2$, and $3=\Delta x$.

If you are asking “what is the difference between two numbers”, visually, your answer is the arrow, $\Delta x$.

If you are asking, as in the previous section, “what do you get if you remove something”, visually, you are removing the arrow and your answer is $x_1$. 
Here are more examples of “subtraction as finding the difference”

“What is the difference between 5 and 3?”  5-3
“start at 3 and count up to 5”
In this case, 3=\(x_1\), 5=\(x_2\), and 2=\(\Delta x\).

```
x_1=3   \Delta x= 2   x_2=5
```

“What is the difference between -4 and 3?” -4-3
means “start at 3 and count down to -4”
In this case, 3=\(x_1\), -4=\(x_2\), and -7=\(\Delta x\).

```
x_2=-4   \Delta x = -7   x_1=3
```

“What is the difference between -4 and -4?” -4-(-4)  “start at -4 and count to -4”
In this case, -4=\(x_1\), -4=\(x_2\), and 0=\(\Delta x\). The arrow has no length.

```
x_1=-4   x_2=-4   \Delta x = 0
```
STOP! Do you absolutely understand the previous 4 examples? Do you “get” why “the difference between -4 and 2” is -6? Are you sure?

Write any notes or thoughts you might have here, especially if you’ve never considered why certain “rules” for subtracting negative numbers are true.

**Wait… what? Subtracting negative numbers?**

What is 9 - -4? That means “start at -4, and count up to 9”, right? Go ahead and draw that right here:

![Number Line Diagram](image)

The difference between 9 and -4 is POSITIVE 13. Get it?

**In the next section, you’ll explore a different way of figuring out how to subtract negative numbers, but, right here, right now, you should be able to figure out how to simplify subtraction with a negative number by thinking of a number line and thinking of the difference between two numbers.**

Just to make sure you got the point: subtraction can describes two different “actions”: either removing a quantity from another, or finding the difference between two quantities.

Just to keep you awake, I’ll drop the \( x_1, x_2, \) and \( \Delta x \) notation, and use other letters:

**Remove B from A:**
\[
A - B
\]

**Take away B from A:**
\[
A - B
\]

**What’s the difference between A and B?**
\[
A - B
\]

**What do you add to B to get to A?**
\[
A - B
\]

**How do you get from B to A?**
A-B
Adding and Subtracting: creating equivalent expressions

Now you, I hope, understand that A-B means “removing B from A”, or finding “how far you have to go to get from B to A”. Be careful! A lot of people want to draw the arrow going from A to B. But, A-B, if it means “the difference between A and B”, is an arrow going from B to A!

But, wait, there’s more! It turns out that A-B will also be the same as ADDING A to the opposite of B. **Think about these three examples:**

- If you have $20 in your bank, and you also have a debt of $7, your total combination would be the $20 + (-$7) = $13.

- On the other hand, if you have $20 in the bank, and then you took out $7 from the bank, you’d also end up with $13. $20 - $7=$13

- And, if you have $20, your brother has $7, the difference between what you have and what your brother has is $20-$7=$13

This ability to create equivalent expressions involving either subtraction or addition is a big deal! You will continue to learn more about it in future chapters.

Example:

Suppose Susie as $10, and Billybob has $15.

“How much money do I have to give Susie (she has $10) so she has the same amount as BillyBob (he has $15)?”

$15-$10 =$5

“What do I get if I add BillyBob’s money to the opposite of what Susie has?”

$15 + -$10 = $5

“What does Billybob end up with if you remove from him the amount of money that Susie has?”

$15-$10 =$5

Now, I’ll make it more abstract:

Suppose Susie has S, and Billybob has B. S and B stand for the amount of money they both have. Although we don’t know how much money they have, we can still write mathematical expressions using these symbols.
The expression that means: “the amount of money I have to give Susie so she has the same amount as BillyBob” is

\[ B - S \]

The expression that means “what I have if I add BillyBob’s money to the opposite of what Susie has” is

\[ B + (-S) \]

The expression that means “What you get if you take away from Billybob the amount of money that Susie has?” is

\[ B - S \]

Are these three expressions always equal?

YES!

Exercises:

17. Suppose Susie has -7, and Billybob has -10.

Write out the expressions that represent the following:

a. “the amount of money I have to give Susie so she has the same amount as BillyBob”.

b. “what I have if I add BillyBob’s money to the opposite of what Susie has”

c. “What you get if you take away from Billybob the amount of money that Susie has?”

The following exercises are “fill-in-the-blank”. The first one is done for you:

18. Fill in the blank: 5 + -4 is the same as 5 - ____ , because both expressions are equal to ______.

19. Fill in the blank: 10 + -8 is the same as 10 - ____ , because both expressions are equal to ______.

20. Fill in the blank: -5 + -4 is the same as -5 - ____ , because both expressions are equal to ______.

21. Fill in the blank: -10 + -8 is the same as -10 - ____ , because both expressions are equal to ______.

22. Fill in the blank: -10 + -10 is the same as -10 - ____ , because both expressions are equal to ______.

23. Fill in the blank: -2 + -8 is the same as -2 - ____ , because both expressions are equal to ______.
24. Fill in the blank: 2 + -5 is the same as 2 - _____ , because both expressions are equal to ________

25. Fill in the blank: -1 + -4 is the same as -1 - _____ , because both expressions are equal to ________

26. Fill in the blank: -10 + -20 is the same as -10 - _____ , because both expressions are equal to ________

27. Fill in the blank: -10 + -1 is the same as -10 - _____ , because both expressions are equal to ________

28. Fill in the blank: 10 + -10 is the same as 10 - _____ , because both expressions are equal to ________

29. Fill in the blank: -10 + -12 is the same as -10 - _____ , because both expressions are equal to ________

30. Fill in the blank: If A + -B =C, then it is true that A - _____ = C

STOP! Do you agree with the statement:

“To help solve problems, I could decide to change an addition problem into a subtraction problem, and I could decide to change a subtraction problem into an addition problem.”

If, “No, I don’t agree”, please go back and read carefully the previous section. If, “Yes, I agree with the statement”, please continue reading

The good news is that if a subtraction problem is hard to think about, you can transform it into an addition problem (and vice versa).

Making connections with addition:

Can you visualize all of these scenarios on a number line?

- Start at 5 and add a 3: 5+3=8
- Remove 3 from 8 to get 5: 8−3=5
- Start at 8 and add a -3 to get 5: 8+-3=5 (-3 is a left-pointing arrow)
- Find the difference between 8 and 3: 8-3=5 (this means start at 3 and go to 8, the difference is how far and in which direction you went)

Making the connection between addition and two different concepts or visualizations of subtraction might be new to you, and it might take some thought for these ideas to sink in.
A concrete example like $5+(-7)=-2$, meaning “combine a 5 and a negative 7” could be visualized like this. I put the -7 in parentheses just to make it easier to notice the negative sign. It might be hard to notice if I simply wrote $5+(-7)=-2$. You can always put parentheses around numbers, it doesn’t change their meaning.

And “the difference between 5 and 7”, meaning $5-7$ and could be visualized like this.

STOP:

If you want to “find the difference between A and B”, you could write $A-B$.

If you wanted to draw this on a number line, would you start at A and go to B, or start at B and go to A?

What do you think? Draw on a number line what $A-B$ would look like, if you are trying to show “finding the difference between A and B”

**If you wrote that the visualization of “the difference between A and B, meaning $A-B$, could be an arrow going from B to A, you are correct. If you wrote something else, please fix your misconception!**

31. Show on a number line “the difference between 3 and 5” and then simplify the expression $3-5$. Pay close attention to the sign of your answer. In this case, the sign is negative, and your arrow should be left-pointing. The “sign” of a number indicates whether it is positive or negative.

32. Draw a number line, and show “the difference between 3 and 2” and then simplify the expression $(3-2)$. Pay close attention to the sign of your answer.

33. Draw a number line, and show on it “the difference between 5 and 2” and then simplify the expression $(5-2)$. Pay close attention to the sign of your answer.
34. Draw a number line, and show on it “the difference between 2 and 3” and then simplify the expression (2-3). Pay close attention to the sign of your answer.

35. Draw a number line, and show on it “the difference between 5 and 9” and then simplify the expression (5-9). Pay close attention to the sign of your answer.

36. Draw a number line, and show “the difference between 9 and 5” and then simplify the expression (9-5). Pay close attention to the sign of your answer.

37. Draw a number line, and show “the difference between -9 and 5” and then simplify the expression (-9-5). Pay close attention to the sign of your answer.

Making Sense of Subtracting Negative Numbers

In the previous exercises, you showed how you can think about subtraction as “finding the difference between two numbers”, and, you showed how you can re-write any subtraction expression as an addition expression. You’ll put those two concepts together to make sense of subtracting negative numbers. You already have done a little of this earlier in this chapter.

For example, if you wanted to know “the difference between 5 and -2”, you could draw a number line, and draw an arrow, starting at -2 and ending at 5. What number does that arrow represent?

38. Draw a number line, and show “the difference between 5 and -2” and then simplify the expression (5 - -2). Remember that when you are showing the difference between A and B on a number line, (A-B), you need to start at B and go to A. Pay close attention to the sign of your answer. So, in this case, you’d start at -2 and go to 5.

39. Draw a number line, and show “the difference between -2 and 5” and then simplify the expression (-2 - 5). Remember that when you are showing the difference between A and B on a number line, (A-B), you need to start at B and go to A. Pay close attention to the sign of your answer.

40. Draw a number line, and show “the difference between 9 and -5” and then simplify the expression (9 - -5). Remember that when you are showing the difference between A and B on a number line, (A-B), you need to start at B and go to A. Pay close attention to the sign of your answer.

41. Draw a number line, and show “the difference between 2 and -3” and then simplify the expression (2 - (-3)). Remember that when you are showing the difference between A and B on a number line, (A-B), you need to start at B and go to A. Pay close attention to the sign of your answer.

42. Draw a number line, and show “the difference between -2 and -3” and then simplify the expression (-2 - (-3)). Remember that when you are showing the difference between A and B on a number line, (A-B), you need to start at B and go to A. Pay close attention to the sign of your answer. In this case, you are starting at -3 and going to -2.
43. Go back and consider how you simplified the following expressions in the previous exercises, when you were thinking about and drawing “finding the difference”

\[(5 - 2) \quad (9 - 5) \quad (2 - 3) \quad (-2 - 3)\]

And compare your results with how you would simplify the following expressions:

\[(5 + 2) \quad (9 + 5) \quad (2 + 3) \quad (-2 + 3)\]

Do you think, in general, that the difference between \(A\) and \(-B\), \((A - -B)\) could be the same number as \((A+B)\), always?

44. For the following, fill in the blank: \(A-B\) is equivalent to \(A+ \_\_\_\_\_\_\_\_\_\_\_\)

45. \(5-2\) is equivalent to \(5+\_\_\_\_\_\_\_\)

46. \(5- (9)\) is equivalent to \(5+\_\_\_\_\_\_\_\_\_\_\)

47. \(5-(-9)\) is equivalent to \(5+\_\_\_\_\_\_\_\_\_\_\)

48. \(-5-(9)\) is equivalent to \(-5+\_\_\_\_\_\_\_\_\_\_\)

49. \(-3 - (-6)\) is equivalent to \(-3+\_\_\_\_\_\_\_\_\_\_\)

50. \(3 - (-3)\) is equivalent to \(3+\_\_\_\_\_\_\_\_\_\_\)

51. \(3 - (3)\) is equivalent to \(3+\_\_\_\_\_\_\_\_\_\_\)

“Subtracting a number is the same as adding its opposite; it doesn’t matter whether the number you are subtracting is positive or negative” (the opposite of a negative is a positive, the opposite of a positive is a negative).

In general, it is always true that adding a number is mathematically equivalent to subtracting its opposite, and subtracting a number is equivalent as adding its opposite (even though the visualizations might be different).

- \(A-B=A+(B)\) subtracting \(B\) is the same as adding the opposite of \(B\)
- \(A+B=A-(B)\) adding \(B\) is the same as subtracting the opposite of \(B\)

The answer to all of the following questions is mathematically equivalent

- \(What is the difference between \(A\) and \(B\)? \((A-B)\)
- \(What is \(A\) added to the opposite of \(B\)? \((A+\_B)\)
- \(What do you get if you remove \(B\) from \(A\)? \((A-B)\)
- \(If you start at \(B\) and go to \(A\), how far have you gone? \((A-B)\)\)
I cannot stress enough how important it is for you to STOP now and wrap your brain around those 4 questions. It would be great if you got to the point where you are 100% sure you can explain them without hesitation. Quiz yourself. Explain them to someone willing to ask questions and demand that you make sense.

Example:

What is the difference between -10 and -3? To visualize this, draw a number line, an arrow starts at -3 and ends at -10. The arrow represents the difference between -10 and -3.

Mathematically, this is written: \(-10 - (-3) = -7\)

Or, you could decide to change the question altogether, merely because subtraction gives you a headache, and you like to change every subtraction problem into an addition problem. You know the answer to the question “What is the difference between -10 and -3?” will be the same as “What is -10 + 3” because subtracting a number will give you the same result as adding its opposite.

\[-10 - (-3) = -10 + 3\]

If a particular problem you are working on involves addition, and you think it might be easier to solve it with subtraction, you can change the question you are answering to a different question that has the same answer. Or, if you are trying to figure out what the result will be when you remove something, you know that you will get the same result if you add the opposite of what you are removing.

Example, what is 5 + -3? Adding a number is the same as subtracting its opposite, so 5 + (-3) is equal to 5 - 3 = 2.

This is a concrete example showing how three different situations are all equal, mathematically

- 5 dollars combined with a 3 dollar debt gives a total of $2.
- If I have $5 and Sue takes away $3, I end up with $2.
If I have $5 and Larry has $3, the difference is $2. In other words, give Larry $2, and he’ll have what I have.

Seeing how one question might have the same answer as another question is at the heart of applied mathematics; this allows us to answer questions that seem difficult by changing the question to one which might be easier to answer and that we know has the same answer. You get to be creative in how you approach and visualize problems.

Read the previous paragraph again.

Now, take a moment to go back and review what you wrote about subtraction at the beginning of this chapter. Do you want to add anything? Most importantly, do you want to correct anything you wrote that you now do not agree with? If that is true, then you should be really excited: if you changed your mind about something, then you are really learning and getting your money’s worth out of this book! Congratulations! If you have nothing to change, then I’m sorry; I hope that you will soon get to a part in the book that changes your thinking.

Exercises:

For each of the following phrases, write a mathematical expression using addition, and, write another, equivalent expression, using subtraction. Simplify each expression (the first one is done for you)

52. Combine -5 with -3
   a. \(-5 + -3 = -8\) or \(-3 + -5 = -8\) are the possible addition problems you could write.
   b. \(-5 - 3 = -8\) or \(-3 - 5 = -8\) are the possible subtraction problems you could write.

53. Combine -3 with 4. (Remember for this and the rest of the exercises, your answers should contain two equivalent expressions, one using subtraction, and one using addition.)
   a. Addition:
   b. Equivalent subtraction:

54. Combine 2 and 5; (hint: adding a number is the same as subtracting its opposite: \(A + B = A - (-B)\) So adding 5 is the same as subtracting negative 5)
   a. Addition
   b. Equivalent subtraction

55. Find the difference between 5 and -3
   a. Subtraction
   b. Equivalent addition
56. Find the difference between -4 and -1
   a. Subtraction
   b. Equivalent addition

57. Find the difference between 4 and 1
   a. Subtraction
   b. Equivalent addition

58. Combine 4 and 1
   a. Addition
   b. Equivalent subtraction

59. Remove 5 from 9
   a. Subtraction
   b. Equivalent addition

60. Take away 3 from 10
   a. Subtraction
   b. Equivalent addition

61. Find the difference between -3 and 0
   a. Subtraction
   b. Equivalent addition

62. Find the difference between 0 and -3
   a. Subtraction
   b. Equivalent addition

63. Find the difference between -3 and 2
   a. Subtraction
   b. Equivalent addition

64. Combine -5 and 5
   a. Addition
   b. Equivalent subtraction
65. Remove -5 from 3
   a. Subtraction
   b. Equivalent addition

66. Remove 3 from -5
   a. Subtraction
   b. Equivalent addition

67. The difference between -5 and 3
   a. Subtraction
   b. Equivalent addition

68. The difference between -13 and -5
   a. Subtraction
   b. Equivalent addition

69. Combine -3 and -2
   a. Addition
   b. Equivalent subtraction

70. Using the symbol for subtraction, write the mathematical expression that means “Remove A from negative B”. Do not simplify.
   Write an expression that is mathematically equivalent to what you wrote above, but using the symbol for addition.

71. a) Using the symbol for subtraction, write the mathematical expression that means “the difference between X and negative Y”
   b) Write an expression mathematically equivalent to what you wrote for above, but using the symbol for addition.
I hope you learned:

• that subtraction represents:
  
  the removal of one quantity from another
  OR
  finding the difference between two quantities

• how to make life easier by knowing when to change some addition problems to subtraction problems that are easier to simplify.

• how to make life easier by knowing when to change some subtraction problems to addition problems that are easier to simplify.

• that some questions that seem completely unrelated may be answered with the help of the same mathematical expressions

As you check your solutions, do not ignore mistakes ... you probably made mistakes because of some mis-remembered “rules” from a math class you had previously, or because you worked too fast and didn’t check yourself. Whatever the reason for your mistakes, write down, after thinking over your mistakes carefully, how you think differently about addition and subtraction.

• What kinds of mistakes did you make?

• How will you change your thinking so you don’t get those mistakes?

• What is the Big Idea (or Big Ideas?) of all of these questions?
Chapter 5: Problem Solving With Addition and Subtraction

In which you will practice, practice, practice

Problem solving with addition and subtraction

In the previous chapters, you learned and thought about addition and subtraction and what those mathematical operations mean. This chapter is devoted to problem-solving using those operations with positive and negative numbers. For now, we won’t introduce decimals or fractions, but the ideas you will play with while answering these questions are the same when used with decimals or fractions.

It is REALLY REALLY important that you do NOT do the following in your head. The numbers are fairly easy to think about, and if you arrive at the answer without writing the corresponding math expression or equation down, you will not have the opportunity to “grow your brain” and make connections among the ideas, skills, and concepts you already think you know with the ideas, skills, and concepts you are still learning. Write down the expressions, and ask yourself “what is the point of this question? Why is it asked here, is it connected to the previous question? What can I learn from this question?”

1.
   a. Write the following scenario using addition (don’t solve if you don’t want to): I have a debt of $200. Pirjo has $500 of cold, hard, cash. If we combine our money, what do we have together?
   b. Re-write an equivalent mathematical expression using subtraction in a way that is easy for you to solve. Solve it.

2.
   a. Write the following scenario using subtraction (don’t solve if you don’t want to): I have a debt of $500. You have a debt of $700. What is the difference between our debts?
b. Re-write an equivalent mathematical expression using addition in a way that is easy for you to solve. Solve it.

c. Re-write an equivalent mathematical expression using subtraction (different from the first one you wrote) in a way that is easy for you to solve. Solve it.

3.

a. Write the following scenario using subtraction (don’t solve if you don’t want to): Tanoko had a total debt of $500, owed to several different people. Josimar decided to forgive the $200 Tanoko owed him; in other words, Josimar removed a debt of $200 from Tanoko’s total. How much money does Tanoko have now (or how much debt?)

b. Re-write an equivalent mathematical expression using addition in a way that is easy for you to solve. Solve it.

c. Re-write an equivalent mathematical expression using subtraction (different from the first one you wrote) in a way that is easy for you to solve. Solve it.

4. I have an unknown number of chocolate candies. You have an unknown number of chocolate candies. Together, write an expression that means the number of candies I have combined with the number of candies you have. Use single-letter variables.

5. I had an unknown amount of mashed potatoes on my plate. When I wasn’t looking, you stole some. Write an expression that means how much mashed potatoes I ended up with. Use single-letter variables.

6. I used to have X apples, and now I have Y apples. What’s the difference between what I have now and what I used to have?

7. There were X apples in a bag, and you then changed the amount in the bag by $\Delta x$. Explain what must be true about $\Delta x$ if the amount of apples in the bag ended up being less than X.

8. There were 7 apples in the bag, and then you changed the amount of apples in the bag by -4. How many ended up being in the bag?

9. What is the difference between 7 and -4?

10. I have 9 dollars and you have a debt of 4 dollars, what is the difference between what I have and what you have?

11. I have a debt of $5. You have a debt of $5. What is the difference between the amount of money we have?

12. I have a debt of $50. You have a debt of $80. What is the difference between what I have and what you have?

13. You are 80 dollars in debt. Someone gives you some money and you end up only $50 in debt. How much money did they give you? (make sure you write this out...)
14.

a. It’s -60 F in Fairbanks Alaska, and 40F in Vancouver Canada. What is the difference between the two cities’ temperatures?

b. It was -60 F in Fairbanks, then it got 40 F colder. What was the final temperature in Fairbanks?

c. What will my total debt be if I combine a debt of $60 with a debt of $40?

d. How is it that the answers to the previous three questions are the same? Can mathematical expressions express different ideas and still be equivalent?

15. In the “number line picture” below, identify $\Delta x$, $x_1$, and $x_2$, and then write TWO equations that describe the picture shown in the number line. Make sure the numbers in your equations are the same as the numbers shown on the number line. For example, if $\Delta x$ is a negative number, make sure your equations have $\Delta x$ being a negative number. In this example, $\Delta x$ is a negative number.

![Number Line Picture]

Equation 1, using this model for addition: $x_1 + \Delta x = x_2$

Equation 2, using this model for subtraction: $\Delta x = x_2 - x_1$

16. In the “number line picture” below, identify $\Delta x$, $x_1$, and $x_2$, and then write TWO equations that describe the picture shown in the number line. Make sure the numbers in your equations are the same as the numbers shown on the number line. For example, if $\Delta x$ is a negative number, make sure your equations have $\Delta x$ being a negative number. In this example, $\Delta x$ is a negative number.

![Number Line Picture]

Equation 1, using this model for addition: $x_1 + \Delta x = x_2$

Equation 2, using this model for subtraction: $\Delta x = x_2 - x_1$

17. The “pictures” representing the two expressions $(4 + -10)$ and $(4 - 10)$ look very different, and yet these expressions are mathematically equivalent. Please
explain this, as if you were explaining to a friend who is new to this way of thinking:

18. In the “number line picture” below, identify \( \Delta x \), \( x_1 \), and \( x_2 \), and then write TWO equations that describe the picture shown in the number line. Make sure the numbers in your equations are the same as the numbers shown on the number line.

![Number Line Diagram](image)

Equation 1, using this model for addition: \( x_1 + \Delta x = x_2 \)
Equation 2, using this model for subtraction: \( \Delta x = x_2 - x_1 \)

19. In the “number line picture” below, identify \( \Delta x \), \( x_1 \), and \( x_2 \), and then write TWO equations that describe the picture shown in the number line. Make sure the numbers in your equations are the same as the numbers shown on the number line. For example, if \( \Delta x \) is a negative number, make sure your equations have \( \Delta x \) being a negative number. In this example, \( \Delta x \) is a negative number.

![Number Line Diagram](image)

Equation 1, using this model for addition: \( x_1 + \Delta x = x_2 \)
Equation 2, using this model for subtraction: \( \Delta x = x_2 - x_1 \)

20. In the “number line picture” below, identify \( \Delta x \), \( x_1 \), and \( x_2 \), and then write TWO equations that describe the picture shown in the number line. Make sure the numbers in your equations are the same as the numbers shown on the number line. For example, if \( \Delta x \) is a negative number, make sure your equations have \( \Delta x \) being a negative number.

![Number Line Diagram](image)

Equation 1, using this model for addition: \( x_1 + \Delta x = x_2 \)
Equation 2, using this model for subtraction: \( \Delta x = x_2 - x_1 \)
21. In the “number line picture” below, identify $\Delta x$, $x_1$, and $x_2$, and then write TWO equations that describe the picture shown in the number line. Make sure the numbers in your equations are the same as the numbers shown on the number line. For example, if $\Delta x$ is a negative number, make sure your equations have $\Delta x$ being a negative number. In this example, $\Delta x$ is a positive number.

Equation 1, using this model for addition: $x_1 + \Delta x = x_2$

Equation 2, using this model for subtraction: $\Delta x = x_2 - x_1$

What did you learn from doing the exercises in this chapter? Especially, what did you learn from your mistakes and struggles?
Chapter 6: What is equal sign?
Equivalent Numbers

In which you will learn:

- The equal sign does not simply mean “the answer”
- What is on the left side of the equal sign is always mathematically equivalent to what is on the right side of the equal sign.

What is an equal sign? part 1: Equivalent numbers

Here’s a review of previous chapters:

- You can combine old toothbrushes and apples, but you can’t simplify the result. 5 old toothbrushes plus 7 apples cannot be simplified. 5 old toothbrushes plus 7 old toothbrushes can be simplified to 12 old toothbrushes.
- Adding two numbers can be written like this: \( x_1 + \Delta x = x_2 \)
  - \( x_1 \) can be visualized as a point on a number line, \( \Delta x \) can be visualized as an arrow on a number line that begins at \( x_1 \), and \( x_2 \) can be visualized as the point on a number line where the arrow ends.
  - Any of the numbers, \( x_1 \), \( \Delta x \) and/or \( x_2 \) might be positive or negative, and visualizing the numbers on a number line will help you simplify addition and become fluent in adding with negative numbers
  - Depending on the context, \( x_1 \) and \( \Delta x \) can be switched, but \( x_2 \) will be the same regardless”
- Subtraction expresses both of the following ideas:
  - the difference between two numbers: \( x_2 - x_1 = \Delta x \)
  - what you get if you remove one number from another.
• Subtraction $A - B$ is mathematically equivalent to $A + (-B)$, meaning that

$A - B$ always equals $A + (-B)$

In the previous chapters, I may have given you the impression that the equal sign means “the answer is” because I’ve introduced addition and subtraction as useful problem-solving tools when you are answering interesting questions like “how much more money does my brother make than I do?” and “If the government and banks forgave my student debt, what would I end up with?”

It’s time to dive more deeply into what the equal sign means.

**The Mighty Equal Sign**

Some people view the “=” to mean “the answer is”. Also, some people look at the two equations:

$$x_1 + \Delta x = x_2$$

and

$$x_2 = x_1 + \Delta x$$

And think they are looking at two different equations.

Write down everything you believe about the meaning of an equal sign, including its usefulness, its meaning, etc… Just brain-dump everything that you remember about those two little parallel lines.

Do you think the next 4 equations are the same or different?

$$3 + 4 = 7 \quad 7 = 3 + 4 \quad 4 + 3 = 7 \quad 7 = 4 + 3$$

What about these equations:

$$x = 7 \quad 7 = x$$

Remember that I answer the question “what the heck is a number?” by saying that a number is anything that could be conceptualized in 1 of 3 ways: point on number line, an arrow with a specific length and direction, or a number of repetitions or piece of something.

How you visualize or conceptualize numbers **does not affect if they are considered equal or not**, so, “the point on a number line representing 3” is equivalent to “an arrow pointing to the right with a length of 3” which also equals “the concept of 3 repetitions”.

Even though the concepts and visualizations might be different, we can represent them mathematically with the symbol 3, and we call them, mathematically, equal.

**Equivalent Expressions with addition**

Recall from the previous chapter than $5 + 1$ could visualized like this
And 10+(-4) could be visualized like this

![Number line visualization](image)

I don’t know about you, but these two visualizations look different to me. However, they both show the following:

- 5+1 is more simply described as 6
- 10+(-4) is more simply described as 6

Therefore, 5+1 is said to be equal to 10+(-4), even though the visualization on a number line of 5+1 and 10+(-4) don’t look the same, if they can be simplified to the same number, they are equal.

\[ 5+1 = 10 + (-4) \]

In general, if mathematical expressions can be simplified to the same number, they are equal. This is true for expressions with addition, subtraction or division, or any other math.

It’s important to note that the symbol = does NOT only mean “and the answer is”.

“The answer” depends on the question.

The “=” symbol means “can be simplified to the same number”, no matter how you are visualizing it. So, 5+2=12+(-5) just means that 5+2 can be simplified to the same number as 12+(-5), namely, 7.

Questions:

1. Write down 3 ways to visualize or conceptualize the number 5, using equations or number lines.
2. Write down 3 ways to visualize or conceptualize the number -2, using equations or number lines.
3. For the following, write down 3 addition problems that simplify to the given number, using a negative number in at least one of your examples.

   For example, if the given number is 2, you could write 2=1+1, 2=-3+5, and 2=5+(-3).

   Draw number lines if it is helpful.

---

1 A student once explained to me that adding a negative 4 to a positive 10 feels psychologically different than adding 1 to 5, but the expressions are mathematically equivalent.

2 If you are learning computer programming, the = symbol has different meanings depending on the computer code you are learning. I am limiting this discussion to what the = symbol means in math classes like algebra, college algebra, trigonometry, and calculus.
a. 5 (write down 3 addition problems that are equal to 5, using a negative number in at least one of your examples).

b. -6
c. -1
d. 1
e. 0

So, you see, the concept of equivalency applies to numbers, no matter how you visualize them, and it also applies to expressions: if two expressions simplify to the same number, then the two expressions are equal.

Now is the time to go back to what you originally wrote about the meaning of the = symbol and make any changes based on thinking you did while you were reading... what thoughts or concepts occurred to you as you read? How did you change your mind about what the symbol = means? Do you need to make corrections to your original ideas?

I hope you learned:

- The equal sign does not simply mean “the answer”
- What is on the left side of the equal sign is always mathematically equivalent to what is on the right side of the equal sign.
Chapter 7: Multiplication

In which you will learn:

- Multiplication means repeating a quantity
  OR
  finding a fraction or piece of a quantity
- Multiplication can mean repeated addition
- Area means how many unit squares can fit on a surface.
- Volume means how many unit cubes can be contained in a space.
- It never matters what order you multiply numbers
- The rules for multiplying negative numbers make sense given the meaning of multiplication

Multiplication

Basic meaning of multiplication

What does multiplication mean to you? Is it memorized rhymes? (I ate and I ate and I’m sick on the floor is my favorite (8 times 8 is 64)). Does it bring up stress because your 4th grade teacher made you stand up in front of the room and recite facts? Did your math learning involve flash cards and public pressure? I need to tell you that it is possible, and quite common, for people to be “math people” (meaning people who enjoy aspects of mathematics) without being able to recite math multiplication facts quickly. I personally have terrible memories of “timed tests” and I promise you, completing those timed tests or not has nothing to do with your mathematical ability. It can be a fun skill, and if you are fast at “math facts”, then good on you, as they say in Australia. But if you have to deliberate for a while if you are asked what 7 time 9 is, that’s ok; you usually have time, unless you work in a fast-paced fish market with customers waiting and no calculator.

What does multiplication mean to you? What memories do you have about learning what multiplication means? What about negative numbers and multiplication? Multiplying large numbers by hand and the process to do that? (578 times 269 for example).

A number can be usefully described as a point on a number line, an arrow with direction and length on a number line, or a “repetition” or “part of” another number. So far, you’ve mostly played with addition and subtraction with the help of visualizing points
on number lines and arrows on numbers lines. Now, I’m finally getting to the third useful way of thinking of numbers: “how many repeats” of some other number, or “some part of, or piece of” some other number. If you are dealing with the type of number that is a “repetition or multiple” or a “piece” then you are probably multiplying or dividing another number.

For example, if I said I had 6 negative three’s, I could draw 6 arrows pointing left, each with length “three”. In this case the number 6 refers to the number of repetitions of arrows. The total length from the beginning of the first arrow to the end of the last arrow is -18. -18 is what you get if you have 6 repetitions of -3.

If I said I had 4 x’s, and x is a negative number, I could draw 4 left-pointing arrows each with length x.

If you repeat a number, then you are multiplying that number. This is why multiplication is called, sometimes, “times”. Remember learning your “times tables”? (sorry if that brought up bad memories!) 7 times 8 means “8 repeated 7 times”. 3x means “x repeated 3 times”. In this perspective on multiplication, one number refers to a “thing” and the other number refers to “the number of that thing you have.” In the phrase “8 repeated 7 times”, “8” is the “thing” and “7” describes how many of that thing you have. “7” is like an adjective, and “8” is like a noun.

7x means we have 7 x’s. X is the “thing” (a noun), and 7 is how many of that thing we have (an adjective).

Take the number 4, and repeat it 3 times.

Next, take the number 3, and repeat it 4 times:
Ok, folks, 4 repeated 3 times, and 3 repeated 4 times will give you the same number. When I realized this when I learned multiplication, it really blew my mind. $3+3+3+3$ is the same as $4+4+4$. Isn’t that crazy?

Whether you think it’s amazing or not, it is true. Maybe you are not as easily amused as I am, but it is a fact that it never matters what order you multiply numbers: 4 times 3 is 3 times 4. By this, I mean, that if you repeat the number $Y$ an $X$ number of times, or if you repeat the number $X$ a $Y$ number of times, you’ll get the same result.

Suppose Jim has 300 dollars. His great aunt Matilda tells him that she will double his money. Jim ends up with $300 \times 2$, or $600$. The “thing” here is $300$, and Jim will get 2 of that thing and have $600$.

Suppose Jane has $2$. Her great uncle Attilio tells her he’ll multiply Jane’s money 300 times! Jane ends up with $2 \times 300$, or $600$. The “thing” is $2$, and Jane will get 300 of those things to end up with $600$, same as Jim in the previous paragraph.

These two scenarios might seem different to you (they seem different to me, too), but the end result is the same. 4 times 3 is always the same as 3 times 4. $X \times Y$ is always the same as $Y \times X$.\footnote{If $X$ and $Y$ are numbers. If they are matrices this is not the case, and you will learn about matrix multiplication in pre-calculus or perhaps linear algebra. Super useful stuff.}

$4 \times x$’s is the same as 4 multiplied to $x$, and we write that as $4x$.

4 repeated $x$ number of times is $x$ multiplied to 4, and we write that also as $4x$.

**How to show you are multiplying**

If you went to elementary school in North America, you probably learned that 4 times 5 is written like $4 \times 5$. In public signs or advertisements, The $X$ is sometimes used to show multiplication, as in “$50$ for $4 \times 2$hour lessons” ($50$ for 4, 2-hour lessons).
Unfortunately, X, X, x, or x is also mathematicians’ favorite symbol for a variable. If you want to show the idea “an unknown number repeated 5 times” as X multiplied to 5, and you write XX5, clearly, you will have some issues. So, right now, right here, make this promise:

If you don’t use an X to show multiplication, what do you use?  
- Use a dot.  4·5=20
- Use parentheses. (4)(5)=20
- Use both a dot and parentheses. (4)·(5) =20

**NOTE!** Be careful when multiplying negative numbers! 3-2 means “3 minus 2”. (3)(-2) means “3 times negative 2”. **Those parentheses make a big difference in meaning, and you have to recognize this!**

- Use nothing if you are multiplying variables. 4 times x is 4x. x times y is xy.

To summarize:

- Multiplication means repetitions of something. It is a shortcut to writing many additions.

So, 4+4+4+4+4 is the same thing as (5)(4).

4x means “4 x’s” or x+x+x+x. AND 4 added to itself an x number of times (or a piece or 4, or x is less than 1… more on this later).

- You will get to the same number if you switch the “thing” with the “number of repeats”. 20 repetitions of $3 will get you the same amount of money as 3 repetitions of $20. It doesn’t ever matter what order you multiply numbers. Wow!

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**Memorizing multiplication for those of us with terrible memories**

7 times 6. That was the one I could never remember. Such stress! It made me sick to my stomach at night thinking I would get that flash card and have to say the answer in

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2 If I had it my way, X would never be used for multiplication. Why teach kids a symbol and then a few years later tell them they should learn a different symbol? Makes no sense to me. But, traditions are hard to break, and you will see X’s to show multiplication in signs, etc..
front of the class. For some reason, I just couldn’t remember 7 times 6. I wish I knew then what I know now: being able to figure math out is more important than memorizing any math. Also, not being able to perform in front of people does not mean you can’t contribute or be successful in solving problems that require math. There are many ways to be successful.

Back to 7 times 6, my nemesis. 7 times six could mean 7 repetitions of 6. 7 repetitions is the same as 2 repetitions combined with 5 repetitions. So 7 times 6 is the same as 2 repetitions of 6 plus 5 repetitions of 6.

STOP! Re-read the previous paragraphs if they made you feel confused or overwhelmed. Read slowly and write notes in the margins if that is helpful.

If you don’t remember 7 times 6, you can search your memory for a “multiplication fact” you do remember and work from there. 2 times 6 is 12, 5 times 6 is 30. 12 plus 30 is 42. Our brains can work out solutions when we forget memorized facts.

Example:

8 times 9. Suppose I don’t remember what that is, but I know that 8 repetitions is the same as 5 + 3 repetitions, and I remember my “5-facts” and my “3-facts”. 5 times 9 is 45, and 3 times 9 is 27. 45 plus 27… hmmm how do I do that again? I can break those numbers into pieces. 40 plus 5 plus 20 plus 7. 40 plus 20 is 60, 5 plus 7 is 12. 60 plus 12 is 72. 8 times 9 is 72. Maybe not the most efficient way, but it works, and you’ve grown your brain while figuring the answer. Now, be honest… did you follow all of that? Unless you read pretty slowly and with a pencil in your hand, I bet you didn’t follow. So, go back, and read it slowly so you follow that logic.

STOP! Make sure you are following - it takes time to read and understand this type of material.

Take a moment to write in your notes on how you could figure 8 times 9 when all you remember is 5 times 9 and 3 times 9, and, to make matters worse, you don’t remember how to add 45 to 27 using the standard algorithm.

8 times 9 again. This time I’m going to think about it as 9 repetitions of 8. 9 repetitions of something is 10 repetitions minus 1 repetition. So, “8 times 9” is “8 times 10 minus 8”.

\[(8)(9) = (8)(10) – (8)(1)\]

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3 Sure, if you have to memorize something for a presentation, go ahead, it’s important. If you have to memorize a procedure, that’s great and useful and necessary. But math is something that relies on logic and understanding and sense-making. I dislike memorizing and that’s one of the reasons I like math! If you like memorizing you will have to get used to logical reasoning and sense-making to be successful in math; memorizing will only get you so far; and, I sincerely apologize for any textbooks and/or teachers who might have told you “here… just memorize this…” when they were trying to help you with math.
80 – 8…. And, honestly, the way I figure this out is by thinking \(80 - 10 + 2\)

\[80 - 8 = 72.\] Follow that?

8 times 9 as 8 times 10 minus 8.

Exercise:

1. Figure out what 9 times 9 is by knowing that \((9)(10)=90\). (Pretend you don’t know what \((9)(9)\) is.

There are so many ways to come to an answer other than memorizing it. The other paths might not be as efficient as having the answer at your fingertips, but they are valid, they are helpful, and they need not be stressful, and considering various ways to answer questions grows your brain like nothing else!

**Multiplication and negative numbers.**

Positive repetitions of a negative number will, I hope it is clear, give you a negative number (if you repeat -5, 10 times, you’ll get -50).

What about “negative repetitions”? What could that possibly mean? Although “4 repetitions of negative 3” makes sense to me, as I hope it does to you as well, what the heck a “negative repetition” means is perhaps not so clear. What if we used the word “opposite” instead of “negative”? (see the chapter on “what is a number” for a review of terminology like “negative” and “opposite”)

If “negative 3 repetitions” means “the opposite of what you get from 3 repetitions”, then “negative 3 repetitions of 4” means “the opposite of 3 repetitions of 4”

\[(-3)(4) = (-12)\]

If you think of “negative” as the “opposite” when dealing with negative numbers, you will be able to figure out how to simplify expressions with negative numbers.

**Exercises:**

2. Figure out what 13 times \(-3\) is, without using a calculator. (hint: 13 repetitions is the same as 10 repetitions combined with 3 repetitions)

3. Figure out what \(-13\) times \(-3\) is, without using a calculator.

4. What is \((-5\) times \((-4)\)?

5. What is \((4\) times \((3)\)?

6. Simplify \(x \cdot -2\), if \(x = -4\) (note the dot next to the negative sign… this is not a subtraction exercise)
7. Simplify \(-3x\), if \(x = 4\)

8. Simplify \(-7x\), if \(x = 99\) (don’t use a calculator; 99 repetitions is the same as 100 minus 1 repetitions)

9. Simplify \(-3x\), if \(x = -99\)

10. Simplify \(-3x\), if \(x = 12\)

11. Simplify \(-3x\), if \(x = 60\)

**Multiplication and Area**

Consider a piece of paper. It is 8 inches long along one side and 11 and a half inches long along the other. What is the area of this paper? How would you figure this out?

STOP! What do you remember about area? Before you read any further, how do you figure out what the area is of a 8 by 11 and a half piece of paper? (and if you have no idea, that’s ok…but you probably do have some idea, so try to work it out)

Did you remember the formula “area = width times length”? Here’s another way to think of area: an area of something is the number of unit squares (“unit” as in equal to one) that fit into a 2-dimensional space. Something is 2-dimensional if it has a length and a width but no height at all. A thing is 1-dimensional if it has only a length but no width. An arrow on a number line is 1-dimensional. A flat piece of paper is sort-of 2-dimensional, but of course it has a height, that height is just very small so we don’t talk about it very much. We experience life in the 3 dimensions of up-and-down, side-to-side, and front-and-back.

A square with a unit area is defined as a square with one side equal to 1 and the other side equal to 1. That’s a unit square. If the units we are using are inches, then a unit square is 1 inch on one side and 1 inch on the other side. If the units we are using are miles, then 1 square mile is a square with one side equal to 1 and the other side equal to 1.

Questions and exercises :

12. Think about a long skinny rectangle, one side is 8 (it could be any unit on length you like, inches, centimeters, miles), and the other side is 1.
   Draw this rectangle in your notes:
   
   How many unit squares are in your rectangle?

13. Draw another rectangle, one side is 8 and the other side is 3.

   How many repetitions of the rectangle you drew for the previous question are represented in this new rectangle?

STOP!
Time to Review a bit: Explain what a “unit area” is here:

Suppose you have a rectangle, the length of one side is equal to $X$ and the length of the other side is equal to $Y$ (until we get to the chapter on fractions, imagine $X$ and $Y$ are both not less than 1). Explain how this can be seen as $X$ repetitions of a rectangle that has one side equal to 1 and the other equal to $Y$.

Show with a picture how, or explain in words how, the number of unit areas in the rectangle is equal to $X$ multiplied to $Y$.

If a piece of paper is 8 inches by 11 and half inches, the number of square inches (unit squares) is 11 and a half repetitions of 8 square inches, or 11 and a half multiplied to 8.

**Volume and multiplication**

Volume is the number of unit cubes in a 3-dimensional space; a unit cube, is (you guessed it), a cube with each side equal to 1 (in whatever units of length you are looking at, inches, feet, miles, kilometers, etc).

A unit cube:
its volume is equal to 1
because each side has a length equal to 1.

Imagine a “box” that is 4 unit-cubes long.

The volume of this box (which is clearly a mashed together bunch of smaller boxes) is 4, because it can be made with 4 cubic units, or unit cubes.

Now, imagine a “box” that is 4 units long and 3 units deep, but only 1 unit tall, made up of littler boxes:
The volume of this “box” is 12, or 3 multiplied to 4 unit cubes. It is 3 repetitions of the box that is just 4 units long.

Questions:

14. Imagine a “box” that is 4 units long and 3 units deep, just like the figure above. This time, however, it is 5 units tall. Explain how this “box” is 5 repetitions of the “box” in the above figure.

15. Explain why “volume of a box” is the same as “area of the box’s bottom” times its height.

Exercises:

16. What is 4+4+4+4+4 using multiplication notation?

17. What is 5+5+5+5 using multiplication notation?

18. What is -3 + -3 + -3 + -3 + -3 using multiplication notation?

19. How many unit squares 1 inch by 1 inch are in a space that is 3 inches long and 5 inches wide?

20. What is the volume of a box if the area of the top (the lid) is 4 square inches and the height is 5 inches?

21. What is the area of a rectangle if one side is 4 and a half and the other side is 3?

22. What is the area of a rectangle if one side is 4 and the other side is a half?

23. What is the volume of a box if one side is 4, the other side is 7, and the other side is a half?

24. What is the volume of a box if one side is 7, the other side is a half, and the other side is a 4?

25. What is the volume of a box if one side is 4, the other side is a half, and the other side is a 7?

26. Consider your answers to the previous three questions. Then answer this: Does it ever matter what order you multiply numbers?

Using symbols and multiplying: understanding coefficients
Remember how you can use a symbol, like the letter $x$, to represent a number? If I write $4x$, that means “4 times $x$”, or “4 multiplied to $x$”, “4 times $x$”. And, now you know that this could mean two different things: 4, repeated some unknown number of times, or $x$, repeated 4 times. Since both of these will end up being the same quantity, it’s usually easiest to think of the number 4 as the number of times you’ve repeated $x$.

$$4x = x+x+x+x$$

In other words, $4x$ means “4 x’s”

Ok… think back to addition, where you learned that you can simplify the combination of old socks with old socks, but you can’t simplify the combination of old socks with baby kittens.

4 baby kittens plus 3 baby kittens will give you 7 baby kittens.

$4 \text{ x’s}$ combined with $3 \text{ x’s}$ will give you $7 \text{ x’s}$.

$$4x + 3x = 7x.$$ 

Get it?

One dollar plus 5 dollars is 6 dollars

$$x + 5x = 6x$$

What about $x + 5$? In this case, can we simplify at all? The expression means “the combination of one $x$ and 5”. Since we don’t know what the number $x$ is, we CANNOT simplify it. Mathematicians call this “combining like terms”, meaning you can simplify the combination of $x$’s with $x$’s, but you can’t simplify the combination of $x$’s with $y$’s. You can simplify the combination of 4 puppies with 5 puppies (9 puppies), but you can’t simplify the combination of 4 puppies and 10 shoes. It just remains 4 puppies $+$ 10 shoes.

Exercises:

Simplify the following expressions, if possible.

27. $x + 5$
28. $x + 5y$
29. $x - 5x$
30. $-x + 5y$
31. $-1 + x$
32. $-x - 5x$
33. $-3x - (-5y)$
34. $-3x + 3x$
35. $4 + 5y$
36. $-x + 5y + 2x - 4y$
37. $-1 + 5y + 2 - 4y$
38. $-x - 5y + 2 - 4y$
39. $-x + 5y - (-2x) - 4y$
40. $-2x + 5y + 2x - 5y$
41. $-x - 5y + 2x + 4y$

Coefficients:

The symbol in front of your unknown variable is called its “coefficient”. In the expression “$5x$”, $5$ is the coefficient of $x$.

The coefficient lets you know how many of, or what part of, the variable you are considering.

So, $5x$, means you have $5$ $x$’s, and $5$ is the coefficient. If your expression is $Cx$, then $C$ is the coefficient of $x$.

Suppose you are combining $5$ $x$’s with $B$ $x$’s. This would look like:

$$5x + Bx$$

If we knew what $B$ is, we could simplify this expression. For example, if $B=8$, then $5x + Bx$ is the same as $13x$, because you can combine the coefficients.

If we don’t know what $B$ is, then $5x + Bx$ would be the same as “$5$ and $B$” $x$’s. To show that the total number “$5$ plus $B$” is together, you use a parentheses.

$$5x + Bx = (5 + B)x$$

In other words, if you are combining like terms, you can combine the coefficients, and, if you like, they can go in a parentheses.

You’ll learn more about parentheses in later chapters.

For now, it’s important to keep in mind the idea of “coefficients” being “the number of repetitions”.
What you learned about multiplication:

- Multiplication means repeating a quantity
  OR
  finding a fraction or piece of a quantity
- Multiplication can mean repeated addition
- Area means how many unit squares can fit on a surface.
- Volume means how many unit cubes can be contained in a space.
- It never matters what order you multiply numbers
- The rules for multiplying negative numbers make sense given the meaning of multiplication
Chapter 8: Division

In which you will learn:

• X divided by Y answers one of the following questions:
  How big is each piece if you cut X into Y equal pieces
  OR
  How many Y’s, plus any leftover fractions of Y, fit into X?
  OR
  What is the ratio of X to Y?
• There are several ways to show division, using a horizontal fraction bar is usually the easiest.
• It is straightforward to explain why division by zero does not make sense given the definitions of division.

Division

To get you thinking about what you already know and believe about division, write what you remember about division. What do you remember about long-division, the (dreaded) fraction division, the meaning of division? When do you use division? What kinds of notation do you remember using for division? This is your space for a complete “brain-dump” of what you remember about division.

What is Division good for?

Asking “what is 12 divided by 4?” is equivalent to both of the following:
• “How big is each piece if you cut 12 into 4 equal pieces?”
  and
• “How many repetitions (or multiples) of 4 fit into 12?”

Ponder those two statements for a while; they represent the two ways we can use division to problem-solve.

Consider this visualization of multiplication

The long arrow has a length of 12. 12 divided by 4 would tell you the answer to the questions
• “How many 4’s are there in 12?”
• “How many arrows are there in this picture that add to 12?”
12 divided by 4 is 3, because 12 divided by 4 means “how many times do you repeat 4 to get to 12?” In the visualization above, there are 3 repetitions of the arrow, and the arrow represents the number 4.

“3 repetitions = an arrow of length 12 divided by an arrow with length 4”

Wait! You might remember learning that division means “cutting into equal pieces”.

In this view, 12 divided by 4 tells you the answer to the question “How big is each piece if 12 is cut into 4 equal pieces?” The previous visualization shows 12 cut into 3 equal pieces, so what’s going on?

12 divided by 4 gives you the answer to the question: “what is each piece if you cut 12 into 4 equal pieces?” In the visualization above, each little arrow has length 3.

“an arrow with length 3 = each piece of an arrow with length 12 cut into 4 equal pieces”

The important thing to remember is that division can answer two seemingly different questions, and can be visualized in two very different ways. If you have a problem to solve, and to solve it you must answer the question “How many x’s fit into y?” you know you’ll get the same answer if you ask the question “How big is each piece if I cut y into x equal pieces?”

In your notes, write the two questions that can be answered with division, and write your reaction to the idea that the same mathematical operation can answer two seemingly different questions.

Division Notation

There are many ways to show division. 12 divided by 4 means “how many 4’s fit into 12” and “the size of each piece if 12 is cut into 4 equal pieces”.

Both of those concepts mean 12 divided by 4 and are written as

\[ 12 \div 4 \]

\[ 4 ) 12 \]
Some of you might have a favorite. And, if you grew up outside of North America, you might be familiar with other notations. However, by far the easiest, most versatile, flexible, and helpful notation for you to use is the one with a horizontal fraction bar. If you are not in the habit of writing division with a fraction bar and are resistant to the idea, I strongly but kindly suggest that you get into that habit.

12 divided by 4 is most useful when it is written as \( \frac{12}{4} \).

I will always write division with a fraction bar, and so should you. I know I don’t like it when people tell me I should do something. However, writing division with a fraction bar will make life much easier for you. If you have to divide multi-digit number by another multi-digit number, please find a calculator or a phone with a calculator app. If you really have to do multi-digit division and you don’t remember how, well, you can use what you know to figure out the answer (I’ll go over some examples), without resorting to remembering procedures you’ve long-since forgotten.\(^1\)

**Division summary:**

\( \frac{T}{B} \) will answer both of these questions\(^2\):

- “How big is each piece, if I cut T into B equal pieces?”
  
  or

- “How many B’s fit into T?” (or “what part of B fits into T?”), or “how many times is B repeated to get to T?”

And, just to review **multiplication**:

B multiplied to X will answer either of the questions:

- “what is B repeated X times?”
  
  or

- “what is X repeated B times?”

**Dividing by Zero**

Think about T divided by B, \( \frac{T}{B} \); what sense could you make of this if B is equal to zero?

- “What do you get if you cut T into 0 equal pieces?” It seems like this question is asking “what do you get if you don’t cut T at all?” but that is NOT what it means; not cutting T at all would mean “keeping T in 1 piece”. Cutting T into 0 pieces just doesn’t make sense. You just can’t cut something into 0 pieces.

---

\(^1\) Youtube is full of excellent videos showing the algorithm for long-division.

\(^2\) T for “top” and B for “bottom”, just to help you keep track. I’ve had many many students get the notation confused, even while they understand the meaning just fine.
• “How many 0’s fit into T?” (or “what part of 0 fits into T?”; or “how many times is 0 repeated to get to T?”) These questions have no number answers. No matter how many times you repeat 0, you’ll never get to T…³

You can’t divide by zero, because it doesn’t make any sense. Dividing by zero doesn’t give you any number.

_A note about tricky words and phrases:_

What is the size of each serving if you divide 18 cups of ice-cream into 9 servings?

Here, I asked “what is the size of a serving if 18 cups are divided into 9 servings?”, and the answer is \( \frac{18}{9} \), which simplifies to 2 cups for each serving.

The structure of the question might make you think that the 18 is going into 9, but 18 going into 9 would be mean 9 divided by 18, or \( \frac{9}{18} \) which simplifies to \( \frac{1}{2} \). (More on this in the chapter on fractions).

Sorry! I didn’t invent the English Language, so don’t blame me.

³ Some people want to say that the only number you can divide by 0 is 0, and that 0 divided by 0 is equivalent to \( \frac{b}{b} = 1 \), (anything divided by itself = 1). But I could argue that if I repeat 0 a thousand times, I’d still get 0, so \( \frac{0}{0} = 1000 \). Using that argument, \( \frac{0}{0} \) could be equivalent to any number! Division by zero just doesn’t make sense.
If you are answering specific questions, you must think carefully about the context of the situation, because small changes in the words can make a big difference.

“What is 10 into 20?” is a sort of old-fashioned way to ask “what is 20 divided by 10?”

\[
\frac{20}{10} = 2.
\]

“What is 10 divided into 20 pieces?” means \(\frac{10}{20} = \frac{1}{2}\) (again, more on this in the chapter on fractions).

So, that “divided into” phrase is quite confusing! I really apologize and the only helpful thing I can say at this point is to pay attention to the situation and try to reword the question in a more helpful way. Given the context of any situation, you will be able to figure out what mathematical operations the question requires; but you \textbf{\textit{will have to think about it and not just start “doing some math” automatically.}}

Questions.

1. Consider an arrow that represents the number -15. How many arrows that represent the number -3 fit into the arrow that represents the number -15? What is the division expression that answers this question? Draw an appropriate number line with these arrows. Is your answer a positive or negative number?

2. Draw an arrow that represents the number -15. Draw 3 equal arrows (repetitions) that combine to equal the number -15 (you are cutting -15 into three equal pieces). What number does each of those three arrows represent? What is the division expression that represents this situation? Is your answer a negative or positive number?

3. What is the opposite of 4 times -3? Write the multiplication expression that answers this question.

4. What is the opposite of 4 times 3? Write the multiplication expression that answers this question.

5. What is the opposite of -4 times 3? Write the multiplication expression that answers this question.

6. What is the opposite of -4 times -3? Write the multiplication expression that answers this question.

7. What is 4 times -3? Write the multiplication expression that answers this question.

8. What is the opposite of negative 3 times 7?

9. How big is each piece if you cut -15 into 3 equal pieces? (is your answer negative or positive?)

10. What is the opposite of the number of the number of 3’s that fit into -15?

11. How many halves fit into 1? Write the division expression that answers this question. I know you haven’t been introduced to fractions yet, but try to answer the question through sense-making.

\textit{Simplifying division expressions, and using division to solve problems.}
You’ll really need a pencil as you read the following. I go through some thought processes that are best read slowly. It is always a good idea to write ideas, thoughts, and questions in the margins. If you are aren’t thinking hard and getting a bit frustrated, then you are wasting your time; you only learn when you see something in a new light, or with a new viewpoint, and you make mistakes.

Example 1
Suppose you are faced with this question:
What is 9 divided by 4?
You think to yourself “how many 4’s fit into 9?” and you write
\[
\frac{9}{4}
\]
You re-write 9 as 4’s, plus whatever is leftover after writing 4’s:
\[
\frac{9}{4} = \frac{4+4+1}{4}
\]
So, you realize that 2 fours fit into 9, with “1 leftover”. That “1 leftover” needs to be part of the division expression, too, 1 divided by 4 is \(\frac{1}{4}\) (more on fractions in the fraction chapter).

In summary, “9 divided by 4” can be written like \(\frac{9}{4}\), and that’s equivalent to \(\frac{4+4+1}{4}\) and that is the same, mathematically, as \(2 + \frac{1}{4}\).

The bottom line is that a meaningful answer to the question, “what is 9 divided by 4?” could be “there are 2 and a quarter repetitions of 4 in 9”.

Repetition is multiplication. Write mathematically what 2 and a quarter repetitions of 4 equals, using the notation for multiplication. This kind of thinking will help you connect division with multiplication later.

Example 2
What is 1200 divided by 40?
You think to yourself, this means “how many 40’s fit into 1200?”
You start by guessing “I think that there are at least 100 40’s in 1200. 100 times 40 is 4000”.
But that’s not right, because 4000 is already bigger than 1200.
Ok… “I think there are at least 10 40’s in 1200 because 10 times 40 is 400. That’s still not big enough, so I’ll add another 10 40’s to get to 800. That will leave 400 left over. I’m about to lose track of what I am thinking so I’d better write this down”
\[
\frac{1200}{40} = \frac{10 \cdot 40 + 10 \cdot 40 + 400}{40}
\]
Now, I take a deep breath and consider what I wrote… does it make sense so far? Yes? I’m going to dive back in now. If not, consider it again to see if it makes sense. Please note, this is just one way to think about this problem… it’s not the best or the most efficient, it’s just one way. If I thought about it again, I’d probably think about it a different way. Math is creative, and there are many paths to success.

My thinking so far looks like this:
1200 divided by 40 will answer the question “how many 40’s fit into 1200?” 1200 is 30 repetitions of 40, so the answer is 30. Are there more efficient ways to answer this question? You bet! Will you answer a similar question in a similar way? Probably, yes, sometimes, probably not other times.

Questions:

12. What will be the size of a piece if you cut 1200 into 30 equal pieces?
13. How many repetitions of 40 fit into 1200?
14. How many repetitions of 30 fit into 1200?
15. What will be the size of each piece if you cut 1200 into 40 equal pieces?

Example:

What is 123 divided by 35?

\[
\frac{123}{35} = \frac{3 \cdot 35 + 18}{35}
\]

First, I take a sort of educated guess: maybe 4 35’s? 4 times 35 is 140, so that’s too many 35’s. Maybe 3 35’s? 3 times 35 is 105

\[
\frac{123}{35} = \frac{3 \cdot 35 + 18}{35}
\]

So, the answer to the question “What is 123 divided by 35?” is 3, with 18 “leftover”. What does that “leftover” mean? I need to divide it by 35 as well if I want to fully simplify the expression (although in some contexts it might not make sense) … so 123 divided by 35 is 3 combined with \( \frac{18}{35} \).

The chapter on fractions will work with this idea more. For now, I’ll leave my answer like this:

\[
\frac{123}{35} = 3 + \frac{18}{35}
\]

because 3 35’s fit into 123, with 18 “leftover”, or “the rest”.

Here, write what you think about this way of simplifying a division problem. Would it be easier to “do division” this way rather than through “long division”? When do you think it would be better to think about “how many fit into” rather than simply doing “long-division”, and when do you think doing “long-division” would be easier?

Exercises: Do the following division expressions by thinking about division as “how many of the denominator (the expression in the bottom of a fraction-bar) “fit into” the

\[4\text{ The “leftover” is sometimes called a “remainder”. You might run into this term in pre-calculus when you might study remainder functions.} \]
numerator (the top of the fraction-bar). It’s important to show all your thinking so you can review this and understand what you wrote. If there is a “remainder”, write your answer to include a fraction.

16. \( \frac{x+x+x}{x} \) Think: how many of the ‘bottom’ fit into the ‘top’?

17. \( \frac{3x}{x} \) remember: "3x" means "3 repetitions of x", or x+x+x.

18. \( \frac{y+x+x+x}{x} \)

19. \( \frac{2x+x+y}{x} \)

20. \( \frac{x+x+x+x}{2x} \)

21. \( \frac{x+x+x+x+y}{2x} \)

22. \( \frac{-10}{-3} \)

23. \( \frac{-100}{3} \)

24. \( \frac{100}{-3} \)

25. \( \frac{124}{5} \)

26. \( \frac{124}{50} \)

27. What is 2 divided by a half? In other words, how many halves fit into 2? Write this as a division expression using a fraction bar (this is like an appetizer problem to get you thinking about fractions, even before we get to that chapter). Answer the question by visualizing it.

28. How many thirds fit into 1? Answer the question by visualizing it, and write this as a division expression using a fraction bar.

29. How many thirds fit into 3? Answer the question by visualizing it, and write this as a division expression using a fraction bar.

Please take a moment to make note of any ideas that were new to you, and how these new ideas fit into your thinking before you read any of this material.

What questions you needed help on. What questions did you have to look up? What questions did you struggle with?

**Ratio**

The ratio of any number, let’s call it \( x \), to another number, let’s call that one \( y \), is \( \frac{x}{y} \).

For example, the ratio of cats to dogs in my house could be \( \frac{3}{5} \). That means that for every 3 cats, there are 5 dogs. It does NOT mean that I have 3 cats and 5 dogs (although I might). I also might have 9 cats and 15 dogs, I might have 30 cats and 50 dogs.

Ratios are usually communicated about as a fraction, not as a mixed number. For example, if you want to mix concrete, the ratio of gravel to sand is usually recommended to be 3 parts gravel to 2 parts sand. You could say, “the ratio of gravel to sand is \( \frac{3}{2} \). Usually, you don’t say “the ratio of gravel to sand is 1 and a half”,

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although you could. It’s easier to think about as a fraction or a statement about division, rather than a sum of a whole number and a fraction.

Often, the word *per* is used to describe a ratio. For example, if you are going 20 miles per hour, the ratio of miles you travel to the time it take to travel is $\frac{20}{1}$, or, simply, 20.

30. What is the ratio of flour to sugar in a recipe that needs 8 cups flour and 5 cups sugar?

31. What is the ratio of flour to sugar in a recipe that needs 8 cups flour and 2 cups sugar?

32. What is the ratio of flour to sugar in a recipe that needs F cups flour and S cups sugar?

33. What is the ratio of flour to sugar in a recipe that needs 1 cup flour and a third cup sugar? Express your answer as a simplified fraction.

34. What is the ratio of flour to sugar in a recipe that needs a third cup flour and 1 cup sugar? Express your answer as a simplified fraction

35. What is the cost per pound of apples if it costs $15 to buy 3 pounds? Express your answer as a simplified fraction

36. What is the ratio of cost per ounce of something if 15 ounces cost $20? Express your answer as a simplified fraction

37. What is the cost per ounce of something if it costs $1 per half an ounce? Express your answer as a simplified fraction

38. If 16 Canadian dollars are worth about 12 US dollars, what is the ratio, expressed as a simplified fraction, of Canadian dollars to US dollars?

39. There are almost exactly 20 miles per 32 km. What is the ratio of miles to km? Express your answer as a simplified fraction.

40. My car used 10 gallons of gas going 300 miles. What was my average miles per gallon? Express the answer as a simplified fraction

41. I walked 8 miles in 2 hours. What was my speed (in miles per hour)? Express your answer as a simplified fraction

42. I earned $500 working 20 hours. How much did I get paid per hour?

43. I earned D dollars working H hours. What did I get paid per hour?

44. My car used G gallons of gas going M miles. What was my average miles per gallon?
Summary:

- Multiplication is repeated addition
- Division is finding out how many times something was repeated to arrive at another number, or, finding out what number was repeated to arrive at another number. Division also expresses ratios. The word “per” means division.
- It never matters what order numbers are multiplied.
- The “rules” for multiplying and dividing negative numbers are a consequence of the meaning of multiplication and division.

In your notes, summarize the misconceptions or struggles that caused you to look up answers or to get answers different from the answer key. If you think you didn’t have any misconceptions, you might not be trying hard enough.

“I used to think…, but now I think….” Is a good way to approach new learning.

I hope you learned:

- X divided by Y answers one of the following questions:
  - How big is each piece if you cut X into Y equal pieces
  - OR
  - How many Y’s, plus any leftover fractions of Y, fit into X?
  - OR
  - The ratio of X to Y
  - OR
  - X per Y

- There are several ways to show division, using a horizontal fraction bar is usually the easiest.
- It is straightforward to explain why division by zero does not make sense given the definitions of division.
Chapter 9: Fractions part 1

In which you will learn:

- How to think about any fraction as some repetitions of unit fractions
- How to make sense of fraction multiplication by thinking about 1) unit fractions and 2) the fact that it never matters what order you multiply numbers 3) the connection between multiplication by a unit fraction and division: $\frac{1}{4}$ of something is the same as that thing divided by 4. So, $\frac{1}{4}$ of $\frac{1}{3}$ is $\frac{1}{12}$
- How to create and recognize equivalent fractions by understanding and recognizing weird forms of the number one.
- How to understand adding and subtracting fractions by 1) thinking about and using weird forms of the number one and 2) understanding that some combinations can be simplified, and some cannot (see Chapter 2, “Addition”)

Fractions, Part 1

For many reasons, thousands of students have a lot of trouble with fractions. In fact, many of my students and friends have reported liking math except for fractions. Many successful adults have told me about their emotionally upsetting memories of trying to make sense of fractions in unsupportive classrooms. Unfortunately, many of those adults eventually gave up on trying to make sense of fractions and forced themselves to practice nonsensical (to them) procedures in order to pass a class or a test.

To examine a concept or idea from a new perspective, it’s often necessary to consider your current perspective carefully, so that new learning can happen when you compare new ideas to your old ideas, like fitting puzzle pieces together and discarding those pieces that don’t fit. If you just jump in without examining your existing understanding and experience, you are less likely to make sense of new ideas. With that in mind, please take a moment to write and reflect on the following prompts:

In your notes, write a short description of your earliest memory of learning fractions. It was probably when you were somewhere between 7 and 15 years old. Write when you felt good about learning and working with fractions, and also when you felt frustrated and perhaps confused.
Based on those early memories of fractions, what conclusions did you come to about yourself and fractions? Are you good with them? Are they impossible to learn? Stressful? Fun? Tedious? Stupid?

What you already understand about fractions might be correct or incorrect, and you may feel good or bad about fractions based on your past experiences. The structure I use for you to explore fractions will connect them with all of the ideas you’ve thought about so far in this book; if you’ve followed along so far, you will find that fractions make sense and you will be able to explain the procedures you perhaps have long-ago memorized (and perhaps forgotten).

You will, however, have to be patient as I review concepts that you may feel you already know. Stay with me and answer the questions anyway, because it may be that you have “gaps” in your understanding that you don’t realize are there unless you answer all the questions.

First, a quick Review of Division:

\[ \frac{T}{B} \]

means \( T \) divided by \( B \) (review the chapter on division briefly to remind yourself).

\[ \frac{T}{B} \]

could be used to answer two questions:

- “How many \( B \)’s fit into \( T \)?”
- “How big is each piece if you cut \( T \) into \( B \) equal pieces?”
- “What is the ratio of \( T \) to \( B \)?”
- “What is \( T \) per \( B \)?”

Fractions: a fresh start with Unit Fractions

A Unit Fraction can be thought of as 1 piece of a whole something: one fourth of a pizza, one fifth of an apple, one third of my salary, one hundredth of the population, are all unit fractions. Think about this question: “if you cut an arrow with length 1 into 3 equal little arrows, what number does each little arrow represent?” This is a question that can be answered with division, one divided by three, or \( \frac{1}{3} \).

Unit fractions are each “piece” if you cut 1 into equal pieces.

1 divided by 5 is one fifth (also known as \( \frac{1}{5} \)), 1 divided by 7 is one seventh (also known as \( \frac{1}{7} \)).

A unit fraction is a number, so, it can be represented as point on a number line or an arrow on a number line. It is also the process of dividing up 1 into equally-sized pieces

Multiple Fractions

If I repeat a unit fraction, what will I get? What is one fourth, repeated 5 times? 5 times \( \frac{1}{4} \), or \((5)\frac{1}{4}\). (Remember, multiplication is repetition).
Can I write \((5)\left(\frac{1}{4}\right)\) in any other way? Read it out loud to yourself: “five fourths”. I would guess many of you would write “five fourths” as \(\frac{5}{4}\).

Watch out for this tricky notation: \(5 \frac{1}{4}\) means “5 plus \(\frac{1}{4}\)” or “5 and \(\frac{1}{4}\)”, but \((5)\left(\frac{1}{4}\right)\) means “5 times \(\frac{1}{4}\)”. Those parentheses make all the difference. To make it even more confusing, any letters used in fraction notation like this \(5 \frac{x}{4}\), indicates multiplication. So, \(5 \frac{x}{4}\) means “(5) times \(\frac{x}{4}\)”, but \(5 \frac{3}{4}\) means “(5) plus \(\frac{3}{4}\)”. Got it? Using a letter instead of a digit makes a difference in meaning with fraction notation.

Could it be that five multiplied to one fourth, five divided by 4, the fraction \(\frac{5}{4}\), and “how many 4’s fit into 5?” and “what each piece is, if you cut 5 into 4 equal pieces” are all equal? Yes!

**What is a fraction?**

First, answer these questions:

1. Show how the following are all equal, using examples, pictures, number lines, anything that helps you make sense of the questions (like pictures of cake or pizza):
   a) five multiplied to one fourth
   b) five divided by 4, both:
      o “how many 4’s fit into 5?”
      o “the size of each piece if you cut 5 into 4 equal pieces”
   c) the fraction \(\frac{5}{4}\)
   d) \(1+\frac{1}{4}\)

So, \(\frac{3}{4}\) (three fourths), is the same as (3) times \(\frac{1}{4}\) and is the same as 3 divided by 4, and is the same as the ratio of 3 to 4.

It seems then, that dividing 3 by 4 is the same as multiplying 3 by \(\frac{1}{4}\)!

In general, any number \(T\) divided by any number \(B\) (as long as \(B\) is not zero) is equal to the fraction \(\frac{T}{B}\), and is also equal to \(T\) multiplied to \(\frac{1}{B}\)

\[
\frac{T}{B} = T \cdot \frac{1}{B}
\]

Remember \(T \cdot \frac{1}{B}\) means \(T\) multiplied to \(\frac{1}{B}\).

The answers to all of the following questions are the same:
“How many B’s fit into T?”

“How big is each piece if you cut T into B equal pieces?”

“What is $\frac{1}{B}$ repeated T times?”

“What is the ratio of T to B?”

Why would you want to look at fractions so many ways? You need to be flexible with how you think and answer questions using fractions, so that you can combine them, subtract them, multiply and divide them in ways that make sense and are accurate.

Remember the chapter on addition and subtraction, where you practiced simplifying adding apples to apples and bananas to bananas, but realized that adding 3 apples to 4 bananas couldn’t be simplified? Adding or subtracting fractions works in a similar way.

Three apples combined with six apples could be simplified to nine apples. But 3 puppies combined with 6 tennis balls cannot be simplified. Three fourths and six fourths can be simplified to nine fourths. Even though three plus six is 9, three sevenths combined with six tenths cannot be easily described as 9 of anything. But, you can simplify it by re-writing the numbers as equivalent fractions… and we’ll get to that soon. Be patient. Before you get to add and subtract any types of fractions, it’s important for you to reflect on what you’ve thought about so far on fractions by doing (and more importantly thinking about) the following exercises:

Write the following phrases using mathematical notation, and simplify the resulting expressions:

2. One fourth, repeated 7 times, combined with one fourth, repeated 6 times.

3. One fourth added to three fourths.

4. The difference between one fifth and negative one fifth.

5. The difference between negative two fifths and negative four fifths.

6. The difference between negative two hundred dollars and negative four hundred dollars.

7. Negative two thirds combined with four thirds.

8. A debt of two dimes combined with four dimes (express your answer as a number of dimes).

9. The size of each serving if you are dividing 5 cookies among 3 children.

10. The difference between negative four fifths and negative three fifths.

11. Write the following as a number times a unit fraction, meaning some number of repetitions of a unit fraction.
    a. $100 \div 3$
    b. $3 \div 100$
c. 3 fourths

d. 2 fifths

e. One half

12. One fourth, repeated 8 times.
13. The number of 4’s that fit into 8.
14. The size of each serving, if I divide 8 cupcakes into 4 servings.
15. 8 divided by 4

STOP! Reflect in your notes on these main ideas:

Any fraction can be understood, and written as, a number times a unit fraction
You can combine fractions easily if the denominator is the same for each.
Any fraction can be written as a whole number times a unit fraction. Being able to
“break up” a fraction in this way allows you to think about multiplying fractions in a
very flexible way, because, (remember?) it never matters what order you multiply
numbers.

Repeating fractions, what does that mean?

Example:

What is 5 times \(\frac{3}{4}\)? In other words, what is 5 repetitions of \(\frac{3}{4}\)?

Since \(\frac{3}{4}\) is the same as \((3)\left(\frac{1}{4}\right)\), the answer to that question “What is 5 times \(\frac{3}{4}\)?” is the
same as the answer to the question: “What is 5 times 3 times \(\frac{1}{4}\)?” Or, “What is 5
repetitions of 3 repetitions of \(\frac{1}{4}\)?”

Remember from the Introduction: part of the power of math is posing a question,
then posing another question that you know has the same answer but is easier.
Figuring out if you should rewrite and re-think about problems before you start
computing is an important part of any mathematical task.

Remember how it doesn’t matter what order you multiply numbers? So, “5 times \(\frac{3}{4}\),
is the same as 5 times 3 times \(\frac{1}{4}\).

What is 15 times \(\frac{1}{4}\)? (stop… where did that 15 come from?)

\[
(15) \left(\frac{1}{4}\right) = \frac{15}{4} \quad \text{(stop… why is this true?)}
\]

\[
\frac{15}{4} = (4)(3) + 3 \frac{3}{4}
\]
\[
\frac{(4)(3)+3}{4} = 3 + \frac{3}{4}
\]

3 four’s fit into 15, with 3 leftover… review division if you have questions on this step

Example:

What is 16 multiplied to \(\frac{7}{8}\)? I can restate the question this way: 16 multiplied to \(\frac{7}{8}\) will give me the same answer as 16 multiplied to 7 multiplied to \(\frac{1}{8}\), and it doesn’t matter what order you multiply those numbers. I’m going to take a moment to decide: what would be the most efficient way to multiply these numbers to save myself some work? (mathematicians like being lazy, but we call it being efficient). If I multiply the (16) to the \(\frac{1}{8}\) first, I’d get \(\frac{16}{8} = 2\)… that seems like an efficient way to do this:

\[
(16) \left( \frac{7}{8} \right) = (16)(7)(\frac{1}{8})
\]

\[
(16)(7) \left( \frac{1}{8} \right) = (16) \left( \frac{1}{8} \right)(7)
\]

because it never matters what order you multiply numbers, I’m being strategic in the order I multiply to make life easier for myself. I can see that multiplying 16 and \(\frac{1}{8}\) would be easier than multiplying 16 and 7. So… that’s what I’m going to do.

\[
(16) \left( \frac{1}{8} \right)(7) = (\frac{16}{8})(7)
\]

make sure you “get” this step! above

\[
(\frac{16}{8})(7) = (2)(7)
\]

why is this true? make sure you can answer “why?”

\[
(2)(7) = 14
\]

So, it turns out that a simpler way to say “16 multiplied to \(\frac{7}{8}\)” is to say “14”. Or, we could say, 14 is equal to 16 multiplied to \(\frac{7}{8}\).

Here is another approach to answering the question “what is 16 multiplied to \(\frac{7}{8}\)?”

\[
(16) \left( \frac{7}{8} \right) = (16)(7)(\frac{1}{8})
\]

\[
(16)(7) \left( \frac{1}{8} \right) = (112)(\frac{1}{8})
\]

\[
(112) \left( \frac{1}{8} \right) = (\frac{112}{8})
\]

Hmmmm…. How many 8’s fit into 112?

\[
(\frac{112}{8}) = \left( \frac{(8)(11)+24}{8} \right)
\]

More than 11 is my first “guess” because 8 times 11 is 88, this leaves 112-88 leftover.
Stop! Why is this true? Where did that “24” come from?

\[
\frac{(9)(11) + (3)(8)}{8} = 14
\]

To divide 112 by 8, I counted how many 8’s fit into 112. I didn’t use a standard “long-division” algorithm, but if you remember how to do that, then go ahead and do it! This was just an example of how to answer this question. There are many ways to figure out that 16 multiplied to \( \frac{7}{8} \) is 14.

Exercises:

Answer the following questions, express your answer in the simplest form you can think of, write your answers as fractions, not decimals. You can check your answers with a calculator, but you must be able to do these without using a calculator.

16. What is a third of 16?

17. What is 16 divided by 3?

18. What is \( \frac{1}{3} \), added to itself 16 times?

19. Simplify: \( \frac{4}{9} \cdot 27 \cdot \frac{1}{2} \) (think carefully!... you can make it difficult or easy depending on your approach. You can rearrange these numbers in many ways!)

20. What is 16 times \( \frac{9}{2} \)?

21. What is 16 times 21, all divided by 7, and all times \( \frac{1}{2} \)? (remember to make life easy for yourself by considering different approaches before you start doing any calculations)

22. What is half of 21, all divided by 7, all times 4? (you should be able to do this without a calculator, and fairly quickly, by noticing that you can divide by 7 before you figure out what half of 21 is)

STOP: Big ideas here were that you can re-arrange multiplication and division to make life easier for yourself, and you can think and write any fraction in multiple ways: as a repetition of unit fractions, or as a division problem

What does “of” mean?

For a quick review, \( \frac{T}{B} \) is equal to \( (T) \cdot \frac{1}{B} \). Multiplication is “repetitions of something”, in this case, \( T \cdot \frac{1}{B} \) is \( T \) repetitions of \( \frac{1}{B} \).

For example, \( \frac{3}{5} = (3) \frac{1}{5} \), and that means I have three repetitions of \( \frac{1}{5} \).

\[
\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}.
\]
Three fifths means I have three repetitions of $\frac{1}{5}$.

But you also know that it doesn’t matter what order you multiply numbers, and that 3 repetitions of 4 will result in the same answer as 4 repetitions of 3.

$$(3+3+3+3)=(4+4+4)$$

$$(4)(3)=(3)(4)$$

Following that same logic, three repetitions of $\frac{1}{5}$ must be the same as $\frac{1}{5}$ repetitions of 3. Huh? What does that mean?

In the chapter on “what is a number”, I wrote “A number could mean “how many repeats” of some other number, or “some part of, or piece of” some other number. If you are dealing with the type of number that is a “repetition,” or a “piece of”, then you are multiplying.

What is $\frac{1}{5}$ of 3? This is the same as 3 divided by 5, right? A fifth of something means the size of each piece if you cut that thing into 5 pieces. So, a fifth of 3 is the same as 3 divided by 5. Ok, but then it has to be the same as 3 repetitions of $\frac{1}{5}$, because you know that $\frac{3}{5}$ is the same as 3 times $\frac{1}{5}$.

“$\frac{1}{5}$ of 3” and “3 repetitions of $\frac{1}{5}$” are equal to the same number.

To summarize what you just read, think about the following questions, all of which have the same answers: $\frac{T}{B}$

“How many B’s fit into T?” Or, “What part of B fits into T?”

“How big is each piece if you cut T into B equal pieces?”

“What is $\frac{1}{B}$ repeated T times?”

“What is $\frac{1}{B}$ th of T?”

To keep it more concrete, here’s an example with numbers instead of the symbols $T$ and $B$. The answer to each of the following questions is $\frac{3}{5}$.

“What part of 5 fits into 3?”

“How big is each piece if you cut 3 into 5 equal pieces?”

“What is $\frac{1}{5}$ repeated 3 times?”

“What is $\frac{1}{5}$ th of 3?”

Here are some multiple choice questions for you to answer; circle your answers, and write notes to yourself explaining your thinking.

23. $\frac{10}{2}$ makes more sense as:
“how many 2’s fit into 10?”

Or

“What part of 2 fits into 10?”

24. \( \frac{2}{8} \) makes more sense as:

“How many 8’s fit into 2?”

Or

“What part of 8 fits into 2?”

25. \( \frac{15}{30} \) makes more sense as:

“How many 30’s fit into 15?”

Or

“What part of 30 fits into 15?”

26. \( \frac{100}{10} \) makes more sense as:

“How many 10’s fit into 100?”

Or

“What part of 10 fits into 100?”

27. \( \frac{15}{2} \) makes more sense as:

“How many 2’s fit into 15?”

Or

“What part of 2 fits into 15?”

Exercises: Try to find the most efficient way to answer the questions by being strategic about what order you use multiplication and/or division. Simplify your answer as much as possible.

28. What is 8 repetitions of \( \frac{3}{4} \)?

29. What is \( \frac{3}{4} \) of 8? (compare your answer to the previous question and think about it for a second)

30. What is 3 repetitions of \( \frac{2}{7} \) combined with 4 repetitions of \( \frac{1}{7} \)?

31. What part of 15 fits into 5?

32. How many 5’s fit into 15? (compare and contrast to the previous question and think about it for a while)

33. What is \( \frac{2}{7} \) of 3 plus 4 repetitions of \( \frac{1}{7} \)?
34. What is \( \frac{2}{7} \) of 3 plus 4 divided by 7?

35. What is \( \frac{2}{7} \) of 3 plus \( \frac{1}{7} \) of 4? [compare this with the previous two questions]

36. What part of 4 fits into 2?

37. What part of 2 fits into 3?

**Pieces of Pieces: multiplying fractions**

Multiplying A to B could mean “A repetitions of B”, it could mean “B repetitions of A”. But, what if A and/or B are not whole numbers? What if one or both of them are fractions?

To review, \((5)\left(\frac{2}{3}\right)\) is 5 repetitions of \(\frac{2}{3}\), or \(\frac{2}{3}\) of 5. Either way you’d like to think about it, you can figure out that \((5)\left(\frac{2}{3}\right) = \frac{10}{3}\), which can be simplified to \(3\frac{1}{3}\).

What about \(\left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\)? In words, this is the question “What is one third of one fourth?” OR, “What is one fourth of one third?” The answers to those questions will be the same, because it doesn’t matter what order you multiply numbers.

38. What do you think a third of a fourth is? How big is each piece if you cut \(\frac{1}{4}\) into three equal pieces?

39. The number one fourth \(\left(\frac{1}{4}\right)\) is drawn as an arrow on the next number line.

a. Imagine a third of that arrow (you cut it into three pieces, each piece is a third of the arrow) … what number would that represent? **Draw a third of the arrow and come up with your answer to “what is a third of a fourth?”**

b. Here is a picture of the number one third \(\left(\frac{1}{3}\right)\), shown as an arrow on a number line. Imagine cutting that arrow into four pieces. What number would each piece represent? **Draw a fourth of this arrow and answer the question “what is a fourth of one third?”**

c. Based on what you wrote above, what do you think about \(\left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\)? It means “a third of a fourth” and, “a fourth of a third”, and “a third divided by 4”
and, “a fourth divided by 3”. With all those meanings rattling around in your head, how do you simplify it? Write your ideas here:

d. Before you read any further, if you can, write a general rule for multiplying unit fractions to each other. For example, what is \( \frac{1}{b} \cdot \frac{1}{c} \), written as one fraction?

Do you remember that one third of something is the same as that thing divided by 3? You know this because you already understand that \( \frac{T}{B} = T \cdot \frac{1}{B} \). A third of a fifth is a fifth divided by 3. Dividing by 3 is the same as finding a third of something. And remember too that it doesn’t matter what order you multiply numbers. This means that the following is true:

\[
\frac{\left( \frac{1}{5} \right)}{3} = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}
\]

AND

\[
\frac{\left( \frac{1}{5} \right)}{3} = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}
\]

is the same as saying any of the following

“If I cut \( \frac{1}{5} \) into 3 equal pieces, each piece will be \( \frac{1}{15} \).”

“One third of \( \frac{1}{5} \) is \( \frac{1}{15} \).”

“If I cut \( \frac{1}{3} \) into 5 equal pieces, each piece will be \( \frac{1}{15} \).”

“One fifth of \( \frac{1}{3} \) is \( \frac{1}{15} \).”

(Remember, \( T \cdot \frac{1}{B} \) means \( T \) times \( \frac{1}{B} \), as opposed to the similar notation with digits: \( 7 \frac{3}{8} \) means \( 7 \ plus \frac{3}{8} \).)

40. Given that \( \frac{T}{B} = T \cdot \frac{1}{B} \), write how you think the expression \( \left( \frac{1}{3} \right) \cdot \frac{1}{4} \) can be written with only one fraction bar, instead of two. Spend some time thinking about it. In this case, \( T = \left( \frac{1}{3} \right) \) and \( B = 4 \).

It turns out that \( \left( \frac{1}{3} \right) \cdot \frac{1}{4} = \left( \frac{1}{3} \right) \cdot \frac{1}{4} \) because dividing by 4 is the same as multiplying by \( \frac{1}{4} \).

In general, this is always true (but remember you can’t divide by zero):

\[
\frac{\left( \frac{T}{C} \right)}{B} = \frac{T}{B \cdot C}
\]

Take a moment and make sure that the above statement makes sense to you. It should NOT be something that you “memorize”, rather, it should be a statement that you understand and can explain, and that you can use to help you simplify expressions.
41. If b and d are not 0, is the following always true? \( \left( \frac{a}{b} \right) \left( \frac{c}{d} \right) = \frac{(a \cdot c)}{(b \cdot d)} \)

Explain. (Hint: you could write each fraction as the numerator times a unit fraction. \( \frac{a}{b} \) can be rewritten as \( a \left( \frac{1}{b} \right) \) (meaning, \( a \) times \( \frac{1}{b} \)). Also, it never matters what order you multiply numbers.

42. What is two thirds of 1 fifth?

43. What is half of 6 fifths?

44. What is 3 fifths of a half?

45. What is 7 times two thirds of three quarters?

46. What is 3 halves times 4 fifths?

STOP: Hooray! You should now understand and be able to make sense of what it means to multiply fractions, not just how to multiply fractions. Congratulations!

**Equivalent Fractions, And, Weird Ways to Write the Number 1.**

For this new section, you’ll start right in answering some questions:

47. How many 5’s fit into 5?

48. What part of 5 fits into 5?

49. If you cut 5 into 5 equal pieces, how big is each piece?

50. What is \( \frac{1}{5} \) of 5?

51. What is \( \frac{1}{5} \), repeated 5 times?

Anything divided by itself is a weird way to write the number 1.

**Adding and Subtracting fractions: making sense of multiplying by weird forms of the number 1.**

If I add 2 fourths to 3 fourths, I get 5 fourths, right? (just like putting 2 lampshades together with 3 lampshades will get me 5 lampshades).

What about adding 2 fifths and 3 fourths? The “2” and the “3” don’t refer to the same things, so I can’t combine them (like putting 2 lampshades together with 3 chairs can’t really be described in a simpler way).

Here’s a question for you: what is one of something? One x is 1x. One 5 is (1)(5), right? One of anything is just itself. So, if I multiply 1 to something, I end up with what I started with.
Suppose I really really want to simplify this expression:

\[
\frac{2}{5} + \frac{3}{4}
\]

\(\frac{2}{5}\) times 1 would be equal to \(\frac{2}{5}\), right?

And

\(\frac{3}{4}\) times 1 is equal to \(\frac{3}{4}\), right?

So, \(\frac{2}{5} + \frac{3}{4} = \frac{2}{5}(1) + \frac{3}{4}(1)\)

OK, so what? I’m leading you down a slow road here for a reason, so stay with me and monitor your thinking to make sure you are following along. It’s important that you don’t just skim.

What is \(\frac{5}{5}\)? What is \(\frac{4}{4}\)? (How many fives fit into five? How many fours fit into four? How big is each piece if you cut 5 into 5 equal pieces?)

Anything divided by itself is a weird way to write the number 1.

If that is true, then it must be true that

\[
\frac{2}{5} \cdot \frac{4}{4} + \frac{3}{4} \cdot \frac{5}{5} = \frac{2}{5} + \frac{3}{4}
\]

It’s also true that

\[
\frac{2}{5} \cdot \frac{1000}{1000} + \frac{3}{4} \cdot \frac{88}{88} = \frac{2}{5} + \frac{3}{4}
\]

But the left hand side of that second equation is not helpful in simplifying the right hand side.

This, however, is helpful:

\[
\frac{2}{5} \cdot \frac{4}{4} + \frac{3}{4} \cdot \frac{5}{5} = \frac{8}{20} + \frac{15}{20}
\]

Now, I can combine those fractions, because 8 twentieths plus 15 twentieths is 23 twentieths.

So, it turns out that if I really wanted to write the combination of \(\frac{2}{5} + \frac{3}{4}\) as one fraction, I could, and it would be \(\frac{23}{20}\), and I figured that out by multiplying each fraction \(\frac{2}{5}\) and \(\frac{3}{4}\) with a weird form of the number 1, so that the fractions I am combining have the same denominator. (The denominator means “the bottom of a fraction”). Some mathematicians call re-writing fractions so they have the same denominator “finding a common denominator”, although it might be more helpful if we called it “finding the same denominator”. In this case, the word “common” is used similarly to how we use it when we say “I have a lot in common with my sister”, as in “the same as”.

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Here’s a summary:

\[
\frac{A}{B} + \frac{C}{D} = \frac{A+D}{B+D} + \frac{C-B}{D-B}
\]

Ok… that’s a lot of letters jumbled together, but take your time till you understand completely what that means, and why it’s useful.

Example:

Express the combination of \(\frac{2}{3}\) and \(\frac{4}{5}\) as a fraction (in other words, simplify \(\frac{2}{3} + \frac{4}{5}\))

\[
\frac{2}{3} + \frac{4}{5} = \frac{2 \cdot 5}{3 \cdot 5} + \frac{4 \cdot 3}{5 \cdot 3}
\]

I multiplied each fraction by a form of the number 1, so they both have the same denominator (bottom number)

\[
\frac{2 \cdot 5}{3 \cdot 5} + \frac{4 \cdot 3}{5 \cdot 3} = \frac{10}{15} + \frac{12}{15}
\]

10 bananas plus 12 bananas is more simply described as 22 bananas, but 22 fifteenths can be written another way.

\[
\frac{22}{15} = \frac{15 + 7}{15}
\]

How many 15’s fit into 22?

\[
\frac{15 + 7}{15} = 1 + \frac{7}{15}
\]

\[
1 + \frac{7}{15} = 1 \frac{7}{15}
\]

The standard way to write 1 + \(\frac{7}{15}\) is 1 \(\frac{7}{15}\).

So, although I can’t really simplify what the combination of 2 pillows and 4 shoes is, I can simplify what 2 thirds combined with 4 fifths is, by writing a number equivalent to 2 thirds and another number equivalent to 4 fifths so that both of the new numbers have the same denominator. I do this by carefully picking a “weird form of the number one” to multiply to the fractions.

**Simplifying fractions: in search of weird forms of the number 1**

To summarize: I can express any fraction in a different way by multiplying it to a weird form of the number one:

\[
\frac{T}{B} = \frac{T}{B} \cdot \frac{A}{A} = \frac{T \cdot A}{B \cdot A}
\]

“A” can be any number except 0, because asking “how many zeros fit into something” doesn’t make sense.

So, for example,

\[
\frac{3}{4} = \frac{15}{20}, \text{ because } \frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5}.
\]

Can we “go backwards” with this process? In other words, how do I get from \(\frac{15}{20}\) to \(\frac{3}{4}\)?

I need to figure out how many “weird forms of the number one” are factors of the fraction \(\frac{15}{20}\).
• **factor** is a number multiplied to another number

• **To factor** is to write a number as numbers multiplied together.

Getting from $\frac{15}{20}$ to $\frac{3}{4}$ is called “simplifying the fraction”, or “reducing the fraction”.

The term “reducing” is a little misleading, because you aren’t making the number smaller at all, you are writing it in a different, equivalent, form. Just like $3+1$ is equivalent to $2+2$ because both expressions simplify to $4$; they are mathematically equivalent. The fractions $\frac{15}{20}$ and $\frac{3}{4}$ do NOT look at all like the same number, but it turns out they are, exactly, the same size! They would be the same point on a number line, or the same arrow on a number line. Looked at another way, if I repeat $\frac{1}{20}$ fifteen times (15 times $\frac{1}{20}$), I’ll end up at the same spot on a number line as I would if I repeated $\frac{1}{4}$ three times (3 times $\frac{1}{4}$).

Ok, then, how do I simplify fractions? It involves factoring both the top number (the numerator) and the bottom of the fraction (the denominator). Remember, to factor a number (or expression) means to express it as other numbers (or expressions) multiplied to each other. I could factor 20 by writing (2)(10). I could also factor it by writing (4)(5).

Understanding factors helps in dealing with fractions.

Example:

Simplify $\frac{15}{20}$

\[
\frac{15}{20} = \frac{3\times5}{5\times4}
\]

I notice that I have a “5” on the top and a “5” on the bottom. What is $5$ divided by $5$?

\[
\frac{3\times5}{5\times4} = \frac{3}{4} \times 1
\]

1 times $\frac{3}{4}$, of course, just $\frac{3}{4}$. (Remember, this is NOT $1\frac{3}{4}$. That would mean $1$ and $\frac{3}{4}$, or $1 + \frac{3}{4}$.

\[
\frac{15}{20} = \frac{3}{4}
\]

Before you read any further, it’s important that you review the ways to look at fractions you’ve played with so far in this chapter. If you cannot explain the reasoning behind all of the statements below, STOP and go back.

$\frac{T}{B}$ means $T$ divided by $B$.

$\frac{T}{B}$ could be used to answer these questions:

“How many $B$’s fit into $T$?”
“How big is each piece if you cut $T$ into $B$ equal pieces?”

$$\frac{T}{B} = (T) \left(\frac{1}{B}\right),$$
and this could mean “repeat $\frac{1}{B}$, $T$ times”, or “one $B$th of $T$” (as long as $B$ is not zero).

$$\left(\frac{a}{b}\right) \left(\frac{c}{d}\right) = \frac{ac}{bd} \quad \text{(as long as $b$ and $d$ are not zero)}$$

$$\left(\frac{a}{b}\right) \cdot \left(\frac{1}{c}\right) \cdot \left(\frac{T}{B}\right)$$

Adding or subtracting fractions is sometimes like adding or subtracting an amount of apples to an amount of apples, and sometimes like adding or subtracting apples and bananas, depending on whether the denominators are the same (called “common denominators”). If you are combining fractions with different denominators, find equivalent fractions that have the same denominator by multiplying by weird forms of the number 1.

Simplify fractions by factoring the numerator (top) and denominator (bottom) and finding weird forms of the number 1.

Exercises:

Simplify, or answer the specific question. If your answer is a fraction, express your fraction in simplified form. If your answer is greater than the number 1, express your answer as a number and a simplified fraction. Try to be strategic in how you solve the problems … Think about your plan before you begin, try to find the simplest, most straightforward way to simplify the expression. The first exercise is done for you, in two different ways.

52. $5 - \frac{2}{3}$

Write $(5) \cdot \left(\frac{2}{3}\right)$, so it is a fraction with the same denominator as $\frac{2}{3}$.

5 is the same as $\frac{15}{3}$, so $(5 - \frac{2}{3}) \text{ is the same as } (\frac{15}{3} - \frac{2}{3})$. And you can simplify $(\frac{15}{3} - \frac{2}{3})$ in the same way you could simplify 15 bananas minus 2 bananas. Furthermore, $\frac{13}{3}$ can be simplified to 4 and $\frac{1}{3}$, or $4\frac{1}{3}$.

OR

I could think that if I’m taking away $\frac{2}{3}$ from 5, then I’ll be left with 4 plus $\frac{1}{3}$, or $4\frac{1}{3}$. STOP! Think about that… get it?

53. $\frac{2}{3} - \frac{3}{3}$

54. $\frac{2}{3} - \frac{3}{5}$
55. $\frac{-2}{3} + \frac{3}{3}
56. \frac{-2}{3} - \frac{-3}{3}
57. \frac{-2}{3} - \frac{-3}{5}
58. \frac{2}{3} - 1
59. \frac{-2}{3} - 2
60. \frac{2}{-3} - 2
61. -2 - \frac{-2}{5}
62. -2 - \left(\frac{\frac{2}{5} \cdot \frac{3}{4}}{}\right)
63. -\left(\frac{\frac{2}{5} \cdot \frac{3}{4}}{}\right) - \left(\frac{\frac{2}{5} \cdot \frac{3}{4}}{}\right) (\text{consider several ways to simplify this BEFORE you begin})
64. \left(\frac{\frac{3}{10}}{}\right) - \left(\frac{\frac{2}{5} \cdot \frac{3}{4}}{}\right)
65. \frac{\frac{2}{x} - \frac{3}{x}}{} (\text{write this as an expression with only one fraction bar, meaning as one fraction})
66. \frac{\frac{2}{x} - \frac{3}{x}}{} (\text{write this as one fraction})
67. \frac{\frac{2}{x} - \frac{3x}{5}}{} (\text{write this as one fraction})
68. \frac{\frac{2}{x} - \frac{3}{5x}}{} (\text{write this as one fraction})
69. 2 - \frac{\frac{3}{5x}}{} (\text{write this as one fraction})
70. x - \frac{\frac{3}{5}}{} (\text{write this as one fraction})
71. 2x - \frac{\frac{3x}{5}}{} (\text{write this as one fraction})
72. -\frac{\frac{2}{x} - \frac{-3}{5x}}{} (\text{write this as one fraction})

Before you go on, check all your answers, and I hope you didn’t easily get them all correct (because that means you may be bored) and if you did, then I hope it was because you had to check the solutions a little and THINK! Pay attention to your errors and to where you needed extra help. Summarize what you learned, and consider this question:

*How will you get to the point where you can do all the exercises without looking at the solutions?*
Use this space to make notes of your errors and where you had to look up solutions. We learn by examining and attending to our errors; this is the most important part! Do NOT ignore your mistakes… scrutinize and learn from them.

Also, be proud of when you worked through complicated ideas.

I hope you learned:

- How to think about any fraction as some repetitions of unit fractions
- How to make sense of fraction multiplication by thinking about
  - unit fractions and
  - the fact that it never matters what order you multiply numbers and
  - the connection between multiplication by a unit fraction and division: $\frac{1}{4}$ of something is the same as that thing divided by 4. So, $\frac{1}{4}$ of $\frac{1}{3}$ is $\frac{1}{12}$
- How to create and recognize equivalent fractions by understanding and recognizing weird forms of the number one.
- How to understand adding and subtracting fractions by 1) thinking about and using weird forms of the number one and 2) understanding that some combinations can be simplified, and some cannot (see Chapter 2, “Addition”)
Chapter 10: Fractions, part 2: Dividing Fractions

In which you will learn how to make sense of dividing fractions, in a way you can explain and re-remember whenever it comes up, for the rest of your life.

Because you have really good ways of thinking about fractions and division available to you:

_Fractions, Part 2 Division_

_First, consider these big concepts:_

1) Any fraction can be thought of as a number times a unit fraction. \( \frac{3}{5} \) is 3 repetitions of \( \frac{1}{5} \)
2) You know that it never matters what order you multiply numbers
3) Division, as in \( \frac{T}{B} \), means
   a. How many B fit into T, OR
   b. The part of B that fits into T OR
   c. The size of each piece if you cut T up into B equal pieces OR
   d. The ratio of T to B
4) You can multiply by the “weird forms of the number 1”

_Dividing fractions:_

Dividing fractions is the beginning of the end for many students in math. If this was the case for you, I apologize, it shouldn’t have been, and it needn’t have been; you can and will make sense of fractions and what it means to divide them, and you will find
that you are able to explain fractions (and division, and ratios and percent) to your peers, your friends and neighbors! Now, THAT, is exciting, right?

Suppose T and/or B are fractions! Yikes!

What is \( \frac{\frac{4}{9}}{\frac{5}{7}} \)? How can that messy thing be simplified? What does it even mean, anyway?

There are many ways to understand how to do this and what it means, and students are often shown different methods, and then practice them until they can effectively (or not) repeat them on some tests. In fact, there is a silly "ditty", the author of which I do not know, that can be used to help students remember the procedure for dividing fractions:

\[ \text{Ours not to reason why,} \]
\[ \text{Just invert and multiply.} \]

However, it turns out that being able to "reason why" is essential to mathematical thinking and being able to use math to solve problems. So… how do you divide fractions? 1

First, calmly think about what you know about division. Go back and look at the beginning of this chapter.

Ok, every fraction, \( \frac{T}{B} \), can be written as \( T \) times the unit fraction \( \frac{1}{B} \)

\[ \frac{T}{B} = T \cdot \frac{1}{B} \]

How is this helpful in dividing fractions? Maybe it will be… keep going.

So, before you do anything else with a fraction, you can express what you have in a different, but equivalent, way. For example:

I can re-write \( \frac{\frac{4}{9}}{\frac{5}{7}} \)

\[ \frac{\frac{4}{9}}{\frac{5}{7}} = \frac{4 \cdot \frac{1}{9}}{5 \cdot \frac{1}{7}} \]

Now, realize that 1) it doesn’t matter in what order you multiply those numbers, and 2) dividing by 5 is the same as multiplying by \( \frac{1}{5} \). So, you can further “break-apart” this expression:

\[ 4 \cdot \left( \frac{\frac{1}{5}}{\frac{5}{7}} \right) \cdot \frac{\frac{1}{3}}{\frac{5}{7}} \]

Dividing by 5 is the same as multiplying by \( \frac{1}{5} \), remember? dividing \( \frac{\frac{1}{3}}{\frac{5}{7}} \) is the same a multiplying by \( \frac{\frac{1}{3}}{\frac{5}{7}} \)

---

1 “keep, change, flip: that’s the action; everybody knows when dividing fractions” is another little ditty that doesn’t help students make sense of what they are doing.
Exercises:

1. Show why $\frac{4}{9}$ and $4 \cdot \frac{1}{9} \cdot \frac{1}{5} \cdot \frac{1}{7}$ are equivalent expressions. Try to do this without looking back at the text, but it’s ok if you do.

2. How many halves fit into 1? In other words, how many half-cookies are there in 1 cookie?

3. How many thirds fit into 1? In other words, how many third-pizzas fit into a pizza?

4. How many halves fit into 5? In other words, how many half-cookies are there in 5 cookies?

5. Simplify each of the following based on your answers to the questions above:
   
   a. $\frac{1}{2}$
   
   b. $\frac{1}{3}$
   
   c. $\frac{5}{7}$

6. Based on what you wrote above, what is a “rule” you could use for dividing a number by a unit fraction? In other words, what is a rule you could use to simplify the following expression (as long as $B \neq 0$)

$$\frac{A}{\frac{1}{B}}$$

Back to the example I started this chapter with, how do I simplify $\frac{4}{9} \cdot \frac{1}{5} \cdot \frac{1}{7}$ so that I can wrap my head around what it might actually mean?

I wrote this as $4 \cdot \frac{1}{9} \cdot \frac{1}{5} \cdot \frac{1}{7}$

Now, you understand that $\frac{1}{7}$ fits into one, 7 times. So we know that $\frac{1}{7} = 7$. So, the expression $4 \cdot \frac{1}{9} \cdot \frac{1}{5} \cdot \frac{1}{7}$ can be re-written as $4 \cdot \frac{1}{9} \cdot \frac{1}{5} \cdot 7$

And, that is an expression you know how to simplify!

$$4 \cdot \frac{1}{9} \cdot \frac{1}{5} \cdot 7 = \frac{28}{45}$$

Example: write this expression so it is not a fraction divided by another fraction:
First, I’m going to “break-apart” each of those fractions

\[
\frac{\frac{A}{B}}{\frac{C}{D}} = \left(\frac{A}{B}\right) \left(\frac{1}{C}\right) \left(\frac{1}{D}\right)
\]

Next, I’m going to write every division as a multiplication.

Remember, dividing by C is the same as multiplying by \(\frac{1}{C}\), and you can make sense of this by thinking “Half of something is the same as that thing divided by 2”.

\[
\frac{\left(\frac{A}{B}\right)}{\left(\frac{1}{C}\right)} \left(\frac{1}{D}\right) = \left(\frac{A}{B}\right) \left(\frac{1}{C}\right) \left(\frac{1}{D}\right)
\]

Next, I realize that \(\frac{1}{D}\) is the same as D. You can make sense of this by thinking “Two halves fit into 1, so \(\frac{1}{2} \times 2 = 2\)”

\[
\left(\frac{1}{C}\right) \left(\frac{1}{D}\right) = \left(\frac{A}{B}\right) \left(\frac{D}{C}\right)
\]

Finally, we get to:

\[
\frac{\frac{A}{B}}{\frac{C}{D}} = \left(\frac{A}{B}\right) \left(\frac{D}{C}\right)
\]

You can work out this “rule” by “breaking apart fractions” and re-writing the expressions.

Mathematical “rules” are there to help you solve problems efficiently, not to cause stress and anxiety by making you think you have so much to memorize:

\[\text{Ours not to reason why,}\]
\[\text{Just invert and multiply.}\]

You ARE here to “reason why”; **personally, I feel insulted when I am told I don’t need to reason why**. Being encouraged to reason is empowering, and I do not like to hang out with people who are not empowering.

Whew! Ok… let’s look at the same question, and look at it another way. Maybe you hated what you just read and your head is spinning and it was too different from anything you’ve seen before.

Try this: \(\frac{\frac{4}{5}}{\frac{7}{7}}\) is like asking “what is the ratio of \(\frac{4}{9}\) to \(\frac{5}{7}\)?” So, simplifying \(\frac{\frac{4}{5}}{\frac{7}{7}}\) is like finding a fraction that is equivalent.

You know all about equivalent fractions from learning about “weird forms of number 1”… So, what weird forms of number 1 could I multiply to the horrible mess \(\frac{\frac{4}{5}}{\frac{7}{7}}\), so that it’s not so horrible?
If I make the numerator, \( \frac{4}{9} \), 9 times bigger, it won’t be a fraction any more, but to keep the ratio the same, I have to make the denominator 9 times bigger too. The weird form of 1 that I’m going to multiply is \( \frac{9}{9} \)

\[
\left( \frac{4}{9} \right) \cdot \frac{9}{9} = \frac{4 \cdot 1}{9}
\]

Exercise:

7. Explain why \( \left( \frac{4}{9} \right) \cdot \frac{9}{9} \) is equivalent to \( \frac{4}{9} \cdot \frac{1}{9} \). Feel free to refer to the text, above.

\( \frac{4}{9} \cdot \frac{1}{9} \) still isn’t simplified. I still have an ugly mess, not a nice ratio of two numbers. But… if the denominator gets 7 times bigger, it won’t be a fraction any more. So….

If I make the numerator and the denominator 7 times bigger, I’ll have an equivalent fraction that isn’t so ugly:

\[
\left( \frac{4}{9} \right) \cdot \frac{1}{9} = \frac{4 \cdot 7}{9} \cdot \frac{1}{9}
\]

\( \frac{4 \cdot 7}{9} \cdot \frac{1}{9} \) simplifies to \( \frac{4 \cdot 7}{9} \cdot \frac{1}{9} \)

Exercise:

8. Explain why \( \frac{4 \cdot 7}{9} \cdot \frac{1}{9} \) simplifies to \( \frac{4 \cdot 7}{5} \cdot \frac{1}{9} \)

9. Simplify the expression: \( \frac{4}{9} \) any way you like, but without using a calculator

Example:

I have 3 and a half cups of yoghurt. A serving is 2 thirds of a cup. How many servings do I have?

This question could be re-phrased: how many 2 thirds fit into 3 and a half? And this type of question can be answered by simplifying this division problem:

\[
\frac{3 + \frac{1}{2}}{\frac{2}{3}}
\]

First, I stare at that for a while. What will my strategy be? Will I first combine the 3 and the \( \frac{1}{2} \), and then divide that by \( \frac{2}{3} \)? Or, will I divide the 3 by the \( \frac{2}{3} \) and add the result
to $\frac{1}{2}$ divided by $\frac{2}{3}$? Based on my experience, I can’t see an advantage one way or the other, so I’m going to combine the 3 and the $\frac{1}{2}$, and then divide the result by $\frac{2}{3}$.

$$3 + \frac{1}{2} \frac{2}{3} = \frac{3 \cdot \frac{2}{2} + \frac{1}{2}}{\frac{2}{3}}$$

$$\frac{3 \cdot \frac{2}{2} + \frac{1}{2}}{\frac{2}{3}} = \frac{\frac{6}{2} + \frac{1}{2}}{\frac{2}{3}}$$

6 halves plus 1 half is 7 halves

$$\frac{\frac{6}{2} + \frac{1}{2}}{\frac{2}{3}} = \frac{\frac{7}{2}}{\frac{2}{3}}$$

Now, I can break up the fraction division into more easily manageable “bites”

$$\frac{\frac{7}{2}}{\frac{2}{3}} = \frac{7 \cdot \frac{1}{2}}{\frac{2}{3}}$$

“How many thirds fit into something, is 3 times that thing” Does this make sense?

$$\frac{7 \cdot \frac{1}{2}}{\frac{2}{3}} = \frac{7 \cdot \frac{1}{2}}{2 \cdot \frac{3}{2}} \cdot 3$$

Dividing by $\frac{2}{3}$ is the same as multiplying by 3.

$$\frac{7 \cdot \frac{1}{2}}{2 \cdot \frac{3}{2}} = 7 \cdot \frac{1}{2} \cdot 3$$

It doesn’t matter what order you multiply numbers!

And dividing by 2 is the same as multiplying by $\frac{1}{2}$.

\[ \text{This is another approach to this exercise: } \frac{\frac{3 + \frac{1}{2}}{\frac{2}{3}}}{\frac{2}{3}} = \frac{3}{\frac{2}{3}} + \frac{\frac{1}{2}}{\frac{2}{3}} \]
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\[ 7 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 3 = \frac{21}{4} \]

This can be simplified because there are a whole bunch of 4’s that fit into 21.

\[ \frac{21}{4} = \frac{(4)(5) + 1}{4} \]

\[ \frac{(4)(5) + 1}{4} = 5 + \frac{1}{4} \]

Bottom line is: There are \(5\frac{1}{4}\) servings of yoghurt in 3 and a half cups, if a serving is \(\frac{2}{3}\) thirds of a cup.

**Example (yes, it’s the same example … look at it from another point of view):**

I have 3 and a half cups of yoghurt. A serving is \(\frac{2}{3}\) thirds of a cup. How many servings do I have?

This question could be re-phrased: how many \(\frac{2}{3}\) thirds fit into 3 and a half? And this type of question can be answered by simplifying this division problem:

\[ \frac{3 + \frac{1}{2}}{\frac{2}{3}} \]

You should be having that “already been there, done that” feeling right now, because this is the same as the previous question. What will my strategy be? I’m going to combine the 3 and the \(\frac{1}{2}\), and then divide the result by \(\frac{2}{3}\).

\[ \frac{3 + \frac{1}{2}}{\frac{2}{3}} = \frac{3 \cdot \frac{2}{3} + \frac{1}{2}}{\frac{2}{3}} \]

\[ \frac{3 \cdot \frac{2}{3} + \frac{1}{2}}{\frac{2}{3}} = \frac{6 + \frac{1}{2}}{\frac{2}{3}} \]

6 halves plus 1 half is 7 halves

\[ \frac{\frac{6}{2} + \frac{1}{2}}{\frac{2}{3}} = \frac{\frac{7}{2}}{\frac{2}{3}} \]

So far, this is exactly what I did before, but now I’m going to combine some steps, this time I’m going to
re-write \( \frac{2}{3} \) as \( 2 \cdot \frac{1}{3} \) and leave everything else the same.

\[
\frac{\frac{7}{2}}{\frac{2}{3}} = \frac{\frac{7}{2}}{2 \cdot \frac{1}{3}}
\]

I know that dividing by \( \frac{1}{3} \) is the same as multiplying by 3, and I’m also dividing by 2 which is the same as taking half of something (or multiplying by \( \frac{1}{2} \)).

\[
\frac{\frac{7}{2}}{2 \cdot \frac{1}{3}} = \frac{\frac{7}{2}}{\frac{1}{2} \cdot 3}
\]

There are no “weird forms of 1” here… so I cannot simplify this fraction by writing it with a different denominator (denominator means the bottom number).

\[
7 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 3 = \frac{21}{4}
\]

Example (yes, it’s the same example … look at it from yet another point of view):

I have 3 and a half cups of yoghurt. A serving is 2 thirds of a cup. How many servings do I have?

3 and a half is equivalent to \( \frac{7}{2} \). So this question is like asking, what is the ratio of \( \frac{7}{2} \) to \( \frac{2}{3} \)?

I need to find an equivalent fraction, by multiplying by weird forms of number 1. The idea is to get rid of the fraction in the numerator, and to get rid of the fraction in the denominator. If I double the numerator and the denominator, I’ll have an equivalent fraction, but the numerator will be a lot easier to deal with:
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\[
\frac{7}{2} \cdot \frac{2}{3} \cdot 2 = \frac{7}{2} \cdot \frac{2}{3} \cdot 2
\]

Exercise:

10. Explain why \( \frac{7}{2} \cdot \frac{2}{3} \cdot 2 = \frac{7}{2} \cdot \frac{2}{3} \cdot 2 \)

It’s still kinda messy, so… if I triple the denominator, it won’t be a fraction any more. So… I need to triple the numerator and the denominator:

\[
\frac{7\cdot3}{2\cdot3\cdot2} = \frac{7\cdot3}{2\cdot2}
\]

\[
\frac{7\cdot3}{2\cdot2} = \frac{21}{4}
\]

Exercise:

11. Explain why \( \frac{21}{4} \) is equivalent to 5 and a quarter

12. How many servings (including fractions of a serving) are there in 3 and a half cups of yoghurt, if a serving is 2 thirds of a cup?

13. What is 2 thirds of 3 and a half cups of yoghurt?

14. If I have 3 and a half cups of yoghurt, and you have 2 thirds of a cup, what’s the ratio of what I have to what you have?

15. If I have 3 and a half cups of yoghurt, and you have 2 thirds of a cup, what do we have combined?

16. If I have 3 and a half cups of yoghurt, and you have 2 thirds of a cup, what’s the difference between what I have and what you have?

17. If I have 3 and a half cups of yoghurt, and you have 2 thirds of a cup, how much more do you need to get so that you would have same amount as I do?

18. What is 3 and a half times more than 2 thirds of a cup?

19. If I have 3 and a half cups of yoghurt, and you have 2 thirds of a cup, what’s the ratio of what you have to what I have?

20. Fill in the blank: 7 divided by 2 is the same as 7 times ____________

21. Fill in the blank: The number of thirds that fit into 9 is the same as 9 times ____________

22. Fill in the blank: The number of threes that fit into 9 is the same as 9 times ____________

23. What is “Nine divided into thirds”?

24. My share of Great Aunt Gertrude’s estate is a third of the total value of the estate. My uncle’s share of the estate is 2 thirds of the total value of the estate. My uncle’s share is how many times bigger than mine? (how many of one share fit into another share?)
25. My share of Great Aunt Sophie’s estate is $\frac{2}{7}$ of the total value of the estate. My uncle’s share of the estate is $\frac{3}{5}$ of the total. My uncle’s share is how many times bigger than mine? (how many of one share fit into another share?)

26. What part of, or how many, $\frac{2}{3}$ fits into $\frac{3}{7}$? Before you do any calculations, ask yourself if the answer should be “greater than one $\frac{2}{3}$ fits into $\frac{3}{7}$” or “only a part of $\frac{2}{3}$ will fit into $\frac{3}{7}$.” In other words, can you figure out which is bigger, $\frac{2}{3}$ or $\frac{3}{7}$, without doing any calculations? (hint: you should be able to).

27. What is $\frac{2x}{3y}$ divided by $\frac{3}{7}$?

28. Summarize how you will remember, for the rest of your life, how to figure out how to divide fractions if you forget the procedure.

29. 
   a. Is memorizing a procedure different from working out solutions that make sense?
   
   b. What are the advantages of memorizing, what are the drawbacks?

30. What is half of two thirds?

31. What is two thirds divided by 2?

32. Simplify: $\frac{3+\frac{1}{2}}{3}$

33. Putting together all you know about multiplication, division, fractions, adding, and subtracting, answer the following questions that refer to the unknown numbers marked on this number line. Remember, negative 1,000 is a smaller number than 5, simply because it is negative.

   a. Which is bigger, $f$ or $(f)(e)$? What is your reasoning, how do you know?
   
   b. Which is bigger, $g$ or $(g)(h)$? What is your reasoning, how do you know?
   
   c. Which is bigger, $f$ or $(f)(g)$? What is your reasoning, how do you know?
   
   d. Which is bigger, $g$ or $(g)(d)$? What is your reasoning, how do you know?
   
   e. Which is bigger, $g$ or $(g)(b)$? What is your reasoning, how do you know?
   
   f. Which is bigger, $(a)$ or $(a)(b)$? What is your reasoning, how do you know?
   
   g. What is bigger, $(a)(b)$ or $(f)$? What is your reasoning, how do you know?
h. What is bigger, (b)(c) or (b)(d)? What is your reasoning, how do you know?

i. What is bigger, (b)(e) or (b)(f)? What is your reasoning, how do you know?

j. What is bigger, -b or (c)(d)? What is your reasoning, how do you know?

k. What is bigger, \( \frac{f}{e} \) or \( \frac{g}{h} \)? What is your reasoning, how do you know?

l. What is bigger, \( \frac{c}{d} \) or \( b \)? What is your reasoning, how do you know?

m. What is bigger, \( \frac{c}{d} \) or \( -b \)? What is your reasoning, how do you know?

n. Of all the labeled numbers, which one is closest to (a)(b)? What is your reasoning, how do you know?

I hope you learned how to make sense of dividing fractions, in a way you can explain and re-remember whenever it comes up, for the rest of your life, not just “procedurally” but in a way that makes sense and is connected to reasoning.

**Remember these big concepts:**

1. Any fraction can be thought of as a number times a unit fraction. \( \frac{3}{5} \) is 3 repetitions of \( \frac{1}{5} \).

2. You know that it never matters what order you multiply numbers.

3. Division, as in \( \frac{T}{B} \), means
   
   a. How many B fit into T,  
   OR
   
   b. The part of B that fits into T
   OR
   
   c. The size of each piece if you cut T up into B equal pieces
   OR
   
   d. The ratio of T to B

4. You can multiply “weird forms of the number 1”
Chapter 11: What is an Equal Sign: Equivalent Equations Revisited

In which you will (re)-learn:

- Once you have created an equation that describes a situation, you might make useful discoveries and statements about that same situation if you “do the same thing to both sides of the equal sign.”
- Finding a solution to an equation means finding the values of any variables in that equation that make the equation a true statement.
  - Some equations have no solutions
  - Some equations have many solutions

What is an Equal Sign? Part 2: Equivalent equations

In the chapter “What is an Equal sign: equivalent expressions”, you learned that even if the scenario, picture, procedures, or questions you are answering are distinct (meaning they are all different), a mathematical expression is equivalent to another expression if they can be simplified to the same number, and a number is equivalent to another number regardless of how you are visualizing it: arrow, point on number line, number of repetitions.

So, $9 - 8$ is equivalent to $1500 - 1499$, and is also equivalent to $-2+3$, (even though the questions “what’s the difference between $9$ and $8$” and “what’s the difference between fifteen thousand dollars and fourteen thousand nine hundred ninety-nine dollars?” and “what do you end up with if you combine a $2$ debt and $3$?” feel like very different questions to some of us… if the answers to the questions can be simplified to the same number, the expressions are equivalent).

Examples:

$9 - 8 = 15,000 - 14,999$. A possible way to describe this equation in words is: “the difference between 9 and 8 is the same as the difference between 15,000 and 14,999, because they both simplify to 1.

And $\frac{126}{6} = \frac{42}{2}$. A possible way to describe this equation in words is: “the number of repetitions of 6 needed to get to 126 can be represented by the same number as each piece if you cut 42 into 2 pieces, namely, 21.”

Solutions to Equations
An equation is a statement that one thing is the same as another thing. As an example, consider the equation $x + 3 = 10$, which is simply the statement that “if I combine $x$ and 3, I’ll get 10”.

If an equation has variables (letters that stand for unknowns), then finding the values of those variables that make the equation true is called “finding the solution to that equation”.

In the above example, the solution to $x + 3 = 10$ is $x = 7$, because if you replace $x$ with 7, you get $7 + 3 = 10$, which is a true statement.

Sometimes, you can find the solutions to equations simply by considering the equation and asking yourself what values of the variables would make the equation true, and sometimes, you get to do some algebra to solve equations. How exciting!

For example, consider this equation:

$$(x)(x) = 25$$

You might know how to find the solutions to this equation through various algebraic techniques, but it might also be faster to ask yourself “what are all the values of $x$ that could make that equation true? Find all the numbers that make this statement true: “this number, if multiplied to itself, will give you the number 25”

For that equation, there are two solutions: $x = -5$ and $x = 5$. You could get to that solution without doing any formal algebra.

**Exercises:**

Find the solutions to the following equations without doing any formal algebra… consider what the equation means and then find the solution.

1. $(x)(x)(x)=1$ (in words: what number, when multiplied to itself 3 times, is 1)
2. $(x)(x)=1$ (in words: what number, when multiplied to itself twice, is 1)
3. $10-x=8$ (If you remove this number from 10, you’d get 8. Or, the difference between 10 and this number is 8)
4. $2x=10$ (in words: twice this number is 10, or, if you double this number, you’d get 10).

**Keeping Your Balance in Math, or “doing same thing to both sides of the equal sign”**

Suppose you have an equation that you know accurately describes the relationship among some quantities. For example, suppose that when your friend Josimar stands on a bathroom scale holding a suitcase the total weight is 210 pounds. So an equation that describes this situation is this

$J + S = 210 \text{lbs}$, where $J$ stands for Josimar’s weight and $S$ stands for the suitcase’s weight.
If Josimar steps off the scale and leaves the suitcase on the scale, the equation that describes that situation is 
\[ S = 210lbs - J. \]

If Josimar picks up his adorable toddler along with the suitcase (Josimar is a strong dude) and stands on the scale, the situation can be described by this equation: 
\[ J + S + T = 210lbs + T \]

The bottom line is that if \((J + S)\) is added to, subtracted from, divided or multiplied to a number, then the result of doing those operations on \((J + S)\) will be equal to doing the same operations on \(210lbs\).

This leads to a really big deal in Algebra, so I’m going to write it in big bold letters

**If an equation correctly describes a situation, any math operation you do to both sides of the equal sign might result in another equation that could also correctly describe the situation.**

The word “might” is important because some mathematical operations, when done to both sides of an equal sign, will not give you an equivalent equation. You’ll think about some of those kinds of mathematical operations in later chapters, and some in later math classes. This is why you must always check your answer in the original equation.

What this means is that problem-solving and answering questions that involve numbers might be a lot easier than you thought, because creating a true equation is often as straightforward as writing a sentence. Once you’ve written that “mathematical sentence” (also known as an equation), you can “do the same thing to both sides of the equal sign” that might make a less complicated equation that will help you understand the situation better.

However, sometimes, you might introduce “extraneous” solutions, and sometimes, you might end up with nonsense. The point is, you have to always consider each step, and ask yourself “does this make sense given this situation?” before you answer specific questions.

Before you read any further, answer this question any way you can:

I can save $50 every week. How many weeks will it take till I have the $200?

There are many ways to approach solving this problem, and, it is likely that however you did it is the most efficient way for you to solve it (the answer is 4 weeks, by the way).
I will show a way of looking at this problem that might not be the most efficient or comfortable way for you to look at it, but if you can understand the process, then solving other, more complicated problems will be much easier for you. Trust me.

$50 every week for some unknown number of weeks means repeating $50 for some amount of weeks. If I call the unknown number of weeks $w$ then, whatever $w$ is, I know that if I repeat $50$ $w$ times, I will arrive at $200$. Repetition is multiplication, remember?

So the equation that describes that situation is $50w = 200$. It’s important to realize that I create this equation before I have any idea what the number $w$ is. After I create an equation that I think describes the situation, I sort of mentally step back and think about it… does it make sense?

Once I have a true equation, I know that I should try to do something to both sides of the equal sign and get another true equation (as long as I don’t divide by 0) … one that is easier to understand.

If my goal is to find out what $w$ equals, then what I want to do with this equation is divide both sides by 50.

$$\frac{50w}{50} = \frac{200}{50}$$

Simplifying both sides of this equation results in the equation

$$w = 4$$

Since I arrived at the equation $w = 4$ by dividing both sides of a true equation by 50, it must be true that $w = 4$.

Make sense? So, to answer questions with numbers, all you have to do is construct a true equation, then do the same thing to both sides of the equal sign until you have another true equation that is more helpful to you. Then, check to see if your answer makes sense. It might be that you did some math that introduced some solutions that don’t fit your original equation, or it might be that you accidentally tried to divide by zero. Of course, different people will come up with equations that look different, but that describe the same situation, but if some mathematical operation could be done to both sides of one equation that results in the other equation, and the two equations have the same solutions, then the two equations are called equivalent, and they both describe the same situation.

**Equivalent Equation Examples**

The equations $2x + 4 = 5x + 7$ and $4 = 3x + 7$ are equivalent because you can get the second equation by subtracting $2x$ from both sides of the first equation, and the value $x = -1$ is a solution to both of these equations (meaning if you replace $x$ with $-1$ in both of those equation, you’d get a true statement).

The equations

$$\frac{3}{x} = \frac{1}{4} \quad \text{and} \quad 3 = \frac{1}{4} \cdot x$$
are equivalent because you can get the second equation by multiplying both sides of
the first equation by $x$, and they have the same solution. If it is true that $\frac{3}{x} = \frac{1}{4}$, then it
is also true that $3 = \frac{1}{4} \cdot x$. In other words, if the first equation is true, then the second
one must be true as well because some operation was done to the first to get the
second, there was no dividing by zero, and both equations have the same solution
($x = 12$).

Exercises

In the following examples, explain what mathematical operation is done to the first
equation to get to the second equation. (you are NOT necessarily solving for $x$, you’ll get to do that later…). Please check your answers as you go because it’s
important you don’t re-enforce wrong ideas!

5. $3x + 3 = 6$  $3x = 3$
6. $3x + 3 = 6$  $x + 1 = 2$
7. $\frac{3}{x} = 9$  $3 = 9x$
8. $x - 3 = 5$  $x = 8$
9. $x - 3 = 5$  $x - 8 = 0$
10. $3 - x = 5$  $3 = 5 + x$
11. $x_1 + \Delta x = x_2$  $x_1 = x_2 - \Delta x$
   (these two equations show the relationship between addition and “removing what was added”)
12. $x_1 + \Delta x = x_2$  $\Delta x = x_2 - x_1$
   (these two equations show the relationship between addition and “finding the difference”)
13. $\frac{4}{5} = x$  $4 = 5x$
14. $\frac{T}{B} = x$  $\frac{T}{B} - x = 0$
15. $\frac{3}{4x} = 2$  $\frac{3}{x} = 8$

Check all of your answers. If you were unsure of your answer, or didn’t get the same
answer as in the solutions, write what you needed help with here. Write what was
confusing. It’s especially helpful to frame your thinking with a prompt like:

“I used to think …., but now I think ….”
Or,

“I didn’t understand that …, but now I understand why that works ….”

**More on equivalent equations: isolating x**

Much of algebra is devoted to “solving for $x$”, where $x$ is a number you don’t know, but you know a true statement that involves $x$

If you know that, then you also know that any equation (as long as you don’t divide by zero) that you can get to by doing the same mathematical operation to both sides of the equal sign might also be true. If your goal is to figure out what number $x$ is, then you need to think about what you can do to both sides so you get an equation that has $x$ all by itself on one or the other side of the equal sign. This is called “isolating $x$”, or, simply “solving for $x$” or “solving the equation”.

Learning efficient ways to isolate $x$, or solve for $x$, takes a lot of practice, and some types of equations are easier to work with than others. In algebra and precalculus classes you will learn various techniques and procedures, but keep in mind they are often simply variations on doing the same thing to both sides of the equal sign, and as long as you are doing that, you won’t be doing anything wrong (but don’t divide by zero). You might not always be efficient unless you practice a lot, but as long as you “do the same thing to both sides of the equal sign” you are doing ok¹, and probably on the right track.

Exercises: Solve for $x$ in the following equations. Some might take multiple steps. Keep “doing the same thing” to both sides of resulting equations until you have isolated $x$.

16. $x + 3 = 6$
17. $\frac{3}{x} = 9$
18. $x - 3 = -5$
19. $x + 2 = -7$
20. $3 - x = 5$
21. $x_1 + \Delta x = x_2$ (solve for $x_1$)
22. $x_1 + \Delta x = x_2$ (solve for $\Delta x$)
23. $\frac{4}{x} = 5$
24. $\frac{x}{B} = A$

¹ But don’t divide by zero, and always check if the solution you come up with is a solution to the original equation, you may have introduced some extra solutions, or overlooked some, as well.
25. $\frac{3}{4x} = 2$

26. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not
   a. Lucy has twice as much money as Carl
   b. Carl has half as much money as Lucy

27. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not
   a. Roxane weighs 40 pounds more than Fred
   b. If you add 40 to Roxane’s weight, you’ll get Fred’s weight

28. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not
   a. If you divide 2 into $x$ equal pieces, each piece will be exactly 7
   b. If you repeat 2 7 times, you get 2

29. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not
   a. Half of Roxane’s money is 3 times more than Melinda’s money
   b. Melinda’s money is exactly a sixth of Roxane’s money

30. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not
   a. Half of Roxane’s money is 3 times more than Melinda’s money
   b. Roxane’s money is exactly 1 and a half times more than Melinda’s money

31. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not
   a. Half of Roxane’s money is 3 times more than Melinda’s money
   b. Roxane’s money is exactly 6 times more than Melinda’s money
32. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. I started out with $30 under my mattress, and each week I added $10, so after 1 week I had $40, after 2 weeks I had $50, after 3 weeks I had $60, etc.…

b. If you take what I have in my mattress and subtract $30, and then divide by $10, you’ll know how many weeks I’ve been putting money in the mattress.

33. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. Pirio’s car get 30 miles on a gallon of gas. Her car uses one gallon of gas to to 30 miles.

b. Pirio’s car uses 4 gallons of gas to drive 120 miles

34. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. 10 divided by x is exactly the same as y divided by 2

b. y divided by x is exactly 20

35. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. 10 divided by x is exactly the same as y divided by 2

b. x divided by y is exactly 20

36. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. 10 divided by x is exactly the same as y divided by 2

b. x divided by y is exactly 1 twentieth

37. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. 10 divided by x is exactly the same as y divided by 2

b. x times y is exactly 20

38. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. 10 divided by x is exactly the same as y divided by 2

b. x times y is exactly 1 twentieth

Before you read any further, check your solutions with the answers. It would be great if you made many errors, because that means you have some learning to do, and
that you are getting your money’s worth out of the time you are putting into your learning. If you made no errors: good job, as well… you must be working carefully.

If you did make ANY errors, the important thing is for you to change your brain based on what you can learn from those errors.

It is VERY important that you pay attention to every mistake. You might notice your mistakes right away and say to yourself “that’s a little mistake, I do get it”. But, dismissing your mistakes casually is dangerous to your learning. Every mistake is important for you.

Here, write down every problem you did incorrectly, analyze your mistakes, and make a strategy for how to avoid each of those mistakes.
I hope you learned (and re-learned, because some of this material is familiar to you from the chapter: “Equivalent Equations in Addition and Subtraction”)

- Once you have created an equation that describes a situation, you might make useful discoveries and statements about that same situation if you “do the same thing to both sides of the equal sign.”

- Finding a solution to an equation means finding the values of any variables in that equation that make the equation a true statement.
  - Some equations have no solutions
  - Some equations have many solutions
Chapter 12: Making Equations Do all the Work for You

In which you will:

• practice answering quantitative questions by
  1) creating an equation that describes the particular situation and then
  2) manipulating the symbols in that equation, by “doing the same thing to both sides of the equal sign,” to answer specific questions about the situation.
• learn about ratios and proportions and how to use them to answer questions
• practice adding, subtracting, multiplying and dividing with negative numbers and fractions in order to answer specific questions.

How to create equations that do all the work for you.

Think about this equation:

\[ \frac{T}{B}x = y \]

where \( T, B, x \) and \( y \) are some numbers (but \( B \) is not 0). If this equation is true, then any mathematical operation I do to both sides of the equal sign will probably result in another true equation (see the chapter on Equivalent Equations).

Keeping that fact in mind, let’s imagine a hungry dog, Al. Suppose little Al has \( \frac{9}{10} \) cups of dog food, which is just \( \frac{2}{3} \) of his regular serving. What is the size of his regular serving?

One way to approach this is to write an equation that I know is true and describes the situation. I don’t need to worry about finding the answer to the question right away. I just need to know that if I write a true equation, then I can “do the same thing to both sides” so that eventually I will be able to answer the specific question being asked.
Ok, back to hungry Al. We know that has \( \frac{9}{10} \) cups of dog food is \( \frac{2}{3} \) of his regular serving. I don’t know the size of a regular serving, so I’ll use the letter \( s \) to stand for regular serving.

**I know that \( \frac{2}{3} \) of \( s \) is equal to \( \frac{9}{10} \) cups.**

Here’s that sentence using mathematical notation instead of English words:

\[
\frac{2}{3} \cdot s = \frac{9}{10}
\]

If that equation is true, then any math operations I do to both sides will most likely result in another true equation (“most likely” because you can’t divide by zero, and you might also introduce “extra” information if you do something like multiply both sides by some other variable… more about this later).

I want to know what \( s \) equals, right? So, how do I get \( s \) by itself?

I could:

- Divide both sides by \( \frac{2}{3} \) and then simplify
  
  OR

- Multiply both sides by 3 and divide both sides by 2, then simplify
  
  OR

- Divide both sides by 2, multiply by 3, then simplify.

I could also do some random stuff, like add 100, or multiply both sides by 9, but that would not help in getting \( s \) by itself.

**Question:**

Why wouldn’t subtracting \( \frac{2}{3} \) from both sides help in isolating \( s \)?

\[
\frac{2}{3} \cdot s - \frac{2}{3} = \frac{9}{10} - \frac{2}{3}
\]

**Answer:**

The \( \frac{2}{3} \) is **multiplied** to the \( s \), and multiplication can’t be undone by subtraction.

The situation is similar to this: “half of $1000 minus half” doesn’t give you $1000. The first word “half” means “half of $1000”, and the second word “half” means “fifty cents”. Get it? “Half of $1000 minus half” would be $500 - $0.50, or $499.50.

Ok… What about that poor dog with not enough dog food? Either option I take, if I isolate \( s \) in the equation \( \frac{2}{3} \cdot s = \frac{9}{10} \), I get \( s = \frac{9}{10} \cdot \frac{3}{2} = \frac{27}{20} = 1 + \frac{7}{20} = 1 \frac{7}{20} \).
You should most definitely solve for \( s \) in that equation and make sure you get the correct answer!

When you are done, ask yourself: is the answer correct? Is two thirds of 1 and 7 twentieths really equal to 9 tenths?

\[
\frac{2}{3} \cdot s = \frac{2}{3} \cdot \frac{27}{20} = \frac{9}{10}
\]

Yup, it is!

Example:

My share of Great Aunt Sophie’s estate is \( \frac{2}{7} \) of the total value of the estate. My uncle’s share of the estate is \( \frac{3}{5} \) of the total. Is my share bigger? Is my uncle’s bigger? How many times is one bigger than the other?

These questions should be familiar to you, you should have answered them and checked your answers in the previous chapter. The first time I asked, the idea was to get you thinking about the mathematical operation of division to help answer questions like “how many of one thing fit into another”.

Here’s another way to think about that question: if my uncle’s share is some number of times bigger than mine, then we can assign that number a variable and create an equation using it. I’ll call the number of times my uncle’s share is bigger than mine \( x \), for lack of imagination.

So, my share, repeated an unknown \( x \) times, will give you my uncle’s share. Since my share is \( \frac{2}{7} \) of the total, and my uncle’s share is \( \frac{3}{5} \), this equation describes this situation:

\[
\frac{2}{7} \cdot x = \frac{3}{5}
\]

This means that \( \frac{2}{7} \), repeated \( x \) times, gets you to \( \frac{3}{5} \).

Get it? If not, slow down and think. Take a break and come back if you want to.

To “do something to both sides so I get an equation where \( x \) is by itself on one side of an equal sign” I could

- divide both sides of the equation by \( \frac{2}{7} \),

  OR

- multiply both sides by 7, then divide by 2

  OR

- divide both sides by 2 and then multiply by 7

NOTE:

- It would NOT be helpful to subtract \( \frac{2}{7} \) from both sides. Get it?
Ratios

The ratio of two numbers is, simply, one number divided by the other. The ratio of \( T \) to \( B \) is \( \frac{T}{B} \). If there are 15 boys and 20 girls in a room, the ratio of boys to girls is \( \frac{15}{20} \); this simplifies to \( \frac{3}{4} \) (because \( \frac{15}{20} = \frac{3 \cdot 5}{4 \cdot 5} \)).

If we know that the ratio between two variables is constant, then we know an equation that is true. For example, if we know that the ratio of sugar to flour in a recipe is \( \frac{1}{2} \), then we know that \( \frac{S}{F} = \frac{1}{2} \), no matter how much sugar, \( S \), or flour, \( F \), we have (more about what to do if \( F=0 \) later, because since you can’t divide by zero, that would create a problem).

Proportions

A proportion is an equation with a fraction on either side of the equal sign. For example, \( \frac{15}{20} = \frac{T}{B} \) is a proportion. If the relationship between two quantities can be described using an equation in which two fractions are equal to each other, the quantities are called proportional.

For example, if the proportion of flour to sugar in a cake recipe is 2 to 1, then it is true that \( \frac{F}{S} = \frac{2}{1} \), and it is also true that \( \frac{S}{F} = \frac{1}{2} \) (as long as \( F \) or \( S \) are not equal to zero, because if you divide by zero, you don’t get a number).

Suppose I get $20 per hour for a job. The ratio of the money I get to the hours I’ve worked is a constant number; this ratio does not change.

I could write
\[
\frac{H}{M} = \frac{1}{20}
\]
where \( H \) is hours I work, and \( M \) is money I’ve earned.

I could also write
\[
\frac{M}{H} = \frac{20}{1}
\]

There is a problem with both of these equations, because dividing by zero doesn’t make sense. And, since it doesn’t make sense to divide by zero, in the first equation it would be impossible for me to make no money, but I know that if I work 0 hours, I’ll make no money, so that doesn’t describe the situation. In the second equation, I couldn’t work 0 hours, but of course I know that it is possible to not work any hours. So, that equation doesn’t precisely describe the situation, either. What I could do is multiply both sides of the equal sign by the problematic variable to get the new equations:

If I multiply both sides of the equal sign by \( M \) in the equation \( \frac{H}{M} = \frac{1}{20} \), I get
\[
H = \frac{1}{20} M
\]
Or if I multiply both sides of the equal sign by \( H \) in the equation \( \frac{M}{H} = \frac{20}{1} \), I get 

\[
M = 20H \quad (\frac{20}{1} \text{ is more simply written as } 20)
\]

According to my mom, the ratio for pie crust is 3-2-1: three parts flour, two parts fat (butter or shortening), and one part water. What are the equations that describe the relationship between flour and fat, between flour and water, and between fat and water? Write them in your notes so that there is no danger of dividing by zero.

These are the possible options you could write. If you wrote something different, what could you do to help your understanding? If you got all of these, good job!

\[
\begin{align*}
\text{flour} &= \frac{3}{2} \times \text{fat}, \\
\text{fat} &= \frac{2}{3} \times \text{flour}, \\
\text{flour} &= 3 \times \text{water}, \\
\text{water} &= \frac{1}{3} \times \text{flour} \\
\text{fat} &= 2 \times \text{water}, \\
\text{water} &= \frac{1}{2} \times \text{fat}
\end{align*}
\]

**Problems/exercises/questions on fractions: putting it all together.** In all of the following, simplify your answer as much as possible. All fractions in your answer should be less than 1, if not, write the number differently (e.g. \( \frac{5}{2} \) is more than 1, so re-write it like \( 2 + \frac{1}{2} \)).

1. What is the size of each serving if you divide 17 cups of ice-cream into 9 servings?
2. What is 17 divided by 9?
3. What part of 17 fits into 9? (Compare this question with the previous one… are they the same?)
4. I have $900 to spend, but I want to buy a boat that costs $1700. What fraction of the total cost to I have?
5. How much flour do I have if I have two thirds of one and a half cups?
6. How much flour do I have if I combine two thirds of a cup with one and a half cups?
7. In the flour bin, there are 9 tenths of a cup of flour. The recipe for cookies requires 2 thirds of a cup to make 10 cookies. How many cookies can I make with 9 tenths of a cup of flour?
8. If triangles have the same angles, then the ratios of the lengths of sides of triangles are always the same. So, for example, the ratios of the lengths of sides for any triangle with angles 90°, 30°, and 60°, will always be the same, no matter how big or small the triangle is. The ratios of the lengths of sides of any triangle with angles equal to 90°, 10°, 80° will always be the same, no matter how big or small the triangle is.

a) Suppose I have a triangle with the following angles: P˚, Q˚, R˚. The length of the longest side is 5, the length of the shortest side is 3, and the length of the middle side is 4.

If I have another triangle with those same angles, and the longest side is 8, what are the lengths of the other sides?

b) Suppose I have another triangle with the following angles: A˚, B˚, C˚. The length of the longest side is 10, the length of the shortest side is 5, and the length of the middle side is approximately $8\frac{2}{3}$ (remember that $8\frac{2}{3}$ is the same as $8 + \frac{2}{3}$).

If I have another triangle with those same angles, but the shortest side is 4, what are the lengths of the other sides?

9. What is $\frac{1}{9}$, repeated 17 times?

10. How many 1 inch by 1 inch squares (also known as square inches) are there in a space that is 2 and a half inches long and 3 and a half inches wide? (you might want to draw a picture).

11. How many unit cubes would fit in a box that is 5 and a half units tall, 2 and a half units wide, and half a unit deep? (you can cut up the unit cubes so they fill up the box completely with no room leftover. In other words, what is the volume?)

12. You have 6 donuts and you want to give 2/3 of them to a friend and keep 1/3 for yourself. How many donuts would your friend get?

13. You will inherit $\frac{5}{6}$ of your great-grandparents’ estate. If the estate is worth twice as much as your own estate, and your estate is worth $300, how much will you inherit?

14. Wendy's hair was originally 10 inches long. She asked her hairdresser to cut 3 inches off. What fraction of her hair did she cut off? What fraction of her original length was she left with?

15. It takes two-thirds of a box of nails to build a birdhouse. If you wanted to build six birdhouses, how many boxes would you need?

16. A bakery used eight and a third cups of flour to make a full size cake. If they wanted to make a cake that was one-quarter the size, how many cups of flour would they need?
17. A fast food restaurant had 9 and a half pounds of flour. If they split the flour evenly among 4 batches of chicken, how much flour would each batch use? Between what two whole numbers does your answer lie?

18. The serving size for the granola that Roxane likes to eat for breakfast is \( \frac{1}{2} \) cup. How many servings are there in a box that holds 5 \( \frac{3}{4} \) cups?

19. A restaurant had 7 days to sell 102 gallons of ice cream before it expired. How much should they sell each day? Which two whole numbers does your answer lie between?

20. A pan of brownies was left out on the counter and \( \frac{1}{4} \) of the pan had already been eaten. Then John came along and ate \( \frac{2}{5} \) of what was left.
   a) What fraction of the total pan of brownies was on the counter before John came along?
   b) What fraction of a whole pan of brownies did John eat?
   c) What fraction of a whole pan was left after John finished eating?
   d) If a whole pan of brownies is 5,000 calories, how many calories did John eat?

21. Write this as one fraction \( \frac{A}{B} - \frac{C}{D} \) (I won’t say “simplify” because it’s not particularly simpler to express this as one fraction, but it is equivalent)

22. I have X dollars, and you have M times what I have.
   e) How many dollars do you have, in terms of X and M?
   f) If M is less than 1, do you have more or less money than I have?
   g) If M is more than 1, do you have more or less money than I have?
   h) If \( (X)(M) \) is greater than X, what do I know about M?
   i) If \( (X)(M) \) is less than X, what do I know about M?

23. How many halves fit into 9? Write a mathematical expression that shows this.

24. A baker is making cakes for a big party. She uses \( \frac{1}{4} \) cup of oil for each cake. How many cakes can she make if she has a bottle of oil that has \( 7 \frac{1}{3} \) cups in it?

25. To make this equation is true, \( \frac{4}{x} = \frac{3}{2} + 2 \) what number must x equal? If \( x \) is not a whole number, between what two whole numbers is it?
26. These next questions are probably the most important questions so far in this book. They require that you put together ideas from all the previous chapters. Pay attention, work slowly and carefully and make sure you understand before moving on to the next problem.

An empty box weighs 7 and a half pounds. After 5 doo-dads are put into the box, it and the doo-dads weigh 8 and two thirds pounds. How much does each doo-dad weigh?

27. An empty box weighs B pounds. After N widgets are put into the box, the box and the widgets together weigh T pounds. How much does each widget weigh (in terms of B, N and T?)

28. An empty box weighs 9 and two thirds pounds. It is filled with some cookies, each cookie weighs 2 fifths of a pound. If the box and cookies altogether weigh 12 and a half pounds, how many cookies (including fractions of cookies) are in the box?

29. An empty box weighs B pounds. It is filled with some cookies, each cookie weighs C pounds. If the box and the cookies altogether weigh T pounds, how many cookies are in the box, in terms if B, C and T?

30. What is an equation that describes the relationship between flour to sugar in a recipe that needs 8 cups flour and 5 cups sugar?

31. What is an equation that describes the relationship between flour to sugar in a recipe that needs 8 cups flour and 2 cups sugar?

32. What is an equation that describes the relationship between flour to sugar in a recipe that needs 1 cup flour and a third cup sugar?

33. What is an equation that describes the relationship between flour to sugar in a recipe that needs a third cup flour and 1 cup sugar?

34. What is an equation that describes the relationship between cost and pounds of apples if it costs $15 to buy 3 pounds?

35. What is an equation that describes the relationship between cost per ounce of something if 15 ounces cost $20?

36. If 16 Canadian dollars are worth about 12 US dollars, what is an equation that describes the relationship of Canadian dollars to US dollars?

37. There are almost exactly 20 miles per 32 km. What is an equation that describes the relationship between miles to km? Express your answer as a simplified fraction.

38. My car used 10 gallons of gas going 300 miles. What is an equation that describes the relationship between gallons I use and miles I drive? Express the answer as a simplified fraction.

39. I walked 8 miles in 2 hours. What is an equation that describes the relationship between the distance I walked and the time I spent walking?
40. I earned $500 working 20 hours. What is an equation that describes the relationship between the time I work and the money I make (assume that I get paid at a constant rate)?

41.
   a. If you drive for 2 hours at 50 miles per hour, how far have you driven?
   b. If you drive for 3 hours at 50 miles per hour, how far have you driven?
   c. If you drive for 10 hours at 50 miles per hour, how far have you driven?
   d. If you drive for 2 hours at 15 miles per hour, how far have you driven?
   e. If you drive for x hours at y miles per hour, how far have you driven, in terms of x and y?
   f. If you drive for H hours at S speed for a total of D miles, what is an equation that describes the relationship between H, S and D? What are all the equations?

42. If you ride your bike with average speed of 18 miles per hour to the store, and it takes you H hours to get there, and you ride your bike at an average speed of 15 miles per hour home, and it takes you T hours to get home, what is a true equation that describes the relationship between H and T (you take the exact same route to the store as home, so the distance you go is the same).

Remember, there is not “a way” to do this. Think about it, makes sense of it, keep track of the meaning of all the symbols you use. The key is to understand the power of using an equation, and to do that you need to know the meanings of addition, subtraction, multiplication, division and the equal sign.

43. This refers to the previous question: if it took you 40 minutes (that’s $\frac{2}{3}$ of an hour) to get to the store, how long did it take you to get home?

44. Think about this: if you have a job that pays $20 per hour, how much do you make in 1 hour? 2 hours? 3 hours? 10 hours? Ok… now you are “primed” to answer the following questions:
   a. You have one job that pays $20 per hour, and another job that pays $25 per hour. If you work H hours on the first job, and T hours on the second job, and you make a total of M dollars, what is an equation that describes the relationship between the numbers H, T and M?
   b. This question refers to the previous question: If you made a total of $17.50 and worked for 30 minutes at the second job, how long did you work at the first job?
   c. Suppose you need to make $510, and you work 7 hours per week on the first job, and, since you are a student, you want to work a total of 15 hours every week. How many weeks (including fractions of weeks) till you can make $510? (just assume you get paid cash).
45.

a. Suppose you start painting a fence and you paint one post every 10 minutes. How many posts do you paint in 20 minutes?

b. How many posts do you paint in 60 minutes?

c. How many posts do you paint in M minutes? (in terms of M)

d. Suppose you paint a total of P posts, and it takes you M minutes. What is an equation that describes the relationship between the number of posts you paint, P, and the minutes you’ve spent painting, M?

e. Suppose Jesse paints one post every 5 minutes. What is an equation that describes the relationship between the number of posts Jesse paints, J, and the time they spend painting, T?

f. If you and Jesse both paint posts, what is an equation that describes the relationship between the total posts painted, Y, and the time you’ve spent painting, X, and the time Jesse has spent painting, T?

g. Suppose you and Jesse both paint posts for the exact same amount of time. How long will it take to get 78 posts painted?

h. Suppose you paint for exactly twice as long as Jesse. How long will each of you work to get exactly 156 posts painted?

Check your answers. Every one of the problems in this chapter is very important and represents important ideas. Hooray for your mistakes, it means you get to learn and grow. Go to your mistakes, analyze them and think about how you have to change your brain in order to avoid those mistakes. Celebrate the fact that you make mistakes, because you CANNOT make any progress without making mistakes, so I hope you made many, and you changed your mind a little bit because of each mistake, and that you are growing your brain and constructing new ideas for yourself.

Here: summarize your mistakes, and note your new understanding or perspectives.

Celebrate your mistakes so you can celebrate your learning. Do not ignore your mistakes. Of course if you don’t learn anything from your mistakes, they are just frustrations.

You learned the power of creating an equation that describes the particular situation and then manipulating the symbols in that equation, by “doing the same thing to both sides of the equal sign,” to answer specific questions about the situation.

learn about ratios and proportions and how to use them to answer questions

You practiced adding, subtracting, multiplying and dividing with negative numbers and fractions in order to answer specific questions.
Chapter 13: Parentheses, Distributing, Factoring, part 1

In which you will learn:

• what parentheses mean.
• when parentheses matter and when they don’t matter.
• how to write an expression that does not have any parentheses that is equivalent to an expression with parentheses.
• that it is sometimes useful to write parentheses in expressions that have division, even when they are not originally written.
• “to factor something” means to write it as numbers or symbols which are multiplied together.
• “to factor something out” means to write an expression as the “something” multiplied to other numbers or symbols.
• drawing boxes is often very useful when multiplying and making sense of the distributive property.

Parentheses, Distributing, Factoring:

Adding, Subtracting, Multiplying and Dividing in the Company of Parentheses, and the Distributive Property.

What do you remember about parentheses? Any handy rhymes someone once taught you? What do they mean? Why are they used? What about “order of operations”? Does that mean anything to you? Write your response in your notes.

Adding and Subtracting

It never matters what order you add numbers: you’ll get the same result.

You may have been told that you should “do what is in the parentheses first.”

I want to examine those instructions:

Suppose my purse is on the table. In my right hand I have 2 pens, and in my left hand I have 4 pens, and there are 3 pens sitting on the table next to my purse. I put the 2 pens from my right hand in my purse, and then I put the 4 pens from my left hand in my purse, and the 3 pens on the table don’t do anything. How many pens I have in total could be written mathematically like this:
(2+4) +3

The parentheses act like my purse, they show that I’ve grouped 2 plus 4 pens by putting them in my purse.

But, it doesn’t matter if my pens are in my purse or on the table: I still have 9 pens! In this case, the parentheses DON’T matter if I’m asking the question “how many pens do I have?”

So, those folks who were telling you to “do what’s in the parentheses first” were, at best, fooling you.

Question:

1. Do parentheses matter when you are adding numbers? For example is the expression
   a + (b+c)
   the same as
   (a+b)+c, and is this the same as
   a+b+c?
   Explain your answer.

Ok… go back to those pens on the table and in my purse. I had a total of 9 pens, right? 3 on the table and (2 +4) in the purse. I have a total of 9 pens. Now, I throw my purse out the window.

Mathematically, this is what happened:

9-(2+4)

Question:

2. Do the parentheses matter in this expression: 9-(2+4) ?

I threw the whole purse out the window, and I have to show the quantities that are grouped together by putting them in parentheses.

Let’s think about that ridiculous situation again: 3 pens on the table and (2+4) pens in the purse, then I throw the purse out the window.

Question:

3. Compare the situation where I am combining (in other words adding) items in my purse to items on the table, to the situation where I combine items in my purse, and then throw my purse out the window. How can you tell when the parentheses matter? Explain why the parentheses are important and necessary in one situation, and not relevant in the other situation.

Think back to when you first learned what parentheses mean: did someone tell you that you need to “do what’s in the parentheses first?”
If you “do what’s in the parentheses first” when simplifying the expression 9-(2+4),
you’d get

\[ 9-(2+4) = 9-6 \]
\[ 9-6=3 \]

But, when you are throwing pens in a purse out the window, do you HAVE to first add
the pens in the purse? You start with 9 pens, throw 2 pens and 4 pens out the window.
You could write this mathematically like:

\[ 9-2-4 = 3 \]

So, you see, if a well-meaning person told you that you HAD to do what’s in the
parentheses first, they were really wrong. If possible, you may do what’s in the
parentheses first, but you don’t have to. When you are throwing pens out the window,
you could first add up all the pens you are throwing out the window, and subtract that
number from the total

\[ 9-(2+4)= 9-6 \text{ “doing what’s in parentheses first”} \]
\[ 9-6=3 \]

OR, you could subtract one handful at a time:

\[ 9-2-4=7-4 \]
7-4=3

You are still left with the same number of pens in the table.

If you have a subtraction sign in front of a parentheses, everything in the parentheses can be thought of as the “purse that is getting thrown out the window”. If there are several items in the purse, they are all getting thrown out. If there is some debt in the purse (meaning some negative numbers), then throwing out the debt is like getting some money.

As in most learning, it’s often easier to learn by doing rather than learn while reading. So, here are some exercises:

Find equivalent expressions that don’t have any parentheses (Do NOT do the arithmetic, just write an equivalent expression with the same amount of numbers.) If the original parentheses mattered, explain what changes you had to make when re-writing the expression. The first one is done for you:

4. 9-(2+4) = 9-2-4. I had to subtract every item in the parentheses.

5. 10-(2+3)

6. x-(a+b)

7. -x+(a-b)

**Multiplying**

Ok… back to the pens on the table, and, my purse. Which turns out, is a magical purse. Yep, it’s now a magical purse, and here’s what makes it magical: whatever I put in there gets doubled. Ha! I really like putting money in there, it’s so great! So, if I put 3 pens in my purse, POP, I’ll have 6 pens in there. If I put 5 pens in there, POP, I’ll have 10 pens. If I put X pens in my purse, POP, I’ll get 2X pens.

Suppose I put 4 pens in my purse: POP, there are (2·4) pens in my purse.

Suppose I put 3 pens on the table and 4 pens in the purse. What’s the difference between the number of pens on the table, and the number of pens in the purse? \( \Delta x = x_2 - x_1 = 3 - (2·4) \)

If you “do what’s in parentheses first” you get

\[
3-(2·4)=3-8
\]

\[
3-8=-5
\]

Remember how to find the difference between two numbers? Visualize a number line and start at 8 and go back to 3, the difference is negative 5 (you’re going backwards to get from 8 to 3).

What if you tried to write the difference between the number of pens on the table (3) and the number of pens in the purse (2·4) without any parentheses?

3-2·4
To simplify this properly, you need to remember that even with no parentheses, multiplication has to be done before addition or subtraction. If I ran the world, we would always write the parentheses to avoid any confusion, but I don’t, so you’ve got to remember that you gotta do the multiplication first! Even if the parentheses aren’t written down!

3 − 2 ⋅ 4 always means 3 − (2 ⋅ 4). To help you out, you might want to write parentheses around any multiplication. So, when you have this:

10 − 9 ⋅ 3

Re-write it like this:

10 − (9 ⋅ 3)

Exercises: Write equivalent expressions without any parentheses. Don’t do any arithmetic, just write equivalent expressions:

8. \((x+y)−a\)
9. \(X−(a+b)\)

10. \(10 −(2 ⋅ 3)\)
11. \(X+(3)(y)\)
12. \(X−(3)(y)\)
13. \(Y−2(x)\)

Back to the magic purse that doubles everything you put into it.

Suppose I grab a handful of chocolates (don’t know how many, so let’s call the amount \(X\)) and throw them into the purse. I grab another handful and throw them into the purse (don’t know how many, so let’s call the amount \(Y\) (I can’t call it \(X\), because that’s the number of chocolates in the first handful, and the handfuls might have different amounts so I need to use different variables)). POP, how many chocolates are in the purse? Mathematically, you could write this as:

\(2X + 2Y\)

What about if the purse takes a moment to double the chocolates? Would that change the total number of chocolates that end up in the purse?

Throw \(X\) chocolates into the purse, then throw \(Y\) chocolates into the purse, then POP, they get doubled:

Mathematically, you could write this as

\((x+y)(2)\) or
\((2)(x+y)\)

Any of those expressions,
expresses this situation mathematically. Any way I write it, it describes the same number of chocolates. The purse could double each handful separately and add the doubled handfuls. OR, the purse could add the handfuls, and then double the result. You get the same answer. 

\[ 2X + 2Y = 2(X+Y) \]

Suppose I have 10 chocolates in my pantry. I put X chocolates and Y chocolates in my magical purse (I didn’t get these chocolates from my pantry… someone gave them to me). What’s the difference between how many chocolates in my pantry and how many in my magical purse? Mathematically, this could be written as:

\[ 10 - 2(X+Y) \]

This is NOT 8(X+Y)…. You gotta do the multiplication first!!

If you want to write 10-2(X+Y) without any parentheses, you could write

\[ 10-2X-2Y \]

Exercises: Write equivalent expressions that do not have any parentheses.

14. \( 2-(x+y) \)
15. \( 3(x+y) \)
16. \( X(2+5) \)
17. \( x+(y+z) \)
18. \( x+2(a+y) \)
19. \( x-2(x+a) \)
20. \( -2(x+y) \)
21. \( 3-2(x) \)
22. \( 5+1(x) \)

Now, my purse! I’m running low on funds, so I throw $5 in my purse, and POP, I get 2($5) in there. I’ve got some time on my hands, so I take the 2($5) out of the purse, and put the 2($5) back in. POP, what do I get? 2($5) gets turned into 2(2($5)). How much money is that?

Let’s “do what’s in the parentheses first, working from inner parentheses to outer” like you may have learned you were supposed to do way back when.

\[ 2(2($5))=2($10) \]
\[ 2($10)=20 \]

But, did I need to do what’s in the inner parentheses first? What if you wrote
2(2($5))

Without any parentheses?

22$5 Well, that’s silly, you need to show the multiplication, so let’s write it again
2·2$5

Doing the arithmetic from left to right, you get
2·2$5 = 4$5
4$5 = $20

Same answer!

Let’s try it one other way: doing the arithmetic, from right to left
2·2$5
2·$10 = $20

Same Answer!

I think you are ready to answer the question:

Do the parentheses matter in this expression 2(2($5))? Nope, they don’t!

What about the parentheses in this expression:

(2)(3·5)=2(15)=30

Write it without parentheses and solve:

(2)(3·5) = 2·3·5

You can go right to left, or left to right, when you are multiplying numbers, it never matters what order you multiply. (Note, if you like math, you can continue with more classes where you will learn about matrix multiplication, where left and right multiplication do matter)

Question:

23. Do parentheses matter if you are only multiplying? Write down three examples that demonstrate your answer.

24. Suppose you have a coupon that let’s you pay only 80% of the listed price of a new purse, and another coupon that let’s you pay only 75%. AND, you are allowed to combine these coupons. (Note, 80% of something means \( \frac{80}{100} \) of it, and 75% of something means \( \frac{75}{100} \) of it. \( \frac{80}{100} \) can be simplified to \( \frac{4}{5} \), and \( \frac{75}{100} \) can be simplified to \( \frac{3}{4} \). So 80% of x is the same as \( \frac{4}{5} \) · x.)

a. Does it matter which coupon you apply first to determine what you pay? Try it both ways and then explain your answer.

b. What percent of the list price do you end up paying?
It turns out that it never matters what order you multiply numbers (but you already knew that!)

25. Without “plugging in” any numbers, decide whether $(3)(xz)$ is always, sometimes, or never, the same as $(3x)(3z)$, and justify your answer. (if you don’t justify your answer, the whole point of this question is lost, so you’d better do so, and your justification had better make sense).

26. Without “plugging in” any numbers, decide whether $(3)(xz)$ is always, sometimes, or never, the same as $(xz)(3)$, and justify your answer. (if you don’t justify your answer, the whole point of this question is lost, so you’d better do so, and your justification had better make sense).

One more note about parentheses and multiplication (especially with fractions):

$(x)(y)$ **always** means $x$ times $y$

So

$(3)\left(\frac{2}{9}\right)$ means $3$ times $\frac{2}{9}$

BUT

$\frac{2}{9}$ means $3$ plus $\frac{2}{9}$

BUT

If any of the numbers are written using variables, it means multiplication. I’m sorry, I didn’t invent these conventions, I’m just telling you like it is.

So, although $\frac{2}{9}$ means $3$ plus $\frac{2}{9}$

$\frac{x}{9}$ means $3$ times $\frac{x}{9}$

Exercises: For the following, write in words what the mathematical expression is showing. The first one is done for you. Don’t do the arithmetic, just write if it is addition or multiplication

27. $(3\frac{1}{2})$  $(3\frac{1}{2})$ means $3$ plus $\frac{1}{2}$

28. $2\frac{x}{y}$

29. $(2)(\frac{1}{3})$

30. $x\frac{1}{3}$

31. $7\frac{1}{3}$

32. $(7)(\frac{1}{3})$

33. $(7\frac{1}{3})$

34. $7\frac{1}{3}$
35. \( \frac{1}{3} (7) \)

**Dividing**

**Division and invisible parentheses:** Any expressions over or under fraction bars are in invisible parentheses (we could get fancy and say they are implicit rather than explicit).

I want to combine my money with my neighbor’s money, and figure out what half of our combined money is.

I’ll call my money \( M \) (for My), and my neighbor’s money I’ll call \( N \) (for Neighbor).

So, I combine these amounts by adding them: \( M + N \), and then I want to find out what half of that is, so I divide by 2, or multiply by \( \frac{1}{2} \).

\[
(M + N) \left( \frac{1}{2} \right) \text{ or } \frac{(M+N)}{2} \text{ or } \left( \frac{1}{2} \right) (M + N)
\]

All of these expressions are equivalent.

What about this expression \( \frac{M+N}{2} \)?

Yep, the expression \( \frac{M+N}{2} \) is equivalent to \( \frac{(M+N)}{2} \).

Anything over or under a fraction bar is in sort of invisible parentheses.

When you are using algebra to help you problem-solve, it is often a very good idea to write those parentheses so you remember they are there, and everything over or under a fraction bar must be grouped together.

**Exercises:** Simplify the following expressions

36. \( \frac{10}{8+2} \)
37. \( \frac{10x}{8+2} \)
38. \( (10) \frac{x}{(8+2)} \)
39. \( (10) \frac{x}{8+2} \)
40. \( (10) \frac{8+2}{5} \)
41. \( \frac{x+2}{x} \) write this as the sum of two fractions
42. \( \frac{x^2}{x} \) (simplify this)
43. Compare the previous two examples.
44. Show how \( \frac{x+2+y}{x+2} \) is equivalent to \( 1 + \frac{y}{x+2} \)

45. Show how \( \frac{b}{c} \) is equivalent to \( \frac{\frac{a}{c}}{\frac{b}{x}} \)

46. Show how \( \frac{b}{c} \) is equivalent to \( (b) \left( \frac{a}{c} \right) \)

47. Show how two thirds of 5 is equivalent to a third of 10.

48. Show how \( \frac{b}{c} \) is equivalent to \( (ab) \left( \frac{1}{c} \right) \)

49. Is it true that, in general, \( \frac{b+x}{c} \) is equivalent to \( \left( \frac{ab}{c} \right) + \left( \frac{ax}{c} \right) \)? Explain why or why not. (assume \( c \) is not equal to zero, because in that case, you’d be dividing by zero, and that doesn’t make sense)

50. Is it true that, in general, \( \frac{b+x}{c+x} \) is equivalent to \( (a) \left( \frac{b}{c+x} \right) + \left( \frac{x}{c+x} \right) \)? Explain why or why not. (assume \( c + x \) is not equal to 0)

Is it true that \( \frac{b+x}{c+x} \) is equivalent to \( (a) \cdot \left\{ \left( \frac{b}{c+x} \right) + \left( \frac{x}{c+x} \right) \right\} \)? Explain why or why not. (assume \( c + x \) is not equal to 0)

51. Is it true that \( \frac{b+x}{c+x} \) is equivalent to \( \left( \frac{ab}{c+x} \right) + \left( \frac{ax}{c+x} \right) \)? Explain why or why not. (assume \( c + x \) is not equal to 0)

This is what that chapter was all about:

- what parentheses mean.
- when parentheses matter and when they don’t matter.
- how to write an expression that does not have any parentheses that is equivalent to an expression with parentheses.
- that it is sometimes useful to write parentheses in expressions that have division, even when they are not originally written.
- “to factor something” means to write it as numbers or symbols which are multiplied together.
- “to factor something out” means to write an expression as the “something” multiplied to other numbers or symbols.
- drawing boxes is often very useful when multiplying and making sense of the distributive property
Box model for multiplication and the distributive property

Multiplication of two quantities can be viewed as finding an area (see the section on multiplication for a review of this); the area of something is how many (or what fraction of) unit areas will fit into that 2-dimensional space.

Consider an area of a rectangle that is 120 inches on one side, and 54 inches on the other side. To find the area you have to multiply 120 to 54

\[
\begin{array}{c}
120 \\
54
\end{array}
\]

Area = (120)(54)

I hope it’s clear that 120 is the same as 100+20, and that 54 is the same as 50+4, so that this area is can be written like (100+20)(50+4). But why would you write = (120)(54) as (100+20)(50+4)? Stay with me...

The area of a rectangle that is 120 by 54 is the big rectangle shown above. It is a combination of 4 smaller rectangles. The areas of those smaller rectangles added together must equal the area of the big rectangle, so, it must be true that (54)(120) is the same as (4)(100) + (4)(20) + (50)(100) + (50)(20).

Each of those separate areas are pretty easy to figure out without a calculator and without having to remember how to multiply multi-digit numbers.

\[
(54)(120) = (400) + (80) + (5000) + (1000).
\]
(400) + (80) + (5000) + (1000)=4 hundred plus 80 plus 5 thousand plus 1 thousand.

Combining the “apples and apples and bananas and bananas”, I get 6 thousand, 4 hundred and 80, or 6,480.

You can use the technique of writing an area to represent any multiplication.

Suppose you had to multiply \((2+x)\) to \((x+y)\).

Think about it like an area of a rectangle, one side has a length \((2+x)\) and the other has a length \((x+y)\). We have no idea what \(x\) and \(y\) are, and in fact they could be negative numbers; and although negative area doesn’t make a lot of sense\(^1\), multiplying by visualizing rectangles is always useful.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{area}=(x)(x))</td>
<td>(\text{area}=(x)(y))</td>
</tr>
<tr>
<td>(\text{area}=2x)</td>
<td>(\text{area}=2y)</td>
</tr>
</tbody>
</table>

This is sometimes called “expanding” the expression \((2 + x)(x + y)\), or writing it in expanded form.

The area of the entire rectangle is the sum (addition) of the individual areas:

\[(x)(x) + xy + 2x + 2y\]

Therefore, it must be true that

\[(x + y)(2 + x) = (x)(x) + xy + 2x + 2y\]

You may have learned other ways to multiply sums. “FOIL” is a common way to help student remember to multiply correctly when multiplying expressions like \((a+b)(c+d)\). FOIL stands for First, Outside, Inside, Last. However, this doesn’t help with multiplying \((a+b)(c+d+f)\).

This process of multiplying sums is called “distribution”. But, now that you understand how to use the visual aid of rectangles to help you multiply, you don’t really need to remember the name of what you are doing, as long as you know how to do it, and why it makes sense!

The tricky part for most students is recognizing a sum. In other words, simplifying this multiplication \((x)(2y)\) does not involve any distribution because there is no

\(^1\) I could imagine a negative area sort of like a science-fiction void area, so that an area of -1, when combined with area +1, together make nothingness.
addition. Expanding \((x)(2 - y)\) does involve distribution because there is a sum, and expanding \((x - 2)(-y - 7 + x)\) is even more complicated because you have to remember to multiply each term of each factor to each term of the other factor. Drawing out rectangles is a REALLY good idea for most students.

Example: Expand \((x - 2)(-y - 7 + x)\)

Even though this expression might come from some problem that has nothing to do with finding an area, you can use the “box model” to expand it.

\[
\begin{array}{ccc}
\text{Length} = -y & \text{Length} = -7 & \text{Length} = x \\
\text{Length} = x & \text{area} = -xy & \text{area} = -7x & \text{area} = xx \\
\text{Length} = -2 & \text{area} = 2y & \text{area} = 14 & \text{area} = -2x
\end{array}
\]

The total area of the rectangle is the sum of all the little areas. Therefore,

\((x - 2)(-y - 7 + x)\) is equivalent to \(-xy - 7x + (x)(x) + 2y + 14 - 2x\)

And, you could say that “the expanded form of

\[(x - 2)(-y - 7 + x)\]

\[-xy - 7x + (x)(x) + 2y + 14 - 2x\]

Exercises: Expand (meaning multiply and write an equivalent expression for) the following expressions. You will have to figure out how to expand expressions with negative numbers. To do this, remember that \(x+y\) is the same as \(x+(-y)\), and that you can have a negative length (just think of a negative length as an arrow with a direction opposite to the direction of a positive number). So the length \(x-3\) can be thought of as a combination of the length \(x\) with the length \(-3\).
53. \((4+x)(x-2+y)\)

54. \((x-y)(a+b-c)\)

55. \((-2x)(3y)\)

56. \((2x-3)(3x)\)
57. \((2x+1)(2x-1)\)

58. \((-2+x)(3y)\)

59. \((-2+x)(3+y)\)

60. \((-2x)(3+y)\)
STOP. Check all your answers … make note of what you did wrong, especially pay attention to patterns in your mistakes. As usual, always celebrate your mistakes and misunderstandings. Think: “oooh… what did I get wrong and what do I get to learn now? How exciting!” Don’t even think about skipping this part of your learning. Reviewing and attending to your mistakes is the ONLY way to learn this material…

61. \((\frac{x}{2} + 3)(-3 + \frac{2}{x})\)
Factoring

In a mathematical expression, a factor is a number, symbol, or group of numbers and symbols multiplied to another number, symbol or group of numbers and symbols. For example, in the expression

\[2xy\]

\[2, \ x \text{ and } y\] are all factors.

The verb phrase “to factor something” means to write it as numbers or symbols which are multiplied together. For example, if I wanted to factor the number 9, I could write \((3)(3)\). Or, I could write \((1\frac{1}{2})(36)\), although usually in textbooks “to factor” a number means to write it as whole numbers multiplied together.

The verb phrase “to factor something out” means to write an expression as the “something” multiplied to other numbers or symbols. For example, if I wanted “to factor out a 3” from the expression 81x, I would write \((3)(27x)\).

Example

Factor out 2x from the expression \((2x + 1)\). To do this, I need to write \((2x + 1)\) as \((2x)\) times something:

\[(2x + 1) = (2x)(1 + \frac{1}{2x})\] (note that for these expressions to be equal, \(x \neq 0\)).

In the previous section, you learned how to expand factored expressions so they are sums by using “box model” visualization. For example, you learned that

\[(2x + 1)(3x + y) = (6x)(x) + 2xy + 3x + y\]

Before you take College Algebra, it is important you understand what factoring means; in College Algebra you will learn how to factor all kinds of expressions. For example, you will learn how to factor the expression \((6x)(x) + 2xy + 3x + y\) so you can write it in factored form \((2x + 1)(3x + y)\).

62. Factor out an x from the expression \((1+2x)\) (assume x does not equal 0)
63. Factor out a 3 from the expression $(3+3x)$

64. Factor out a 2 from the expression $(2x+4y)$

65. Simplify the following expression by factoring the top and bottom (numerator and denominator) of the fraction and then simplifying further if possible.

\[
\frac{2x+8}{16x+8}
\]
66. Simplify the following expression by factoring the top and bottom (numerator and denominator) of the fraction and then simplifying by finding “weird forms of the number 1” (assume that \(x \neq -4\) and \(x \neq -\frac{1}{2}\), because if \(x\) equaled either of those values, you’d be dividing by 0)

\[
\frac{(2x+8)}{(16x+8)(x+4)}
\]
67. For each expression in the left column, find ALL of the equivalent expressions in the right column. If there are more than 1 matches, **WRITE ALL OF THE MATCHES.** Write the corresponding letter next to the expression in the left column.

<table>
<thead>
<tr>
<th>( \frac{x+2y}{x} )</th>
<th>A. ( 2y + 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{(x)(2y)}{x} )</td>
<td>B. ( x+2y )</td>
</tr>
<tr>
<td>( \left( \frac{xy}{x} \right) \left( \frac{1}{x} \right) (2) )</td>
<td>C. ( x(3z) )</td>
</tr>
<tr>
<td></td>
<td>D. ( \frac{(2y)}{x} )</td>
</tr>
<tr>
<td></td>
<td>E. ( (2y) )</td>
</tr>
<tr>
<td></td>
<td>F. ( 1 + 2y )</td>
</tr>
<tr>
<td></td>
<td>G. ( 1 + \frac{2y}{x} )</td>
</tr>
<tr>
<td></td>
<td>H. ( 2(x+y) )</td>
</tr>
</tbody>
</table>

68. For each expression in the left column, find ALL of the equivalent expressions in the right column. If there are more than 1 matches, **WRITE ALL OF THE**
**MATCHES.** Write the corresponding letter next to the expression in the left column.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Letter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3(xz)$</td>
<td>A. $(3x)(3z)$</td>
</tr>
<tr>
<td>$3(x + z)$</td>
<td>B. $(3x)(z)$</td>
</tr>
<tr>
<td></td>
<td>C. $3xz$</td>
</tr>
<tr>
<td></td>
<td>D. $3x + 3z$</td>
</tr>
<tr>
<td></td>
<td>E. $3x + z$</td>
</tr>
<tr>
<td></td>
<td>F. $(3x + z)(3)$</td>
</tr>
</tbody>
</table>

69. The length of one side of a room is 4 feet less than $2x$ (in other words, the length is $2x-4$). The length of the other side of the room is $c$ less than the sum $a+b$ (in other words, the length is $a+b-c$). Write an expression for the area of the room that does not have any parentheses.
STOP!

Check your work. What have you learned about factoring? What mistakes did you make? Write your reflections on factoring and the usefulness of using factoring to simplify expressions.

I hope you learned:

- what parentheses mean.
- when parentheses matter and when they don’t matter.
- how to write an expression that does not have any parentheses that is equivalent to an expression with parentheses.
- that it is sometimes useful to write parentheses in expressions that have division, even when they are not originally written.
- “to factor something” means to write it as numbers or symbols which are multiplied together.
- “to factor something out” means to write an expression as the “something” multiplied to other numbers or symbols.
- drawing boxes is often very useful when multiplying and making sense of the distributive property.
Chapter 14: Decimal Notation and Exponents

In which you will learn:

- what exponential notation means, including negative exponents, fraction exponents, and zero exponents
- how to make sense of the rules of exponents, so that you don’t have to memorize them, but you can work them out on your own.
- how the root symbol, $\sqrt{}$, is related to exponential notation

Decimal Notation and Exponents

Numbers, points on number line, arrows, repetitions, multiples, fractions, etc…, are represented with digits (digits are the symbols 0,1,2,3,4,5,6,7,8, and 9), and the placement of those digits gives numbers their meaning. 123 means 1 hundred, 2 tens, and 3 ones. The location of the digits tells you the meaning of a number.

The number 12.3 is different from the numbers 123 or .123, although the digits in all of these numbers are the same. The position of digits tells us what the number means.

For some positions, we have words; hundred, thousand, million, billion, trillion, quadrillion, all are words the tell us the meaning of a digit.

3 hundred means something different than 3 thousand, and that means something different than 0.0003. We could use words, or we could use position (300, 3000, 0.0003) to let us know the quantity we mean.

After that little introduction, take a moment to write down what you know about decimal notation.

How do you multiply, divide, add, and subtract numbers that include the decimal point (or decimal comma if you grew up in a place that uses a comma)?

Write examples, also (if you remember or know), that show how you convert numbers that are written as fractions into decimal notation, and how you convert some decimal numbers into fractions.

It’s important to connect the new ideas you read about in this book with ideas and understanding from your past experiences with math; you need to connect new ideas
to old ones (some are silver, some are gold, some are rubbish and need to be discarded). So don’t skip these questions, answer them before you read any further.

To the left of the decimal point (or a comma in some parts of the world), are place values that are less than -1 (meaning more negative than -1) or larger than 1, and each place indicates another multiple of 10.

**Examples:**

423 means 4 hundreds plus 2 tens plus 3 ones. Another way of writing this is (4 times 10 times 10) + (2 times 10) plus (3 times 1).

\[ 423 = (4 \cdot 10 \cdot 10) + (2 \cdot 10) + (3 \cdot 1) \]

6 million, 5 hundred thousand, 2 hundred and thirty five. Think about those words:

- “million” means “10 times itself 6 times”
- “hundred thousand” can be picked apart to mean “hundred times a thousand”

\[ 6,500,235 = (6 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) + (5 \cdot 10 \cdot 10 \cdot 10 \cdot 10) + (2 \cdot 10) + (3 \cdot 10) + 5 \]

**Examples:**

\[-54 = -5(10) + -4 \]
\[-345 = -3(10 \cdot 10) + -4(10) + -5 \]

Every term in the expanded form of those negative numbers is negative!

To the right of the decimal point each place indicates a division by multiples 10, or multiplication by multiples of \( \frac{1}{10} \).

For example, 0.423 means 4 tenths plus 2 hundredths plus 3 thousandths. Another way of writing this is (4 times \( \frac{1}{10} \)) plus (2 times \( \frac{1}{10} \) times \( \frac{1}{10} \)) plus (3 times \( \frac{1}{10} \) times \( \frac{1}{10} \) times \( \frac{1}{10} \)).

**Examples:**

\[-0.045 = -4 \left( \frac{1}{10} \cdot \frac{1}{10} \right) + -5 \left( \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \right) \]
\[-0.901 = -9 \left( \frac{1}{10} \right) + -1 \left( \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \right) \]
\[ -500,024.901 = -5(10)(10)(10)(10)(10) + -2(10) + -4 + -9 \left( \frac{1}{10} \right) \]
\[ + -1\left( \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \right) \]

Ok, right now, I hope you are wondering why I would write numbers in this expanded way, and that it’s very tedious. I couldn’t agree more. I broke up those numbers for two reasons:
1. So you see that there are always equivalent ways to write numbers: numbers
are always “break-apart-able”

2. You are about to see how writing numbers using decimal notation can make
life easier. But first, you have to learn about exponential notation.

**Exponential notation**

To save space, ink, and to make life easier, mathematicians write repeated
multiplication using a superscript. For example, \((2)(2)(2) = 2^3\), and \((x)(x)(x)(x) = x^4\)

People say this expression in many ways (sorry!):

“x to the fourth”

“x to the fourth power”

“x raised to the fourth”

“x raised to the fourth power”

“x exponent 4”

STOP! What do you remember about exponential notation? Where (if anywhere)
have you seen it before? Do you remember when you were first introduced to this
notation? Have you ever used scientific or engineering notation (perhaps in a science
class), and did you connect that idea to repeated multiplication of 10 or \(\frac{1}{10}\)?

Here, summarize what you remember about exponential notation. What does it
mean? How can you simplify exponential notation? What do you remember about
how to add, subtract, multiply or divide expressions that are written using
exponential notation?

**Exercises:**

*Note: these exercises lead you down a path to understanding. They MUST be done
in the order they are written or you won’t build the understanding they are designed
to create. If you get stuck, don’t go on. Stop and figure it out, or you’ll go down the
wrong path!*

1. Write \((x)(x)(x)(x)(x)\) in exponential notation.

2. Write \(x^3 \cdot x^5\) in expanded form (as x’s multiplied to each other)

3. Write \({(x)(x)(x)} \cdot {(x)(x)(x)(x)}\) in exponential notation

4. Based on your answers from the previous two questions, what do you think
is a general rule for simplifying expressions like \(x^A \cdot x^B\)? (express your rule
as an equation)

5. Write \(\frac{x^5}{x^3}\) in expanded form, and simplify.
6. Write $\frac{x^6}{x^3}$ in expanded form, and simplify.

7. Based on your answers from the previous two questions, what do you think is the general rule for simplifying expressions like $\frac{x^A}{x^B}$? (express your rule as an equation)

8. Write $\frac{x^3}{x^6}$ in expanded form, and simplify.

9. Write $\frac{x^2}{x^7}$ in expanded form, and simplify.

10. In exercise 7, you came up with a general rule for simplifying expressions like $\frac{x^A}{x^B}$. Use that rule to simplify the expressions $\frac{x^3}{x^6}$ and $\frac{x^2}{x^5}$.

compare your answer when you used the rule, to your answers for the previous two questions.

11. Based on your answer above, what is another way to write expressions with negative exponents, as in the expression $x^{-5}$?

12. Write $\frac{x^5}{x^7}$ in expanded form, and simplify.

13. Write $\frac{x^3}{x^3}$ in expanded form, and simplify.

14. Use the general rule you came up with for simplifying expressions like $\frac{x^A}{x^B}$ to simplify $\frac{x^5}{x^5}$ and $\frac{x^3}{x^3}$.

look back and compare your answers to the previous two questions.

15. Based on your answer to the previous question, what do you think is another way to write an expression zero exponent, as in $x^0$?

16. Write $(x^3)^4$ in expanded form, and then write the result in the form $x$ to a power

17. Write $(x^4)^3$ in expanded form, and then write the result in the form $x$ to a power

18. Write $(x^2)^3$ in expanded form, and then write the result in the form $x$ to a power

19. Based on your answers to the previous questions, what do you think is a general rule for simplifying expressions like $(x^A)^B$? (write your rule as an equation)

20. Write $(xy)^3$ in expanded form, and then write the result in the form $(x$ to a power) times $(y$ to a power). [hint: $xy$ means $x$ times $y$, and it does not matter what order you multiply numbers]

21. Write $(xy)^5$ in expanded form, and then write the result in the form $(x$ to a power) times $(y$ to a power)

22. Based on your answers to the previous questions, what do you think is a general rule for “distributing” exponents in expressions like $(xy)^A$? (write your rule as an equation)
23. Write $\frac{x^5}{x^4}$ in expanded form, then simplify.

24. Use the general rule you came up with for simplifying expressions like $\frac{x^a}{x^b}$ to the expressions $\frac{x^5}{x^4}$ and $\frac{x^7}{x^6}$ and then explain what $x^1$ means.

25. Write $(a + x)^2$ in expanded form, then expand by multiplying (feel free to use the box model to help).

26. Write $(ax)^2$ in expanded form, then simplify.

27. Explain what is different between the two previous exercises that allows you to distribute the exponent in one of them but not the other.

28. Write an equation that corresponds to this sentence: there is a number $x$, such that if you square it (meaning raise it to the power 2), you’ll get the number $y$.

STOP!

What do you remember about the concept of square-root? For example, do you remember how to simplify $\sqrt{9}$?

If $x = \sqrt{25}$, what can $x$ be?

If you have never seen this notation before, that’s ok. But if you have, write what you remember it means in your notes:

29. In the equation $x^2 = 4$, will both $x = 2$ and $x = -2$ make the equation true? In other words, can $x = 2$ or $x = -2$? Are both of these numbers solutions to this equation?

30. How many solutions are there to the equation $x^2 = 4$?

31. How many solutions are there to the equation $x^3 = 27$? [hint: $3 \cdot 3 \cdot 3 = 27$]

32. How many solutions are there to the equation $x^4 = 16$? [hint: $2 \cdot 2 \cdot 2 \cdot 2 = 16$]

33. What is the rule for determining how many solutions there are to equations of the form $x^a = B$? [describe your rule in words, not an equation]

34. If you “do the same thing to both sides of an equation, you might get another equivalent equation (but don’t divide by zero).” If you wanted to solve for $x$ in the equation $x^2 = 9$, you could raise both sides to the $\frac{1}{2}$ power. You will get $(x^2)^{\frac{1}{2}} = 9^{\frac{1}{2}}$. Simplify the left-hand side of that equation to solve for $x$.

35. How many solutions are there to the equation $x^2 = 9$? What are they?

36. Explain how the solution to $x^2 = 9$ is $9^{\frac{1}{2}}$ and $-9^{\frac{1}{2}}$, and that, therefore, $9^{\frac{1}{2}}$ must be another way to write the number 3, and $-9^{\frac{1}{2}}$ must be another way to write the number -3.
37. \( x^3 = 27 \). Solve for \( x \) by raising both sides to the \( \frac{1}{3} \) power and expressing the number 27 as \( 3^{(3)} \) (This procedure lets you figure out what number gives you 27 if you multiply it to itself 3 times!)

38. \( x^2 = 4 \). Find a solution to this equation by raising both sides to the \( \frac{1}{2} \) power, and expressing 4 as \( (\pm 2)^2 \) and figure out what number gives you 4 if you multiply it to itself twice.

**Exponents that are also a fractions**

**The square-root symbol:** \( \sqrt{\quad} \).

The square-root of \( x \) is written like this

\[ \sqrt{x} \]  

“The square-root of \( x \).”

It means “find all the numbers that, if you square them, will be equal to \( x \).”

So, \( \sqrt{9} = +3 \text{ and } -3 \), because if you square -3 you get 9, and if you square 3, you also get 9:

\[(\pm 3)^2 = 9 \text{, therefore } \pm 3 = \sqrt{9}. \] In other words, \( \sqrt{9} \) is the same as 3 or -3.

What would happen if I took the equation \((\pm 3)^2 = 9\) and raised both sides to the \( \frac{1}{2} \) power?

\[(\pm 3^2)^{\frac{1}{2}} = 9^{\frac{1}{2}}\]

\[\pm 3 = 9^{\frac{1}{2}}.\]

\[9^{\frac{1}{2}} = \sqrt{9} \]

**Exercises, continued:**

39. Based on what you just read and thought about, how is the expression \( x^{\frac{1}{2}} \) related to the expression \( \sqrt{x} \)? Going back and reviewing is probably a good idea as you answer this question.

It turns out that “roots” are like “unit fraction exponents”. Personally, for me, computation using unit fraction exponents is much easier than using the root symbol, so I always change any root symbols to fraction exponents. For example, \( \sqrt{x} = x^{\frac{1}{2}} \), and in general

\[ n\sqrt{x} = x^{\frac{1}{n}} \]

If the “root” symbol has no number on it, it is equivalent to \( x^{\frac{1}{2}} \), \( \sqrt{x} = x^{\frac{1}{2}} = \sqrt{x} \)

40. Write the expression using exponential notation: \( \sqrt{x} \)
41. Write the expression using exponential notation: \((\sqrt[7]{x})^8\)
42. Write the expression using exponential notation: \(8^{\frac{7}{x}}\)
43. Write the expression using exponential notation: \(7^{\sqrt{x}}\)
44. Write the expression using exponential notation: \((\sqrt[7]{x})^7\), and then simplify
45. Simplify (without a calculator) \(\sqrt[4]{x^2}\)
46. Simplify as much as possible (without a calculator), and write your answer using exponential notation \(\sqrt[9]{2^2}\)
47. Simplify (without a calculator) \(\sqrt[7]{x^2}\)
48. Simplify (without a calculator), and write your answer with exponential notation \(\sqrt[9]{4^2}\)
49. Explain whether or not this equation is true for any positive values of \(x\): \(\sqrt[7]{(x^B)^A} = (\sqrt[7]{x})^B\) In other words, are the expressions on either side of the equal sign equivalent? (hint, re-write both sides using exponential notation, and then simplify)
50. To solve for \(x\) in the equation \(x^7 = A\), George raises both sides of the equation to the \(\frac{1}{7}\) power, and then gets the equation \((x^7)^{\frac{1}{7}} = A^{\frac{1}{7}}\), and because George remembers the rule that \((x^m)^n = x^{m\cdot n}\), he knows that \((x^7)^{\frac{1}{7}} = x^1 = x\). So, finally, George says that if \(x^7 = A\), then \(x = A^{\frac{1}{7}}\).
Lisa looks at the equation \(x^7 = A\) and solves for \(x\) by taking the seventh root of both sides of the equation: \(\sqrt[7]{x^7} = \sqrt[7]{A}\) and says that if \(x^7 = A\) then \(x = \sqrt[7]{A}\).
George and Lisa get into an argument over who is correct. How can you help settle their argument?
51. \(x^3 = 2^{-3}\) (how many solutions are there and why?)
52. \(x^4 = 3^{-4}\) (how many solutions are there and why?)

STOP! Make sure you go back and think about everything you wrote so far in this chapter… pay particular attention to any mistakes you made, or confusion you had (Hooray for mistakes: you’re learning!). The next page is for you to complete and “put it all together.”
53. Answer the following questions (hint: see your answers to the **Bold** questions. You already have the answers to these questions, but it will be convenient if you have them here in one place for reference.)

**What is a general rule for simplifying expressions like** \(x^A \cdot x^B\)? **Explain where this rule comes from**

**What is the general rule for simplifying expressions like** \(\frac{x^A}{x^B}\)? **Explain where this rule comes from**

**What is another way or writing a negative exponent, as in the expression** \(2^{-3}\)? **Explain where this rule comes from**

**What is the meaning of a zero exponent, as in** \(2^0\)? **Explain where this rule comes from**

**What is a general rule for simplifying expressions like** \((x^A)^B\)? **Explain where this rule comes from**

**What is a general rule for “distributing” exponents in expressions like** \((x \cdot y)^A\)? **Explain where this rule comes from**

**Explain what** \(x^1\) **means. Explain where this rule comes from**

**How you can decide whether to distribute exponents in these two expressions:** \((xy)^A\) and \((x + y)^A\)?

**What is the rule for determining how many solutions there might be to equations of the form** \(x^a = B\)? **Explain where this rule comes from.**

**What does a fraction exponent mean, and how is it useful for solving for** \(x\) **in equations like** \(x^a = B\)?

I hope you learned::

- what exponential notation means, including negative exponents, fraction exponents, and zero exponents
- how to make sense of the rules of exponents, so that you don’t have to memorize them, but you can work them out on your own.
- how the root symbol, \(\sqrt{}\), is related to exponential notation
Chapter 15: Really big numbers and really small numbers

In which you will learn how to use exponential notation and place value to compare, add, subtract, multiply and divide really big numbers and really small numbers without a calculator. You’ll also practice the skills and use the concepts from previous chapters.

Understanding really, really, really, big numbers and really, really, really, small numbers using powers of 10.

What is 0.5? It means 5 tenths, or $(5)\left(\frac{1}{10}\right)$.

(I hope it is clear that this is the same as $\frac{1}{2}$)

What about 0.3? This means $(3)\left(\frac{1}{10}\right)$, or $\frac{3}{10}$.

If you want to approximate the fraction $\frac{1}{3}$ with decimals, you can divide 1 by 3 to get $\frac{1}{3} = 0.\overline{3}$. The line over the digits indicate that they continue forever. $\frac{1}{3}$ is an exact number, and 0.33 is an approximation of that number. 0.33333333 is a better approximation, but still an approximation. No matter how many digits you write, it’s not exact. It is a common misconception that 0.3 is the same as $\frac{1}{3}$. But, 0.3 is very definitely NOT the same as $\frac{1}{3}$: 3 is not a factor of 10, so you can’t express $\frac{1}{3}$ as an amount of tenths or hundredths or thousandths, etc….

Ok… back to 0.5. Another way to write that is $5\left(\frac{1}{10}\right)$

What about 0.05? That would be $5\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)$ (otherwise known as $5\left(\frac{1}{100}\right)$)

How about 0.0005? Same thing as $5\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)$

---

1 Now THAT is a weird, mind-blowing idea … what is forever?
Look at this number: 0.0000000000000005
It makes me have a headache to figure out what, exactly, that number actually is because there are so many zeros. Each place to the right of the decimal point represents a division by 10. So, the number

0.0000000000000005 is the same as

\[ 5 \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \].

Well, to me, that’s not easier to understand than a bunch of zeroes in a row. It’s still just difficult to figure out how many \( \frac{1}{10} \)'s there are. There must be a better way!

Question: What is

\[ 5 \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) \] using exponential notation for all those \( \frac{1}{10} \)'s multiplied to each other?

Write that number again, this time using the rule you constructed for the meaning of negative exponents:

Do you agree that it’s easier to understand what 0.0000000000000005 is, if we write it like \((5)(10^{-16})\)?

Example:

What is 20 thousand dollars plus half a million dollars? ²


20 thousand is \(20 \cdot 10^3\), and half a million is \(0.5 \cdot 10^6\), or \(\frac{1}{2} \cdot 10^6\).

So, if you’ve got 20 thousand combined with half a million, you’ve got this:

\((20)(10^3) + (0.5)(10^6)\)

How can you combine those numbers? You need to express them with the same “nouns”, so we are adding apples to apples.

\((20)(10^3) + (0.5)(10^3)(10^3) = 20\text{ thousand} + 0.5\text{ thousand thousand}\)

What is 0.5 thousand? There are many ways to think about this, but here is a way:

\(0.5\text{ thousand} = (5)(10^{-1})\text{ thousand}\)

\((5)(10^{-1})(10^3) = (5)(10^2)\)

\((5)(10^2) = 500.\) Did you follow that? If not, go back and read it again and make sure you follow along.

Here’s another way to think about 0.5 thousand:

² Wouldn’t it be nice to have to answer this question while reviewing personal finances?
0.5 thousand is \( \frac{1}{2} \) thousand

Half of a thousand is 500

Again, did you follow that? If not, go back and read it again and make sure you follow along.

So, what you have realized is that 20 thousand plus half a million is the same as 20 thousand plus 500 thousand, so now we can combine apples and apples to get 20 thousand + 500 thousand = 520 thousand.

Bottom line: 20 thousand dollars plus half a million dollars is 520 thousand dollars.

Example:

Calculate the following without using a calculator:

\[
\frac{(0.000009)(0.000003)}{(270,000,000)}(4,500,000)
\]

Hint: write each number in parentheses as an “easy” number times a power of 10. Then use the rules of exponents to simplify the powers of 10.

\((45)(10^{-14})\) is one way to write this answer. If you write it out in decimal form, good luck keeping track of your zeros!

You might say that you could have answered this question without using exponents. I don’t doubt it. But, using exponential notation to help you problem-solve is a very powerful tool.

There are many approaches to the previous problem, and you had many choices about what you could do. The answer, however, is only one number (but of course there are several ways to write that number…). Here’s one way to figure out the answer (but, make sure you work out your own path as well, one that makes sense to you, and that uses powers of 10)

\[
\frac{(0.000009)(0.000003)}{(270,000,000)}(4,500,000) = \frac{(9)(10^{-6})(3)(10^{-6})}{(27)(10^7)} (45)(10^5)
\]

First, I re-arrange the numbers to make the calculation easier:

\[
\]

At this point, you need to think about how to make life easier for yourself, where can you find those “weird forms of the number 1” so you can make this division easier? DON’T just march ahead without thinking about your strategy!

Here is one path to this solution (but, again, make sure you work out your own path as well, one that makes sense to you)

\[
\frac{(9)(3)(45)(10^5)(10^{-6})(10^{-6})}{(27)(10^7)} = \frac{(9)(3)(45)(10^{5+6+6-7})}{(27)} 1
\]
\[
\frac{(9)(3)(45)(10^{5+6-6-7})}{(27)} = \frac{(27)}{27}(45)(10^{-1+6-7})
\]
\[
(1)(45)(10^{-1+6-7}) = (45)(10^{-7+7})
\]
\[
(45)(10^{-7+7}) = 45(10^{-14})
\]

**Exercises:** Solve the following equations for \( x \), without using a calculator. Of course, some of these exercises require the skills and thinking from the chapters.

1. \( x \cdot 10^3 = 2 \cdot 10^{-3} \)
2. \( 10^4 = 500x \cdot 10^{-4} \) (how many solutions are there and why?)
3. What would you rather have, 500 thousand dollars of half a million dollars? Explain your answer.
4. What would you rather have, 250 thousand or a quarter million dollars? Explain your answer.
5. What would you rather have, 18 hundred dollars or 0.18 million dollars? Explain your answer.
6. Which is more money, \( 10^9 \) pennies or a million dollars?
7. The total weight in a container of widgets is 20,000 pounds. Each widget weighs 0.00004 pounds.
   a. Are there a lot of widgets or just a few widgets in the container? (just think about it)
   b. How many widgets are there in the container? Express your answer two different ways: as a whole number times a power of 10 AND as a number in standard form.
8. Simplify the expression \((2^{16})\left(\frac{1}{2}\right)\). Write your answer with no exponents or root symbols. (Don’t use a calculator!)
9. Simplify the expression \((4 \cdot 10^8)\left(\frac{1}{2}\right)\). Write your answer with no exponents or root symbols. (Don’t use a calculator!)
10. Simplify and write your answer in standard form: \(10,000(10^{-3} + 10^{-3})\)
11. Simplify and write your answer in standard form: \(10,000(10^{-3} \cdot 10^{-3})\)
12. Simplify and write your answer in standard form: \((2^{-3} + 2^{-3})\)
13. Simplify and write your answer in standard form: \((2^{-3} \cdot 2^{-3})\)
14. Simplify and write your answer in standard form: \(16(2^{-3} + 2^{-3})\)
15. Simplify and write your answer in standard form: \(16(2^{-3} \cdot 2^{-3})\)
16. Simplify and write your answer in standard form: \((2)(-3)(2)(-3)\)
STOP! Check all your answers, make sure you understand and attend to any difficulties you had, or ideas that you had trouble fitting into what you already understood about decimals, exponents, and roots. Celebrate what you learned and pat yourself on the back. You are brilliant!

Make note of anything particular you want to remember or had to work hard to understand and connect to:

I hope you learned how to use exponential notation and place value to compare, add, subtract, multiply and divide really big numbers and really small numbers without a calculator. You’ll also practice the skills and use the concepts from previous chapters.
Chapter 1.
Exercises:

First, take a look back and review what you wrote in your notes above, and then do the following exercises. The solutions are at the back of the book, so please check yourself as you go and examine your mistakes. It’s important that you work slowly and thoughtfully. This is NOT a timed test. The most important favor you can do yourself is to celebrate any errors you make. If you don’t make any errors, you are probably not learning. So, celebrate your errors, examine them, understand what went wrong. Never ever ever ignore them or hope they will go away on their own. When you make an error, think “hooray… now I have a mystery to solve”.

For each of the following, identify the number that is being represented or described. Feel free to use a number line to help you, but write your answer as a positive or negative number:

2. An arrow with a length of 3 pointing to the left. 
   \[\text{NEGATIVE 3, OR -3}\]

3. The number represented by this arrow:
   \[\text{NEGATIVE 4, OR -4}\]

4. The number of times this unknown negative number is repeated:
   \[4\]

5. The point on a number line that is half-way between negative 4 and positive 2.
   \[\text{NEGATIVE 1, OR -1}\]

6. An arrow that starts at 8 and goes to 10. 
   \[2\]

7. An arrow that starts at 10 and ends at 8. 
   \[-2\]

8. An arrow that starts at -3 and goes to -1. 
   \[2\]

9. An arrow that starts at -1 and goes to -3. 
   \[-2, \text{ OR NEGATIVE 2}\]
10. The number of times the arrows are repeated: 5
11. The number of times the arrows are repeated: 4
12. The point on a number line that is halfway between -2 and 0. -1
13. The point on a number line that is halfway between -9 and -7. -8
14. The point on a number line that is halfway between positive 5 and negative 5. 0
15. The opposite of the number -3. 3
16. The opposite of the number 3. -3
17. If x = -8, what is the opposite of x? 8
18. If x=7, what is the opposite of negative x? 7
19. If x=-5, what is negative x? NEGATIVE X = 5, OR -X=5 (PLEASE NOTE THAT THIS MEANS NEGATIVE X IS A POSITIVE 5, AND X IS A NEGATIVE 5)
20. If negative x=5, what is x? X= -5
21. If negative x=3, what is x? X= -3
22. If the opposite of x=8, what is x? X=8
23. If the opposite of x=-3, what is x? X=3
24. If x is -10, is “negative x” a positive or a negative number? “negative x” really should be called “the opposite of x”. The opposite of -10 is 10. So –x is a positive number.

25. \( x + y = y + x \) numbers in general, no exceptions
26. \( x - 1 = x + \left(-1\right) \) numbers in general, no exceptions
27. \( x - 1 = 10 \) the solution list is, \( x=11 \)
28. \( 2x = 10 \) the solution list is just one specific number, \( x=5 \)
29. \( 2x = y \) a relationship between quantities
30. \( 2x = x + x \) numbers in general, no exceptions
31. \( 2x + y = 2x + 3 \), the solution list is a specific number, \( y=3 \) for one variable, but \( x \) can be any number so the values that make this equation true are not list-able.

32. \( x^2 = (x)(x) \) numbers in general
33. \( x^2 = 9 \) specific numbers: \( x=3 , x=-3 \)
34. \( x^2 = y \) a relationship
35. \( x + x + x = 2x + x \) numbers in general
36. \( x + x = 4 \) specific number, \( x=2 \)
37. \( x + x = 2 \) specific number, \( x=1 \)
38. \( x - y = x + \left(-y\right) \) numbers in general
39. \( x + 2 = y \) relationship
40. \( x = y \) relationship
41. \( x = 2 \) specific number
42. \( x = 1 \) specific number
43. \( x = x \) numbers in general
Adding

Exercises with adding “like terms”

If possible, simplify the following combinations so your result is 1 number and 1 noun. You might have to change how you describe some of the objects. For example, a dollar could be described as 10 dimes or 4 quarters.

1. 2 dollars and 5 dollars CAN BE COMBINED TO BE 7 DOLLARS

2. 2 dollars and 3 dimes CAN BE COMBINED TO BE 23 DIMES OR 2.3 DOLLARS

3. 2 dollars and 10 dimes CAN BE COMBINED TO BE 3 DOLLARS OR 30 DIMES

4. 2 dollars and 16 dimes CAN BE COMBINED TO BE 36 DIMES, OR 3.6 DOLLARS

5. 2 pencils and 7 chairs CAN’T BE COMBINED, UNLESS YOU COMBINE THEM TO MAKE FIREWOOD

6. 2 dollars 12 dimes and 120 pennies CAN BE COMBINED TO MAKE 20+12+12 = 44 DIMES, OR 4.4 DOLLARS, OR 440 PENNIES

7. 3 hundred plus 9 hundred IS THE SAME AS 12 HUNDRED, ALSO KNOWN AS 1 THOUSAND 2 HUNDRED

8. 23 hundred plus 8 hundred CAN BE COMBINED TO MAKE 31 HUNDRED, ALSO KNOWN AS 3 THOUSAND 1 HUNDRED

9. 2 thousand plus 3 hundred plus 8 hundred CAN BE DESCRIBED AS 20 HUNDRED PLUS 3 HUNDRED PLUS 8 HUNDRED = 31 HUNDRED, ALSO KNOWN AS 3 THOUSAND 1 HUNDRED. COMPARE THIS TO PREVIOUS QUESTION.

Questions:

10. Do “23 hundred” and “2 thousand 3 hundred” refer to the same number? YES

11. Do “2 tens plus 8 ones” and “28” refer to the same quantity? YES
12. Do “3 tens plus 8 tens” and “11 tens” refer to the same quantity? YES, BUT NO ONE SAYS “11 TENS”, WE SAY 1 HUNDRED AND TEN.

13. What is a more common way to refer to the quantity “11 tens”? SEE ANSWER ABOVE

14. 2 tens plus 6 tens IS THE SAME AS 8 TENS

15. 2 tens plus 4 ones IS THE SAME AS 24 ONES

16. 2 x’s plus 3 x’s (x refers to some quantity that you don’t know or don’t feel like specifying) IS THE SAME AS 5 X’S.

17. 2 x’s plus 3 y’s (x and y refer to some quantities that you don’t know, or don’t feel like specifying) CAN’T BE SIMPLIFIED WITHOUT KNOWING WHAT X AND Y ARE, OR HOW THEY RELATE TO EACH OTHER. IF WE KNEW HOW THEY RELATED TO EACH OTHER WE COULD SIMPLIFY THIS, LIKE ADDING DIMES AND DOLLARS. BUT X MIGHT BE DIMES, AND Y MIGHT BE BLADES OF GRASS.

18. My daughter has $-10, and my son has $50.
   a. Show on this number line how much money you would have to give my daughter so that she ends up with the same amount of money my son has.
      
      DRAW AN ARROW GOING FROM -10 TO 50. THE ARROW IS POINTING TO THE RIGHT, SO IT’S POSITIVE, AND IT HAS LENGTH OF 60. THE ARROW REPRESENTS THE NUMBER POSITIVE 60

   b. Write an addition equation that shows the relationship among the three numbers: -10, Δx, and 50

      \[-10 + Δx = 50\]

   c. What is the number Δx? Is Δx a positive or negative number?

      SEE ANSWER TO A, ABOVE.

19. My daughter has D amount of money, and my son has S amount of money.
If D is less than S, you’d have to give my daughter a positive Δx amount of money for her to end up with what my son has. See the picture below.
If D is more than S, you’d have to give her a debt (add a negative amount to the money she has) for her to end up with the same amount of money as my son. See the picture below.

a. Write an equation that shows the relationship among the three numbers: D, Δx, and S.

$D + \Delta x = S$ IS THE ONLY EQUATION THAT SHOWS THIS RELATIONSHIP USING ADDITION. YOU COULD WRITE IT USING SUBTRACTION LIKE THIS $S-D = \Delta x$, OR $\Delta x = S-D$, BUT WE’LL LEARN ABOUT THAT IN THE NEXT CHAPTER.

b. Do you understand that it is the same equation whether or not my daughter has more or less money than my son? Explain in your own words how that can be the same equation (it has something to do with the idea that the sign of a number is included in the variable)

YES, IT IS THE SAME EQUATION REGARDLESS OF WHO HAS MORE MONEY. THERE IS NOT A SUBTRACTION SIGN OR A NEGATIVE SIGN IN SIGHT, BECAUSE THE NEGATIVE SIGNS ARE PART OF THE SYMBOLS REPRESENTING THE NUMBERS

c. If S is 50 dollars and D is -30 dollars, what is Δx? In other words, what do I give my daughter so she ends up with what my son has? Write the equation that gives you your answer.

$-30 + \Delta x = 50$

$\Delta x = 80$. I’D HAVE TO GIVE MY DAUGHTER 80 DOLLARS
d. If S is -10 dollars and D is -5 dollars, what is \( \Delta x \)? In other words, what do I give my daughter so she ends up with what my son has? Write the equation that gives you your answer.

\[-5 + \Delta x = -10\]

I’D HAVE TO GIVE MY DAUGHTER A DEBT OF 5 DOLLARS. I’D HAVE TO GIVE HER NEGATIVE 5 DOLLARS.

Write the specific addition problems that are represented visually in the following number lines. (even if you can represent what is written as a subtraction problem, write it as an addition problem.)

20. \( x_1 + \Delta x = x_2 \)

\[3 + 2 = 5\]

21. \( x_1 + \Delta x = x_2 \)

\[2 + 3 = 5\]

22. \[-5 + 2 = -3\]
23. 
\[ 2 + (-5) = -3 \]

24. 
\[ 2 + (-2) = 0 \]

25. 
\[ (-2) + (-7) = -9 \]

26. 
\[ 7 + (-3) = 4 \]
27. 

\[ -3 + 7 = 4 \]

28. Write your reaction to realizing that problems that you can consider to be subtraction are actually addition problems using negative numbers. Some common reactions are: “in my head, I change an addition to a subtraction so I can actually solve the equation”

Answers will vary... If you are doing this work as part of a class, bear in mind that your answers to this question, and other open-ended questions may be graded and collected, even though there is no correct answer.

29. Suppose I am combining some assets (represented by positive numbers) and debts (represented by negative numbers) to figure out how much money I have in total. To look at this like a mathematician (and to look at it in a way that will help you in math class) I want to look at this combination through the mental lens of this structure: \( x_1 + \Delta x = x_2 \)

First, I think about the money I own, because I like to start out with happy thoughts. Suppose I have $200. Then, I combine that with my debt of -$300.

\[ x_1 = 200 \]
\[ \Delta x = -300 \]

What is \( x_2 \)? Draw a number line representing this addition.

30. Next, I am going to go about this a different way.

First, I think about my debt, because it’s really on my mind: -$300. Then, I combine that with the money I have: $200

\[ x_1 = -300 \]
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Solution: Addition
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\[ \Delta x = $200 \]

What is \( x_2 \)? Draw a number line.

31. Question: Do you end up with the same result if you add numbers in different order?  It never matters what order you add numbers. The “number line picture” will look different, but, in mathematics, the result of combination does not depend on the order you combine (this is not true of cooking! Sometimes you have to combine the eggs with the milk before you add the sugar, for example!)

32. Question: Does it EVER matter in what order you add numbers? What about when the numbers are both negative or both positive?  It never matters!

33. If I add \( X \) (an unknown number) to 7, will the result be more or less than 7? (or, is it impossible to tell?)

It is impossible to tell. If \( X \) is a positive number adding 7 to \( X \) will result in a number larger than 7. If \( X \) is a negative number, adding 7 to \( X \) will result in a number smaller than \( X \).

34. If I add \( X \) (an unknown number) to 7, will the result be more or less than \( X \)? (or, is it impossible to tell?) The result will be more than \( X \). 7 more than \( X \) is bigger than \( X \), no matter what \( X \) is.

35. I add \( Y \) to -2 and I get the number \( N \). Is \( N \) less than or bigger than -2, or, is it impossible to tell?) It is impossible to tell.

36. I add \( Y \) to -2 and I get the number \( N \). Is \( N \) less than or bigger than \( Y \), or, is it impossible to tell?) If I add a negative number, the result will be less than what I started with, no matter what number I started with. \( N \) will be less than \( Y \).

37. How could you write “adding \( Y \) to -2 is \( N \)” as an equation? (choose all that are correct)
   a. \(-2 + Y = N\)
   b. \( Y -2 = N\)
   Both are correct
Question:

All of the exercises above were addition exercises. There was an initial value and it is combined with another value. However, some of the exercises included negative numbers. What was your reaction to the negative numbers? Did you try to re-think the problems as subtraction? Why? What do you remember about that from past classes? (if your tendency is to change addition problems to subtraction problems, that’s ok, but if you did, it’s important that you understand the idea of addition as

\[ x_1 + \Delta x = x_2 \]

**And that any of these three symbols might represent negative numbers.**

Here, write your reaction to solving the problems with negative numbers.
Chapter 3: Solution What is an Equal Sign? Equivalent Equations in Addition and Subtraction

Exercises
In the following examples, explain what mathematical operation is done to the first equation to get to the second equation. (you are NOT necessarily solving for \(x\), you’ll get to do that later…). Please check your answers as you go because it’s important you don’t re-enforce wrong ideas!

1. \(2 + 3 = 5\) \(\quad 2 = 5 - 3\)
   \textit{Subtract 3}

2. \(x - 3 = 5\) \(\quad x = 8\)
   \textit{Add 3}

3. \(x - 3 = 5\) \(\quad x - 8 = 0\)
   \textit{Subtract 5}

4. \(3 - x = 5\) \(\quad 3 = 5 + x\)
   \textit{Add} \(x\)

5. \(x_1 + \Delta x = x_2\) \(\quad x_1 = x_2 - \Delta x\)
   (these two equations show the relationship between addition and “removing what was added”)
   \textit{Subtract \(\Delta x\)}

6. \(x_1 + \Delta x = x_2\) \(\quad \Delta x = x_2 - x_1\)
   (these two equations show the relationship between addition and “finding the difference”)
   \textit{Subtract \(x_1\)}

Exercises: Solve for \(x\) in the following equations. Some might take multiple steps. Keep “doing the same thing” to both sides of the equal sign of resulting equations until you have isolated \(x\).

7. \(x + 3 = 6\) \(\quad x = 3\)

8. \(x - 3 = -5\) \(\quad x = -2\)

9. \(x + 2 = -7\) \(\quad x = -9\)

10. \(3 - x = 5\) \(\quad x = -2\)

11. \(x_1 + \Delta x = x_2\) \(\quad \text{(solve for} x_1)\quad x_1 = x_2 - \Delta x\)

12. \(x_1 + \Delta x = x_2\) \(\quad \text{(solve for} \Delta x)\quad \Delta x = x_2 - x_1\)
13. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The difference between Fred and Melinda’s money is $100
      \[ F - M = 100 \]
   b. If you add 100 to Fred’s money, you’ll have the same as Melinda’s money
      \[ F + 100 = M \]
   The two equations are not equivalent. You can’t “do something” to both sides of the equal sign of one of the equations and end up with the other equation. The statements are not equivalent.

14. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The difference between Fred and Melinda’s money is $100
      \[ F - M = 100 \]
   b. If you add 100 to Melinda’s money, you’ll have the same as Fred’s money
      \[ M + 100 = F \]
   If you subtract 100 from both sides of the equal sign in the second equation, you get the first equation. Or, if you add \( M \) to both sides of the equal sign in the first equation, you get the second equation. These statements are equivalent.

15. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The difference between Fred and Melinda’s money is $100
      \[ F - M = 100 \]
   b. If you subtract 100 from Melinda’s money, you’ll have the same as Fred’s money
      \[ M - 100 = F \]
      Nope… they are not equivalent. Fred has more money.

16. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The difference between Fred and Melinda’s money is $100
      \[ F - M = 100 \]
   b. If you subtract 100 from Fred’s money, you’ll have the same as Melinda’s money
      \[ F - 100 = M \]
      Yes, if you add 100 and subtract \( M \) from both sides of the equal sign of the second equation you can get the first. Or, if you add \( M \) and subtract 100 from both sides of the equal sign the first equation, you get the second equation.
      These statements are equivalent. They are two different ways of saying the same thing.
17. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The temperature in Fairbanks, Alaska is 15 degrees more than the temperature in Seattle, Washington
      \[ F = S + 15 \]
   b. Fairbank’s temperature is Seattle’s temperature minus 15 degrees
      \[ F = S - 15 \]
      These statements are not equivalent

18. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The temperature in Fairbanks, Alaska is 15 degrees more than the temperature in Seattle, Washington
      \[ F = S + 15 \]
   b. Seattle’s temperature is Fairbanks’ temperature minus 15 degrees
      \[ S = F - 15 \]
      Yeah, these are equivalent statements. If you subtract 15 from both sides of the equal sign of the first equation, you get the second equation.

19. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. Seattle’s temperature is Fairbanks’ temperature minus 15 degrees
      \[ S = F - 15 \]
   b. Fairbanks’ temperature is Seattle’s temperature plus 15 degrees
      \[ F = S + 15 \]
      Yes, these are equivalent. Add 15 to both sides of the equal sign of the first equation, get the second equation

20. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The difference between Fairbanks’ temperature and Seattle’s temperature is 15
      \[ F - S = 15 \]
   b. Fairbanks’ temperature is Seattle’s temperature plus 15 degrees
      \[ F = S + 15 \]
      Yes, these are equivalent. Add S to both sides of the equal sign of the first equation

21. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other
   a. The difference between Fairbanks’ temperature and Seattle’s temperature is 15
      \[ F - S = 15 \]
   b. Seattle’s temperature is 15 degrees more than Fairbanks’
      \[ S = 15 + F \]
      Nope, you can’t get from one of these equations to another
22. Are the following statements equivalent? Explain how you know by showing whether the equations for each statement are equivalent to each other

   a. Fairbanks’ temperature is Seattle’s temperature plus 15 degrees
      \[ F = S + 15 \]

   b. Fairbanks’ temperature minus 15 gives you Seattle’s temperature.
      \[ F - 15 = S \]

   Yep... these are equivalent statements. If you subtract 15 from both sides of the equal sign of the first equation, you get the second equation.

I hope you learned:

- Once you have created an equation that describes a situation, you might make useful discoveries and statements about that same situation if you “do the same thing to both sides of the equal sign.”
- Subtraction “undoes” addition
- Finding a solution to an equation means finding the values of any variables in that equation that make the equation a true statement.
1. **a. Show on this number line “remove 3 from 10”, 10 − 3. Remember: 10 is the result of some addition, \( x_2 \). 3 is the arrow that ends at 10, it is \( \Delta x \). You are trying to figure out what \( x_1 \) is. \( x_1 \) is where the arrow begins.**

   \[
   x_1 = 7 \quad \Delta x \text{ is a positive number} \quad x_2 = 10
   \]

   **b. Simplify this expression: 10 − 3 = 7**

2. **Show on a number line “Combine 10 and −3”, and simplify the expression: 10 + (−3) = 7**

   \[
   \Delta x \text{ is} \ -3
   \]

3. **Show on this number line “remove 8 from 12”, 12 − 8. Remember: 12 is the result of addition. 8 is the arrow that ends at 12, it is \( \Delta x \). You are trying to figure out what \( x_1 \) is. \( x_1 \) is where the arrow begins. 8 is a POSITIVE number, it’s an arrow pointing to the right.**

   \[
   x_1 = 4 \quad \Delta x = 8 \quad x_2 = 12
   \]
4. Show on a number line the combination of 12 and \(-8\), in other words simplify \(12 + -8 = 4\)

5. Show on this number line “remove 9 from 15”, \(15 - 9\). Remember: 15 is the result of some addition, \(x_2\). 9 is the arrow that ends at 15, it is \(\Delta x\). You are trying to figure out what \(x_1\) is. \(x_1\) is where the arrow begins.

6. Show on this number line the combination of 15 and \(-9\), in other words, show \(15 + -9\) on this number line.

Simplify \(15 + -9 = 6\)

7. Back up!
   
a. Write down the equations for question 1 and question 2, what do you notice?
   
   \[10 - 3 = 7\]
   
   \[10 + -3 = 7\]

   What do you notice? Yep, both expressions are equal to 7.
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Solutions: Subtraction
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b. Write down the equations for question 3 and question 4, what do you notice?
   \[ 12 - 8 = 4 \]
   \[ 12 + -8 = 4 \]
   What do you notice? Both are equal to 4.

c. Write down the equations for question 5 and question 6, what do you notice?
   \[ 15 - 9 = 6 \]
   \[ 15 + -9 = 6 \]
   What do you notice? Both are equal to 6

d. Do you think that if you need to simplify a subtraction problem it might sometimes be easier to transform it into an addition expression? And, sometimes, if you have an addition problem, it might be sometimes easier to transform it into a subtraction problem that you know has the same answer?

   YES! I THINK THIS IS A VERY BIG DEAL! I DON’T HAVE TO DO THE PROBLEM GIVEN TO ME, I CAN DO AN EASIER PROBLEM IF I WANT TO !!!

8. Show on this number line “remove 9 from 6”, \( 6 - 9 \). Remember: 6 is the result of some addition, \( x_2 \). 9 is the arrow, \( 9 = \Delta x \). You are trying to figure out what \( x_1 \) is. \( x_1 \) is where the arrow begins. This one is a bit trickier than the previous exercises.

\[
\begin{align*}
x_1 &= -3 \\
\Delta x &= 9 \\
x_2 &= 6
\end{align*}
\]

9. Wait.. what? That previous quation might seem kinda of weird. How can you remove a 9 from 6 if 9 is bigger than 6? Hmm… Suppose you have $6 because you combined $9 and a $3 debt; if you take away $9, you’ll end up with a $3 debt. Are you thinking “this approach is making something that
should be simple (subtraction) more complicated that it needs to be!”. Memorizing rules might seem simple, but if you don’t make sense of those rules, they can’t help you solve problems outside of a narrow context of math class.

a. Why do you think I might be asking you to think deeply about “taking away” rather than giving you a set of rules on how to subtract numbers?

I’ll leave the answer up to you. Whether or not this homework is getting collected by a teacher, you should answer it.

b. If your goal is to quickly answer many subtraction problems, then this approach is probably more time consuming than what you have been taught. But, if your goals include understanding how to use subtraction to make sense of numbers, and answering questions involving “removing one thing from another thing”, this approach is what you need. In your life, no one will ever, outside of a math class, ask you to answer many subtraction problems. Being able to make sense of the idea of “removing something” is useful, do you agree?

Again, the answer depends on you. Please write down your response whether someone is collecting this homework or not. Thinking about why you are learning is one of the most important parts of getting an education.

10. Show on this number line: combine 6 and $-9$. The expression is $6 + -9 = -3$

11. What do you end up with if you remove 10 from -6. Use a number line to show subtraction as “removing the $\Delta x$”.

Hint: the expression that means “remove 10 from $-6$" is $-6 - 10$
12. What do you get if you combine 10 and $-6$? Show on the number line and write the expression that means “combine 10 with $-6$”: $10 + (-6) = 4$

13. What do you get if you remove 5 from $-11$? To make sense of this, imagine that you have a total of a debt of $5$. You got there by having $5$ and some other amount. What was that other amount that you had to combine with $5$ to end up with a debt of $5$? $-5 - 5 = -10$

14. What do you get if you combine $-5$ and $-5$? $-5 + (-5) = -10$

15. So far this chapter, I’ve been leading you down a garden path, and it’s time for you to STOP and look around and figure out where I’ve led you.

   a. Write the expression that means “remove 3 from 10”, and then simplify that expression:

   $$10 - 3 = 7$$

   Write the expression that means “add 10 and $-3$”, and then simplify
10 + \(-3\) = 7

b. Write the expression that means “remove 8 from 12”, and then simplify that expression

\(12 - 8 = 4\)

Write the expression that means “add 12 and \(-8\)”, and then simplify

\(12 + (-8) = 4\)

c. Write the expression that means “remove 9 from 15”, and then simplify that expression

\(15 - 9 = 6\)

Write the expression that means “add 15 and \(-9\)”, and then simplify

\(15 + (-9) = 6\)

d. Write the expression that means “remove 9 from 6”, and then simplify that expression

\(6 - 9 = -3\)

Write the expression that means “add 6 and \(-9\)”, and then simplify

\(6 + (-9) = -3\)

e. Write the expression that means “remove 10 from \(-6\)” and then simplify that expression

\(-6 - 10 = -16\)

Write the expression that means “add \(-6\) and \(-10\)”, and then simplify

\(-6 + (-10) = -16\)

f. Write the expression that means “remove 5 from \(-5\)”, and then simplify that expression

\(-5 - 5 = -10\)

Write the expression that means “add \(-5\) and \(-5\)”, and then simplify

\(-5 + (-5) = -10\)

16. Mathematical equivalency is a BIG DEAL!! It helps you answer questions that are hard, by changing to question to one you know the answer to, or at least is easier for you to figure out.
17. Suppose Susie has -7, and Billybob has -10.

Write out the expressions that represent the following:

a. “the amount of money I have to give Susie so she has the same amount as BillyBob”.

\[-10 - (-7) \quad \text{BE CAREFUL WITH THIS ONE... THINK ABOUT IT. YOU NEED TO COUNT FROM -7 TO -10.}\]

b. “what I have if I add BillyBob’s money to the opposite of what Susie has”

\[-10 + -(-7) \quad \text{OR} \quad -10 + 7\]

c. “What you get if you take away from Billybob the amount of money that Susie has?”

\[-10 - (-7)\]

The following exercises are “fill-in-the-blank”. The first one is done for you:

18. Fill in the blank: 5 + -4 is the same as 5 - _4__, because both expressions are equal to __1__.

19. Fill in the blank: 10 + -8 is the same as 10 - _8__, because both expressions are equal to __2__

20. Fill in the blank: -5 + -4 is the same as -5 - _4__, because both expressions are equal to __-9__

21. Fill in the blank: -10 + -8 is the same as -10 - _8__, because both expressions are equal to __-18__

22. Fill in the blank: -10 + -10 is the same as -10 - _10__, because both expressions are equal to __-20__

23. Fill in the blank: -2 + -8 is the same as -2 - _8__, because both expressions are equal to __-10__
24. Fill in the blank: $2 + -5$ is the same as $2 - 5$, because both expressions are equal to $-3$

25. Fill in the blank: $-1 + -4$ is the same as $-1 - 4$, because both expressions are equal to $-5$

26. Fill in the blank: $-10 + -20$ is the same as $-10 - 20$, because both expressions are equal to $-30$

27. Fill in the blank: $-10 + -1$ is the same as $-10 - 1$, because both expressions are equal to $-11$

28. Fill in the blank: $10 + -10$ is the same as $10 - 10$, because both expressions are equal to $0$

29. Fill in the blank: $-10 + -12$ is the same as $-10 - 12$, because both expressions are equal to $-22$

30. Fill in the blank: If $A + -B = C$, then it is true that $A - B = C$

31. Show on a number line “the difference between 3 and 5” and then simplify the expression $3 - 5$. Pay close attention to the sign of your answer. In this case, the sign is negative, and your arrow should be left-pointing. The “sign” of a number indicates whether it is positive or negative.

3-5 WOULD BE AN ARROW STARTING AT 5 AND ENDING AT 3. SO IT’S LEFT-FACING. $3 - 5 = -2$

32. Draw a number line, and show “the difference between 3 and 2” and then simplify the expression $(3 - 2)$. Pay close attention to the sign of your answer.

AN ARROW STARTING AT 2 AND ENDING AT 3. A RIGHT-FACING ARROW. $3 - 2 = 1$. 
33. Draw a number line, and show on it “the difference between 5 and 2” and then simplify the expression (5-2). Pay close attention to the sign of your answer.

   AN ARROW STARTING AT 2 AND ENDING AT 5. A RIGHT-FACING ARROW. 5-2 = 3

34. Draw a number line, and show on it “the difference between 2 and 3” and then simplify the expression (2-3). Pay close attention to the sign of your answer.

   ARROW STARTS AT 3 AND ENDS AT 2. 2-3 = -1. THE ARROW IS LEFT-POINTING.

35. Draw a number line, and show on it “the difference between 5 and 9” and then simplify the expression (5-9). Pay close attention to the sign of your answer.


36. Draw a number line, and show “the difference between 9 and 5” and then simplify the expression (9-5). Pay close attention to the sign of your answer.

   THIS IS THE OPPOSITE OF THE PREVIOUS QUESTION. START AT 5 AND GO TO 9. 9 - 5 = 4

37. Draw a number line, and show “the difference between -9 and 5” and then simplify the expression (-9-5). Pay close attention to the sign of your answer.


38. Draw a number line, and show “the difference between 5 and -2” and then simplify the expression (5-(-2)). Remember that when you are showing the difference between A and B on a number line, (A-B), you need to start at B and go to A. Pay close attention to the sign of your answer. So, in this case, you’d start at -2 and go to 5.

   5- (-2). START AT -2 AND GO TO 5. THE DIFFERENCE IS SHOWN BY A RIGHT-POINTING ARROW WITH LENGTH 7.

   5- (-2) = 7
39. Draw a number line, and show “the difference between -2 and 5” and then simplify the expression \((-2 - 5)\). Remember that when you are showing the difference between A and B on a number line, \((A - B)\), you need to start at B and go to A. Pay close attention to the sign of your answer.

START AT 5 AND GO TO -2. THE DIFFERENCE IS SHOWN BY THE LEFT-POINTING ARROW WITH LENGTH 7.

\(-2 - 5 = -7\)

40. Draw a number line, and show “the difference between 9 and -5” and then simplify the expression \((9 - (-5))\). Remember that when you are showing the difference between A and B on a number line, \((A - B)\), you need to start at B and go to A. Pay close attention to the sign of your answer.

START AT -5 AND GO TO 9. THE DIFFERENCE IS SHOWN BY THE RIGHT-POINTING ARROW WITH LENGTH 14.

\(9 - (-5) = 14\)

41. Draw a number line, and show “the difference between 2 and -3” and then simplify the expression \((2 - (-3))\). Remember that when you are showing the difference between A and B on a number line, \((A - B)\), you need to start at B and go to A. Pay close attention to the sign of your answer.

START AT -3 AND GO TO 2. THE DIFFERENCE IS SHOWN BY THE RIGHT-POINTING ARROW WITH LENGTH 5

\(2 - (-3) = 5\)

42. Draw a number line, and show “the difference between -2 and -3” and then simplify the expression \((-2 - (-3))\). Remember that when you are showing the difference between A and B on a number line, \((A - B)\), you need to start at B and go to A. Pay close attention to the sign of your answer. In this case, you are starting at -3 and going to -2.

START AT -3 AND GO TO -2. THE DIFFERENCE IS SHOWN BY THE RIGHT-POINTING ARROW WITH LENGTH 1

\(-2 - (-3) = 1\)

43. Go back and consider how you simplified the following expressions in the previous exercises, when you were thinking about and drawing “finding the difference”:

\[
\begin{align*}
(5 - -2) & = 7 \\
(9 - -5) & = 14 \\
(2 - -3) & = 5 \\
(-2 - -3) & = 1 \\
\end{align*}
\]
And compare your results with how you would simplify the following expressions:

\[(5 + 2) \quad (9 + 5) \quad (2 + 3) \quad (-2 + 3)\]

\[7 \quad 14 \quad 5 \quad 1\]

Do you think, in general, that the difference between A and \(\neg B\), \((A \neg B)\) could be the same number as \((A+B)\), always?

*I’M BEGINNING TO SUSPECT THEY COULD ALWAYS BE THE SAME.*

44. For the following, fill in the blank: \(A-B\) is equivalent to \(A+\ _{-B}\__________\)
45. \(5-2\) is equivalent to \(5+\ _{-2}\__________\)
46. \(5- (9)\) is equivalent to \(5+\ _{-9}\__________\)
47. \(5-(9)\) is equivalent to \(5+\ _{-9}\__________\)
48. \(-5-(9)\) is equivalent to \(-5+\ _{-9}\__________\)
49. \(3- (-6)\) is equivalent to \(3+\ _{6}\__________\)
50. \(3- (-3)\) is equivalent to \(3+\ _{3}\__________\)
51. \(3 – (3)\) is equivalent to \(3+\ _{-3}\__________\)
52. Combine \(-5\) with \(-3\)
   a. \(-5 + -3 = -8\) or \(-3+ -5=-8\) are the possible addition problems you could write.
   b. \(-5-3=-8\) or \(-3-5=-8\) are the possible subtraction problems you could write.
53. Combine \(-3\) with \(4\). (Remember for this and the rest of the exercises, your answers should contain two equivalent expressions, one using subtraction, and one using addition.)
   a. Addition: \(-3 + 4 = 1\)
   b. Equivalent subtraction: \(4-3 = 1\), OR \(-3 – (-4) =1\)
54. Combine 2 and \(5\); (hint: adding a number is the same as subtracting its opposite: \(A+B=A-(-B)\). So adding 5 is the same as subtracting negative 5)
   a. Addition \(2 + 5 = 7\)
   b. Equivalent subtraction \(2 – (-5)=7\) OR \(5 – (-2) = 7\)
55. Find the difference between 5 and \(-3\)
   a. Subtraction \(5 – (-3)=8\)
   b. Equivalent addition \(5 + 3=8\)
56. Find the difference between \(-4\) and \(-1\)
   a. Subtraction \(-4 – (-1) = -3\)
   b. Equivalent addition \(-4 + 1 = -3\)
57. Find the difference between 4 and \(1\)
   a. Subtraction \(4-1=3\)
   b. Equivalent addition \(4 + (-1) = 3\)
58. Combine 4 and \(1\)
   a. Addition \(4+1=5\)
   b. Equivalent subtraction \(4-(-1)=5\)
59. Remove 5 from 9
   a. Subtraction 9-5=4
   b. Equivalent addition 9 + (-5) =4

60. Take away 3 from 10
   a. Subtraction 10-3=7
   b. Equivalent addition 10 + (-3) =7

61. Find the difference between -3 and 0
   a. Subtraction -3 -0 = -3
   b. Equivalent addition -3 +0(0)= -3 (note, -0 and 0 are the same number, so this question is a little weird.)

62. Find the difference between 0 and -3
   a. Subtraction 0 - (-3) = 3
   b. Equivalent addition 0+3=3

63. Find the difference between -3 and 2
   a. Subtraction -3-2 = -5
   b. Equivalent addition -3 + -2 = -5

64. Combine -5 and 5
   a. Addition -5 + 5
   b. Equivalent subtraction -5 – (-5) =0 OR 5-5=0

65. Remove -5 from 3
   a. Subtraction 3 - (-5) = 8
   b. Equivalent addition 3 + 5 = 8

66. Remove 3 from -5
   a. Subtraction -5 -3 = -8
   b. Equivalent addition -5 + -3 = -8

67. The difference between -5 and 3
   a. Subtraction -5 – 3 = -8
   b. Equivalent addition -5 + -3 = -8

68. The difference between -13 and -5
   a. Subtraction -13 – (-5) = -8
   b. Equivalent addition -13 + 5 = -8

69. Combine -3 and -2
   a. Addition -3 + -2 = -5
   b. Equivalent subtraction -3 -2=-5

70. Using the symbol for subtraction, write the mathematical expression that means “Remove A from negative B”. Do not simplify.-B-A
   Write an expression that is mathematically equivalent to what you wrote above, but using the symbol for addition.-B + -A

71. 
   a) Using the symbol for subtraction, write the mathematical expression that means “the difference between X and negative Y”X – (-Y)
   b) Write an expression mathematically equivalent to what you wrote for above, but using the symbol for addition. X+Y
Problem solving with addition and subtraction

It is REALLY REALLY important that you do NOT do the following in your head.

For each of these, the point is to ask yourself “what is there to be learned from this question?”

1.
   a. Write the following scenario using addition (don’t solve if you don’t want to): I have a debt of $200. Pirjo has $500 of cold, hard, cash. If we combine our money, what do we have together?

   \[-200 + 500 = 300\]

   b. Re-write an equivalent mathematical expression using subtraction in a way that is easy for you to solve. Solve it.

   \[500 + (-200) = 300\] (I KNOW TO DO THIS BECAUSE I KNOW IT DOESN’T MATTER WHAT ORDER I ADD NUMBERS)

2.
   a. Write the following scenario using subtraction (don’t solve if you don’t want to): I have a debt of $500. You have a debt of $700. What is the difference between our debts?

   \[-500 - (-700)\]

   b. Re-write an equivalent mathematical expression using addition in a way that is easy for you to solve. Solve it.

   \[-500 + 700 = 200\]

   c. Re-write an equivalent mathematical expression using subtraction (different from the first one you wrote) in a way that is easy for you to solve. Solve it.

   \[700 - 500 = 200\]

3.
   a. Write the following scenario using subtraction (don’t solve if you don’t want to): Tanoko had a total debt of $500, owed to several different people. Josimar decided to forgive the $200 Tanoko owed him; in other words, Josimar removed a debt of $200 from Tanoko’s total. How much money does Tanoko have now (or how much debt?)

   \[-500 - (-200)\]

   b. Re-write an equivalent mathematical expression using addition in a way that is easy for you to solve. Solve it.

   \[-500 + 200 = -300\]

   c. Re-write an equivalent mathematical expression using subtraction (different from the first one you wrote) in a way that is easy for you to solve. Solve it.
4. I have an unknown number of chocolate candies. You have an unknown number of chocolate candies. Together, write an expression that means the number of candies I have combined with the number of candies you have. Use single-letter variables.

\[ C + S \] (YOU CAN USE ANY LETTERS YOU LIKE, OF COURSE)

5. I had an unknown amount of mashed potatoes on my plate. When I wasn’t looking, you stole some. Write an expression that means how much mashed potatoes I ended up with. Use single-letter variables.

\[ X - Y \]

6. I used to have X apples, and now I have Y apples. What’s the difference between what I have now and what I used to have?

\[ Y - X \]

7. There were X apples in a bag, and you then changed the amount in the bag by \( \Delta x \). Explain what must be true about \( \Delta x \) if the amount of apples in the bag ended up being less than X.

\[ X + \Delta x \] IS LESS THAN X, SO \( \Delta x \) MUST BE A NEGATIVE NUMBER.

8. There were 7 apples in the bag, and then you changed the amount of apples in the bag by -4. How many ended up being in the bag?

\[ 7 + (-4) = 3 \]

THERE ENDED UP BEING 3 APPLES IN THE BAG

9. What is the difference between 7 and -4?

\[ 7 - (-4) = 11 \]

10. I have 9 dollars and you have a debt of 4 dollars, what is the difference between what I have and what you have?

\[ 9 - (-4) = 13 \]

THE DIFFERENCE BETWEEN WHAT I HAVE AND WHAT YOU HAVE IS 13 DOLLARS. SOMEONE WOULD HAVE TO GIVE YOU 13 DOLLARS IF YOU WANTED TO END UP WITH WHAT I HAVE.

11. I have a debt of $5. You have a debt of $5. What is the difference between the amount of money we have?

\[ -5 - (-5) = 0 \]

THERE IS NO DIFFERENCE. WE HAVE THE SAME AMOUNT OF MONEY.

12. I have a debt of $50. You have a debt of $80. What is the difference between what I have and what you have?
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-50 – (-80) = 30
YOU CAN FIGURE THIS OUT BY DRAWING AN ARROW ON A NUMBER LINE FROM -80 TO -50 (MAYBE YOU COULD JUST DRAW IT IN YOUR HEAD). YOU COULD ALSO RE-WRITE IT AS AN ADDITION LIKE THIS:
-50 + 80.
FROM THERE, YOU CAN RE-ARRANGE BECAUSE IT NEVER MATTERS WHAT ORDER YOU ADD NUMBERS:
80 + -50 = 80-50 = 30
THE DIFFERENCE IS 30 DOLLARS. THAT’S 30 POSITIVE DOLLARS.

13. You are 80 dollars in debt. Someone gives you some money and you end up only $50 in debt. How much money did they give you? (make sure you write this out…)
-80 + X = -50. WHAT IS X?
WHAT’S THE DIFFERENCE BETWEEN -50 AND -80?
-50 – (-80) = -50 + 80
80 + (-50) = 80-50
80-50 = 30.
THAT NICE PERSON GAVE YOU 30 DOLLARS.

14.

a. It’s -60 F in Fairbanks Alaska, and 40F in Vancouver Canada. What is the difference between the two cities’ temperatures?
-60 – 40 = -100
THE DIFFERENCE IS NEGATIVE 100 DEGREEES.

b. It was -60 F in Fairbanks, then it got 40 F colder. What was the final temperature in Fairbanks?
-60 + (-40) = -100
THE FINAL TEMPERATURE WAS -100

c. What will my total debt be if I combine a debt of $60 with a debt of $40?
-60 + (-40) = -100
THE TOTAL WILL BE NEGATIVE 100

d. How is it that the answers to the previous three questions are the same? Can mathematical expressions express different ideas and still be equivalent?
YES, THIS IS SO COOL. IF I DO NOT KNOW HOW TO WRAP MY HEAD AROUND THE MEANING OF AN EXPRESSION, I CAN RE-WRITE IT TO LOOK LIKE SOMETHING A LOT SIMPLER TO UNDERSTAND.

15. In the “number line picture” below, identify Δx, x₁, and x₂, and then write TWO equations that describe the picture shown in the number line. Make sure the numbers in your equations are the same as the numbers shown on the
number line. For example, if \( \Delta x \) is a negative number, make sure your equations have \( \Delta x \) being a negative number. In this example, \( \Delta x \) is a negative number.

\[
\begin{align*}
\text{Equation 1, using this model for addition: } & \quad x_1 + \Delta x = x_2 \\
& \quad 4 + (-10) = -6 \\
\text{Equation 2, using this model for subtraction: } & \quad \Delta x = x_2 - x_1 \\
& \quad -10 = -6 - 4
\end{align*}
\]

16. In the “number line picture” below, identify \( \Delta x \), \( x_1 \), and \( x_2 \), and then write TWO equations that describe the picture shown in the number line. Make sure the numbers in your equations are the same as the numbers shown on the number line. For example, if \( \Delta x \) is a negative number, make sure your equations have \( \Delta x \) being a negative number. In this example, \( \Delta x \) is a negative number.

\[
\begin{align*}
\text{Equation 1, using this model for addition: } & \quad x_1 + \Delta x = x_2 \\
& \quad 10 + (-6) = 4 \\
\text{Equation 2, using this model for subtraction: } & \quad \Delta x = x_2 - x_1 \\
& \quad -6 = 4 - 10
\end{align*}
\]

17. The “pictures” representing the two expressions \((4 + -10)\) and \((4 - 10)\) look very different, and yet these expressions are mathematically equivalent. Please explain this, as if you were explaining to a friend who is new to this way of thinking:
YOU’LL HAVE TO USE YOUR OWN WORDS. BE SURE TO CONNECT WITH ACTUAL MEANING, NOT “RULES” ABOUT FLIPPING SIGNS.

18. In the “number line picture” below, identify \( \Delta x, x_1, \) and \( x_2, \) and then write TWO equations that describe the picture shown in the number line. Make sure the numbers in your equations are the same as the numbers shown on the number line.

Equation 1, using this model for addition: \( x_1 + \Delta x = x_2 \)

\[-4 + 13 = 9\]

Equation 2, using this model for subtraction: \( \Delta x = x_2 - x_1 \)

\[13 = 9 - (-4)\]

19. In the “number line picture” below, identify \( \Delta x, x_1, \) and \( x_2, \) and then write TWO equations that describe the picture shown in the number line. Make sure the numbers in your equations are the same as the numbers shown on the number line. For example, if \( \Delta x \) is a negative number, make sure your equations have \( \Delta x \) being a negative number. In this example, \( \Delta x \) is a negative number.

Equation 1, using this model for addition: \( x_1 + \Delta x = x_2 \)

\[-4 + (-2) = -6\]

Equation 2, using this model for subtraction: \( \Delta x = x_2 - x_1 \)

\[-2 = -6 - (-4)\]
20. In the “number line picture” below, identify $\Delta x$, $x_1$, and $x_2$, and then write TWO equations that describe the picture shown in the number line. Make sure the numbers in your equations are the same as the numbers shown on the number line. For example, if $\Delta x$ is a negative number, make sure your equations have $\Delta x$ being a negative number.

Equation 1, using this model for addition: $x_1 + \Delta x = x_2$

$-5 + 8 = 3$

Equation 2, using this model for subtraction: $\Delta x = x_2 - x_1$

$8 = 3 - (-5)$

21. In the “number line picture” below, identify $\Delta x$, $x_1$, and $x_2$, and then write TWO equations that describe the picture shown in the number line. Make sure the numbers in your equations are the same as the numbers shown on the number line. For example, if $\Delta x$ is a negative number, make sure your equations have $\Delta x$ being a negative number. In this example, $\Delta x$ is a positive number.

Equation 1, using this model for addition: $x_1 + \Delta x = x_2$

$-6 + 2 = -4$

Equation 2, using this model for subtraction: $\Delta x = x_2 - x_1$

$2 = -4 - (-6)$
What is an equal sign? part 1: Equivalent numbers

Questions:

1. Write down 3 ways to visualize or conceptualize the number 5, using equations or number lines.

   ANSWERS WILL VARY

2. Write down 3 ways to visualize or conceptualize the number -2, using equations or number lines.

   ANSWERS WILL VARY

3. For the following, write down 3 addition problems that simplify to the given number, using a negative number in at least one of your examples. For example, if the given number is 2, you could write \(2=1+1, 2=-3+5, \) and \(2=5+(-3)\).
   Draw number lines if it is helpful.

   ANSWERS WILL VARY

HMMM… THE SOLUTIONS TO THESE QUESTIONS WERE NOT VERY HELPFUL.
I’VE NO DOUBT YOU CAN FIGURE IT OUT, THOUGH. 😊
Multiplication

1. Figure out what 9 times 9 is by knowing that (9)(10)=90. (Pretend you don’t know what (9)(9) is.
   
   Lots of ways to look at this. Here’s one way: (9)(10)=90, so (9)(9) is 9 less than 90. 90 minus 9 is 90 minus 10, plus 1. 80 + 1.  
   
   (9)(9)=81

2. Figure out what 13 times -3 is, without using a calculator. (hint: 13 repetitions is the same as 10 repetitions combined with 3 repetitions)
   
   (13)(-3)= ten repetitions of (-3) plus 3 repetitions of (-3). -30 + -9 = -39

3. Figure out what -13 times -3 is, without using a calculator.
   
   (-13)(-3). The opposite of repetitions of a negative number will be a positive number. So, the answer, whatever it ends up being, is the same as (13)(3). Go through the same thought process as the previous question to get (-13)(-3) = 39.

4. What is (-5) times (-4)? 20

5. What is (4) times (3)? 12

6. Simplify $x \cdot -2$, if $x = -4$ (note the dot next to the negative sign… this is not a subtraction exercise)
   
   (-4)(-2)= (4)(2)=8

7. Simplify $-3x$, if $x = 4$ (-3)(4)= - 12

8. Simplify $-7x$, if $x = 99$ (don’t use a calculator; 99 repetitions is the same as 100 minus 1 repetitions) (-7)(99) = (-7)(100) –(-7)(1) = -700 + 7 = -693. STOP! This is EXACTLY the kind of question that will be on a quiz and on the final. And, you will have to “show your thought process”.

9. Simplify $-3x$, if $x = -99$ (-3)(-99) = (3)(99) = (3)(100) – (3)(1)=300-3=297

10. Simplify $-3x$, if $x = 12$
    
    Here’s one way to solve this in your head:
    
    (-3)(12)=(-3)(10) + (-3)(2) = -30 + -6 = -36
11. Simplify \(-3x\), if \(x = 60\):
\[
(-3)(60) = (-3)(6)(10) = (-18)(10) = -180
\]

Questions and exercises:

12. Think about a long skinny rectangle, one side is 8 (it could be any unit on length you like, inches, centimeters, miles), and the other side is 1. Draw this rectangle here:

How many unit squares are in your rectangle? 8

13. Draw another rectangle, one side is 8 and the other side is 3.

How many repetitions of the rectangle you drew for the previous question are represented in this new rectangle? 3 repetitions.

If a piece of paper is 8 inches by 11 and half inches, the number of square inches (unit squares) is 11 and a half repetitions of 8 square inches, or 11 and a half multiplied to 8.

**Volume and multiplication**

Volume is the number of unit cubes in a 3-dimensional space; a unit cube, is (you guessed it), a cube with each side equal to 1 (in whatever units of length you are looking at, inches, feet, miles, kilometers, etc).

A unit cube:
its volume is equal to 1 because each side has a length equal to 1.

The volume of this “box” is 12, or 3 multiplied to 4 unit cubes. It is 3 repetitions of the box that is just 4 units long.
Questions:

14. Imagine a “box” that is 4 units long and 3 units deep, just like the figure above. This time, however, it is 5 units tall. Explain how this “box” is 5 repetitions of the “box” in the above figure.

*You’ll have to explain this in your own words...*

15. Explain why “volume of a box” is the same as “area of the box’s bottom” times its height.

*In your own words, something about “stacking” or “repeating” the bottom layer to make the whole “bigger box”*

16. What is 4+4+4+4+4 using multiplication notation? (4)(5) or (5)(4)

17. What is 5+5+5+5 using multiplication notation? (5)(4) or (4)(5)

18. What is -3 + -3 + -3 + -3 + -3 using multiplication notation? (-3)(5) or (5)(-3)

19. How many unit squares 1 inch by 1 inch are in an area that is 3 inches long and 5 inches wide?

15 unit squares (draw a picture and count them, if you like!)

20. What is the volume of a box if the area of the top (the lid) is 4 square inches and the height is 5 inches? The volume is (4 square inches) times (5 inches.)

20 cubic inches is the volume.

21. What is the area of a rectangle if one side is 4 and half and the other side is 3?

(4 plus half) repeated 3 times is 4 repeated 3 times plus half of 3. 12 plus 1 and a half. 13 and a half. The area is 13 and a half. You might solve this in other ways.

22. What is the area of a rectangle if one side is 4 and the other side is a half?

Half of 4 is 2. The area is 2.

23. What is the volume of a box if one side is 4, the other side is 7, and the other side is a half?

4 times 7 times half. 28 times half. Half of 28 is 14. There are other ways to do this, but the answer is 14.

24. What is the volume of a box if one side is 7, the other side is a half, and the other side is a 4?

Half of 7 is 3 and a half. 4 repetitions of 3 and a half is 4 repetitions of 3 plus 4 repetitions of half. 12 plus 2 is 14. The volume is 14.

25. What is the volume of a box if one side is 4, the other side is a half, and the other side is a 7?

Half of 4 is 2. 7 repetitions of 2 is 14. The volume is 14.
26. Does it ever matter what order you multiply numbers? *Nope. See the previous three questions and compare them.*

Simplify the following expressions, if possible.

27. \(x + 5\) *nope*
28. \(x + 5y\) *nope*
29. \(x - 5x = -4x\)
30. \(-x + 5y\) *nope*
31. \(-1 + x\) *nope*
32. \(-x - 5x = -6x\)
33. \(-3x - (-5y)\) *nope*
34. \(-3x + 3x = 0\)
35. \(4 + 5y\) *nope*
36. \(-x + 5y + 2x - 4y = x + y\)
37. \(-1 + 5y + 2 - 4y = 1 + y\) (note this is NOT the same as the previous question)
38. \(-x - 5y + 2 - 4y = -x - 9y + 2\)
39. \(-x + 5y - (-2x) - 4y = x + y\)
40. \(-2x + 5y + 2x - 5y = 0\)
41. \(-x - 5y + 2x + 4y = x - y\)
42. What does the word “coefficient” mean?
   *Answers will vary, but, basically, the coefficient means “how many repetitions” or “what part of” the variable is being represented. In the expression \(5x + 22y\), the coefficient of \(x\) is 5.*
Chapter 8
Solutions: Division

p. 1

Division

Questions.

1. Consider an arrow that represents the number -15. How many arrows that represent the number -3 fit into the arrow that represents the number -15? What is the division expression that answers this question? Draw an appropriate number line with these arrows. Is your answer a positive or negative number?

   Draw an arrow! 5 repetitions of (-3) fit into (-15).

   The division expression is \[ \frac{-15}{-3} \]

   The answer is positive 5

2. Draw an arrow that represents the number -15. Draw 3 equal arrows (repetitions) that combine to equal the number -15 (you are cutting -15 into three equal pieces). What number does each of those three arrows represent? What is the division expression that represents this situation? Is your answer a negative or positive number?

   Draw an arrow! If you cut (-15) into 3 pieces, each piece is -5

   The division expression is \[ \frac{-15}{3} \]

   The answer is negative 5

3. What is the opposite of 4 times -3? Write the multiplication expression that answers this question.

   \[ -(4)(-3) = \text{the opposite of negative 12 = 12} \]

4. What is the opposite of 4 times 3? Write the multiplication expression that answers this question.

   \[ -(4)(3) = -12 \]

5. What is the opposite of -4 times 3? Write the multiplication expression that answers this question.

   \[ -(-4)(3) = 12 \]

6. What is the opposite of -4 times -3? Write the multiplication expression that answers this question.

   \[ -(-4)(-3) = -12 \]

7. What is 4 times -3? Write the multiplication expression that answers this question.

   \[ (4)(-3) = -12 \]

8. What is the opposite of negative 3 times 7?

   \[ -(-3)(7) = 21 \]

9. How big is each piece if you cut -15 into 3 equal pieces? (is your answer negative or positive?)

   \[ \frac{-15}{3} = -5. \text{ The answer is negative 5} \]

10. What is the opposite of the number of the number of 3’s that fit into -15?

    \[ \frac{-15}{3} = 5. \text{ The answer is positive 5.} \]

11. How many halves fit into 1? Write the division expression that answers this question. I know you haven’t been introduced to fractions yet, but try to answer the question through sense-making.

    \[ \frac{1}{2}. \text{ There are 2 halves in 1. So, } \frac{1}{\frac{1}{2}} \text{ must be equal to 2.} \]
Questions:

12. What will be the size of a piece if you cut 1200 into 30 equal pieces?
   \[
   \frac{1200}{30} = 40
   \]

13. How many repetitions of 40 fit into 1200?
   \[
   \frac{1200}{40} = 30
   \]

14. How many repetitions of 30 fit into 1200?
   \[
   \frac{1200}{30} = 40
   \]

15. What will be the size of each piece if you cut 1200 into 40 equal pieces?
   \[
   \frac{1200}{40} = 30
   \]

The point of these isn’t to memorize the multiplication or division facts, but to think about the kinds of questions can be answered using multiplication and division.

Exercises: Do the following division expressions by thinking about division as “how many of the numerator (the expression in the bottom of a fraction-bar) fit into” the numerator (the top of the fraction-bar). It’s important to show all your thinking so you can review this and understand what you wrote. If there is a “remainder”, write your answer to include a fraction.

16. \[
\frac{x+x+x}{x} \quad \text{Think: how many of the ‘bottom’ fit into the ‘top’?}
\]
   3 of them. The answer is 3.

17. \[
\frac{3x}{x} \quad \text{remember: "3x" means "3 repetitions of x", or x+x+x.}
\]
   3

18. \[
\frac{y+y+y+y}{x} \quad \frac{y}{x} + 3
\]

19. \[
\frac{2x+x+y}{x} \quad 2 + 1 + \frac{y}{x} = 3 + \frac{y}{x}
\]

20. \[
\frac{x+x+x+x+x}{2x} \quad 2
\]

21. \[
\frac{x+x+x+x+x}{2x} \quad 2 + \frac{y}{2x}
\]

22. \[
\frac{-10}{-3} = \frac{(-9)+(-1)}{-3} = 3 + \frac{-1}{-3} = 3 + \frac{1}{3} = 3 \frac{1}{3}
\]

23. \[
\frac{-100}{3} = \frac{(-99)+(-1)}{3} = -33 + \frac{-1}{3} = -33 + \frac{-1}{3} = -33 \frac{1}{3}
\]

24. \[
\frac{100}{-3} = \frac{(99)+(1)}{-3} = -33 + \frac{1}{-3} = -33 + \frac{-1}{3} = -33 \frac{1}{3}
\]

25. \[
\frac{124}{5} = \frac{(120)+(4)}{5} = 20 + 4 + \frac{4}{5} = 24 \frac{4}{5}
\]
26. \( \frac{124}{50} = \frac{(120)+4}{50} = \frac{(100)+(20)+(4)}{50} = 2 + \frac{20}{50} + \frac{4}{50} = 2 + \frac{24}{50} = 2 + \frac{(2)(12)}{(2)(25)} = 2 + \frac{12}{25} \)

\( = 2 \frac{12}{25} \) (in all of these, there are many paths to the solution. If your path is different than what I wrote, great! BUT, understand each step of my path as well)

27. What is 2 divided by a half? In other words, how many halves fit into 2? Write this as a division expression using a fraction bar (this is like an appetizer problem to get you thinking about fractions, even before we get to that chapter). Answer the question by visualizing it.

\( \frac{2}{\frac{1}{2}} \) By thinking about it, I know that the answer must be 4. There are 4 half-cookies in 2 cookies.

28. How many thirds fit into 1? Answer the question by visualizing it, and write this as a division expression using a fraction bar.

\( \frac{1}{3} \) = 3

29. How many thirds fit into 3? Answer the question by visualizing it, and write this as a division expression using a fraction bar.

\( \frac{3}{\frac{1}{3}} = (3)(3) = 9 \)

30. What is the ratio of flour to sugar in a recipe that needs 8 cups flour and 5 cups sugar? \( \frac{8}{5} \)

31. What is the ratio of flour to sugar in a recipe that needs 8 cups flour and 2 cups sugar?

4 to 1, or \( \frac{4}{1} \), or simply, 4

32. What is the ratio of flour to sugar in a recipe that needs F cups flour and S cups sugar? \( \frac{F}{S} \)

33. What is the ratio of flour to sugar in a recipe that needs 1 cup flour and a third cup sugar? \( \frac{3}{1} \)

34. What is the ratio of flour to sugar in a recipe that needs a third cup flour and 1 cup sugar? \( \frac{1}{3} \)

35. What is the cost per pound of apples if it costs $15 to buy 3 pounds? $5 per pound.

36. What is the ratio of cost per ounce of something if 15 ounces cost $20? \( \frac{45}{3oz} \)

37. What is the cost per ounce of something if it costs $1 per half an ounce? \( \frac{25}{1oz} \)

38. If 16 Canadian dollars are worth about 12 US dollars, what is the ratio, expressed as a simplified fraction, of Canadian dollars to US dollars? \( \frac{45}{3CAD} \)

39. There are almost exactly 20 miles per 32 km. What is the ratio of miles to km? Express your answer as a simplified fraction. \( \frac{5miles}{8km} \)

40. My car used 10 gallons of gas going 300 miles. What was my average miles per gallon? Express the answer as a simplified fraction. \( \frac{30miles}{1gal} \)

41. I walked 8 miles in 2 hours. What was my speed (in miles per hour)? 4 miles per hour

42. I earned $500 working 20 hours. How much did I get paid per hour? 25 dollars per hour

43. I earned D dollars working H hours. What did I get paid per hour? \( \frac{D}{H} \)

44. My car used G gallons of gas going M miles. What was my average miles per gallon? \( \frac{M}{G} \)
**Fractions, Part 1**

Write the following phrases using mathematical notation, and simplify the resulting expressions:

1. Show how the following are all equal, using examples, pictures, number lines, anything that helps you make sense of the questions (like pictures of cake or pizza):
   - five multiplied to one fourth
   - five divided by 4, both:
     - “how many 4’s fit into 5?”
     - “the size of each piece if you cut 5 into 4 equal pieces”
   - the fraction \( \frac{5}{4} \)
   - \( 1 + \frac{1}{4} \)

2. One fourth, repeated 7 times, combined with one fourth, repeated 6 times.
   
   Here is one path to the solution. You might find another path... be sure you can find your own path to the solution, as well as follow this one:
   
   \[
   \left( \frac{1}{4} \right) (7) + \left( \frac{1}{4} \right) (6) = \\
   \left( \frac{1}{4} \right) (13) = \\
   \left( \frac{13}{4} \right) = \\
   \left( \frac{8+4+1}{4} \right) = \\
   2 + 1 + \left( \frac{1}{4} \right) = \\
   3 + \left( \frac{1}{4} \right) = \\
   3 \frac{1}{4}
   \]

3. One fourth added to three fourths.
\[
\left( \frac{1}{4} \right) + \left( \frac{3}{4} \right) = \\
\left( \frac{4}{4} \right) = 1
\]

4. The difference between one fifth and negative one fifth.

\[
\left( \frac{1}{5} \right) - \left( - \left( \frac{1}{5} \right) \right) = \\
\left( \frac{1}{5} \right) + \left( \frac{1}{5} \right) = \frac{2}{5}
\]

5. The difference between negative two fifths and negative four fifths.

\[
- \left( \frac{2}{5} \right) - \left( - \left( \frac{4}{5} \right) \right) = \\
- \left( \frac{2}{5} \right) + \left( \frac{4}{5} \right) = \\
\left( \frac{4}{5} \right) - \left( \frac{2}{5} \right) = \frac{2}{5}
\]

6. The difference between negative two hundred dollars and negative four hundred dollars.

You could do this: count up from negative 400 to negative 200.... You have to
go up 200.

\[
-200$ - (-400$)= \\
-200$ + 400$= \\
400$ - 200$= \\
200$
\]

7. Negative two thirds combined with four thirds.
\[-\left(\frac{2}{3}\right) + \left(\frac{4}{3}\right) = \]

\[\left(\frac{4}{3}\right) - \left(\frac{2}{3}\right) = \]

\[\left(\frac{2}{3}\right)\]

8. A debt of two dimes combined with four dimes (express your answer as a number of dimes).

\[-2\text{dimes} + 4\text{dimes} = 2 \text{ dimes}\]

9. The size of each serving if you are dividing 5 cookies among 3 children.

\[\left(\frac{5}{3}\right) = \frac{3 + 2}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{2}{3} = 1\frac{2}{3}\]

10. The difference between negative four fifths and negative three fifths.

\[-\left(\frac{4}{5}\right) - \left(-\frac{3}{5}\right) = -\left(\frac{4}{5}\right) + \left(\frac{3}{5}\right) = -\left(\frac{1}{5}\right)\]

11. Write the following as a number times a unit fraction, meaning, some number of repetitions of a unit fraction:

   a. 100 divided by 3 \(\left(100\right)\left(\frac{1}{3}\right)\)

   b. 3 divided by 100 \(\left(3\right)\left(\frac{1}{100}\right)\)

   c. 3 fourths \(\left(3\right)\left(\frac{1}{4}\right)\)

   d. 2 fifths \(\left(2\right)\left(\frac{1}{5}\right)\)

   e. One half \(\left(1\right)\left(\frac{1}{2}\right)\)

12. One fourth, repeated 8 times.
\[
\frac{1}{4} \cdot (8) = \frac{8}{4} = 2
\]

*Or*

\[
\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{8}{4} = 2
\]

13. The number of 4’s that fit into 8.

\[
\frac{8}{4} = 2
\]

*COMPARE THIS WITH THE PREVIOUS QUESTION*

14. The size of each serving, if I divide 8 cupcakes into 4 servings.

\[
\frac{8}{4} = 2
\]

*COMPARE THIS WITH THE PREVIOUS TWO QUESTIONS*

15. 8 divided by 4

\[
\frac{8}{4} = 2
\]

*COMPARE THIS WITH THE PREVIOUS THREE QUESTIONS*
Answer the following questions, express your answer in the simplest form you can think of, write your answers as fractions, not decimals. You can check your answers with a calculator, but you must be able to do these without using a calculator.

16. What is a third of 16?

   Here is one path to the solution, not necessarily the most efficient path:

   \[
   \frac{1}{3} (16) = \frac{16}{3} = \frac{9 + 1 + 6}{3} = 3 + \frac{1}{3} + 2 = 5 + \frac{1}{3} = 5\frac{1}{3}
   \]

17. What is 16 divided by 3?

   Here is one path to the solution. Compare this to the previous question

   \[
   \frac{16}{3} = \frac{15 + 1}{3} = 5 + \frac{1}{3} = 5\frac{1}{3}
   \]

18. What is \(\frac{1}{3}\), added to itself 16 times?

   I could write \(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} \ldots\) for a long time, but that’s silly.

   How’s this instead:

   \[
   \frac{1}{3} (16) = 5\frac{1}{3}
   \]

   Compare this to the previous two questions.

19. Simplify: \(\frac{4}{9} \cdot 27 \cdot \frac{1}{2}\) (think carefully about the easiest way to simplify this expression … you can make it difficult or easy depending on your approach)

   Answering this question the easiest way depends upon you knowing that it doesn’t matter what order you multiply numbers, and that \(\frac{\text{T}}{B} = (\text{T})(\frac{1}{B})\)

   \[
   \frac{4}{9} \cdot 27 \cdot \frac{1}{2} = \frac{1}{9} \cdot 4 \cdot 27 \cdot \frac{1}{2} =
   \]
20. What is 16 times \( \frac{9}{2} \)?

Answering this question the easiest way depends upon you knowing that it doesn’t matter what order you multiply numbers, and that \( \frac{T}{B} = (T') \left( \frac{1}{B} \right) \)

\[
16 \cdot \frac{9}{2} = \\
16 \cdot 9 \cdot \frac{1}{2} = \\
9 \cdot \frac{16}{2} = 9 \cdot 8 = \\
\text{if you remember your times tables, you can be done now. But let’s suppose you’ve forgotten what 9 times 8 is so you need to figure it out} \\
9 \cdot 8 = (10 \cdot 8) - (1 \cdot 8) = 80 - 8 = 72
\]

21. What is 16 times 21, all divided by 7, and all times \( \frac{1}{2} \)? (remember to make life easy for yourself by considering different approaches before you start doing any calculations)

\[
\left( \frac{16 \cdot 21}{7} \right) \left( \frac{1}{2} \right) = \\
\left( \frac{16}{2} \right) \left( \frac{21}{7} \right) = 8 \cdot 3 = 24
\]

22. What is half of 21, all divided by 7, all times 4? (you should be able to do this without a calculator, and fairly quickly, by noticing that you can divide by 7 before you figure out what half of 21 is)

\[
\left( \frac{21}{2} \right) \left( \frac{1}{7} \right) 4 = \left( \frac{21}{7} \right) \left( \frac{1}{2} \right) = (3)(2) = 6 = \\
\]

Here are some multiple choice questions for you to answer; circle your answers, and write notes to yourself explaining your thinking.

23. \( \frac{10}{2} \) makes more sense as “how many 2’s fit into 10?” or “what part of 2 fits into 10?”
24. \( \frac{2}{8} \) makes more sense as “how many 8’s fit into 2?” or “what part of 8 fits into 2?”

25. \( \frac{15}{30} \) makes more sense as “how many 30’s fit into 15?” or “what part of 30 fits into 15?”

26. \( \frac{100}{10} \) makes more sense as “how many 10’s fit into 100?” or “what part of 10 fits into 100?”

27. \( \frac{15}{2} \) makes more sense as “how many 2’s fit into 15?” or “what part of 2 fits into 15?”
Exercises: For each of the following questions, write a mathematical expression that can be used to answer the question, then simplify your answer, perhaps by changing the numbers so they “look” different. The new arrangement should make answering the question easier. Try to find the most efficient way to answer the questions by being strategic about what order you use multiplication and/or division.

28. What is 8 repetitions of $\frac{3}{4}$?

$$8 \cdot \frac{3}{4} = \frac{8}{4} \cdot 3 = 2 \cdot 3 = 6$$

29. What is $\frac{3}{4}$ of 8?

*Compare to previous question*

$$\frac{3}{4} \cdot 8 = \frac{8}{4} \cdot 3 = 2 \cdot 3 = 6$$

30. What is 3 repetitions of $\frac{2}{7}$ combined with 4 repetitions of $\frac{1}{7}$?

$$3 \cdot \frac{2}{7} + 4 \cdot \frac{1}{7} =$$

$$\frac{6}{7} + \frac{4}{7} = \frac{10}{7} =$$

$$\frac{7+3}{7} =$$

$$1 + \frac{3}{7} = 1 \frac{3}{7}$$

31. What part of 15 fits into 5?

$$\frac{5}{15} = \frac{5}{(5)(3)} = \left(\frac{5}{5}\right) \left(\frac{1}{3}\right) = (1) \left(\frac{1}{3}\right)$$
\[ \frac{1}{3} \text{ of } 15 \text{ is the same as } 5 \]

32. How many 5’s fit into 15?

*Compare this to the previous question*

\[ \frac{15}{5} = \frac{(5)(3)}{(5)} = \left(\frac{5}{5}\right)(3) = (1)(3) \]

*3 fives fit into 15*
33. What is \( \frac{2}{7} \) of 3 plus 4 repetitions of \( \frac{1}{7} \)?

\[
\frac{2}{7} \cdot 3 + 4 \cdot \frac{1}{7} = 6 \cdot \frac{1}{7} + 4 \cdot \frac{1}{7} = 10 \cdot \frac{1}{7} = \frac{7+3}{7} = 1 + \frac{3}{7} = 1 \frac{3}{7}
\]

34. What is \( \frac{2}{7} \) of 3 plus 4 divided by 7?

\[
\frac{2}{7} \cdot 3 + \frac{4}{7} = 6 \cdot \frac{1}{7} + \frac{1}{7} = 10 \cdot \frac{1}{7} = \frac{7+3}{7} = 1 + \frac{3}{7} = 1 \frac{3}{7}
\]

35. What is \( \frac{2}{7} \) of 3 plus \( \frac{1}{7} \) of 4?

\[
\frac{2}{7} \cdot 3 + \frac{1}{7} \cdot 4 = 6 \cdot \frac{1}{7} + 4 \cdot \frac{1}{7} = 10 \cdot \frac{1}{7} = \frac{7+3}{7} = 1 + \frac{3}{7} = 1 \frac{3}{7}
\]

36. What part of 4 fits into 2?

\[
\frac{2}{4} = \frac{1}{2}
\]

"Half of 4 fits into 1 half."

37. What part of 2 fits into 3?

\[
\frac{3}{2} = \frac{2+1}{2} = 1 + \frac{1}{2}
\]

"One and half two’s fit into 3."

38. What do you think a third of a fourth is? How big is each piece if you cut \( \frac{1}{4} \) into three equal pieces?

"Your answers will vary... there is no wrong answer to the question "what do you think"."
Before you go on, think about this carefully.
39. On the next number line, one fourth \(\frac{1}{4}\) is drawn as an arrow on a number line.

a. Imagine a third of that arrow (you cut it into three pieces, each piece is a third of the arrow) … what number would that represent? **Draw a third of the arrow and come up with your answer to “what is a third of a fourth?”** A third of a fourth is a twelfth.

\[\begin{array}{c}
\text{0} \\
\text{\hspace{1cm}} \\
\text{1}
\end{array}\]

b. Here is a picture of one third \(\frac{1}{3}\), shown as an arrow on a number line. Imagine cutting that arrow into four pieces. What number would each piece represent? **Draw a fourth of this arrow and answer the question “what is a fourth of one third?”** A fourth of one third is a twelfth.

\[\begin{array}{c}
\text{0} \\
\text{\hspace{1cm}} \\
\text{Δx} \\
\text{\hspace{1cm}} \\
\text{1}
\end{array}\]

c. Based on what you wrote above, what do you think about \(\frac{1}{3} \times \frac{1}{4}\)? It means “a third of a fourth” and, “a fourth of a third”, and “a third divided by 4” and, “a fourth divided by 3”. With all those meanings rattling around in your head, how do you simplify it? Write your ideas here: \(\frac{1}{3} \times \frac{1}{4} = \frac{1}{3 \cdot 4} = \frac{1}{12}\)

d. Before you read any further, if you can, write a general rule for multiplying unit fractions to each other. For example, what is \(\frac{1}{b} \cdot \frac{1}{c}\), written as one fraction? \(\left(\frac{1}{b}\right) \left(\frac{1}{c}\right) = \left(\frac{1}{b \cdot c}\right)\)

40. Given that \(\frac{T}{B} = T \times \frac{1}{B}\), write how you think the expression \(\frac{1}{3} \times \frac{1}{4}\) can be written with only one fraction bar, instead of two. Spend some time thinking about it. In this case, \(T = \frac{1}{3}\) and \(B = 4\).

\[\frac{\frac{1}{3}}{4} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}\]
41. If b and d are not 0, is the following always true? \( \left( \frac{a}{b} \right) \left( \frac{c}{d} \right) = \frac{(a \cdot c)}{(b \cdot d)} \)

Explain. (Hint: you could write each fraction as the numerator times a unit fraction.

For example, \( \frac{a}{b} \) can be rewritten as \( a \left( \frac{1}{b} \right) \) (meaning, \( a \) times \( \frac{1}{b} \)). Also, it never matters what order you multiply numbers.

*Your explanation will depend on you, but, yes, this is always true. Even if you are not doing this work for a class, or this homework is not being turned in, writing an explanation here that makes sense is probably one of the most important exercises you can complete in this section.*

42. What is two thirds of 1 fifth?

\[
\frac{2}{3} \div \frac{1}{5} = \frac{2}{15}
\]

43. What is half of 6 fifths?

\[
\frac{1}{2} \div \frac{6}{5} = \frac{6}{2} \div \frac{1}{5} = 3 \cdot \frac{1}{5}
\]

*Half of 6 fifths, is the same as half of 6, multiplied to a fifth, or 3 repetitions of a fifth, of 3 fifths.*

44. What is 3 fifths of a half?

\[
\frac{3}{5} \div \frac{1}{2} = \frac{3}{10}
\]

45. What is 7 times two thirds of three quarters?

\[
7 \cdot \frac{2}{3} \cdot \frac{3}{4} it\ doesn't\ matter\ what\ order\ I\ multiply
\]

\[
7 \cdot \frac{3}{4} = 7 \cdot \frac{1}{2} = \frac{7}{2}
\]

46. What is 3 halves times 4 fifths?

\[
\frac{3}{2} \cdot \frac{4}{5} = \frac{3}{2} \cdot \frac{1}{5} = \frac{6}{5}
\]

47. How many 5’s fit into 5? 1

48. What part of 5 fits into 5? 1, *Exactly all of it.*

49. If you cut 5 into 5 equal pieces, how big is each piece? 1
50. What is $\frac{1}{5}$ of 5? 1

51. What is $\frac{1}{5}$, repeated 5 times? 1

Exercises:

Simplify, or answer the specific question. If your answer is a fraction, express your fraction in simplified form. If your answer is greater than the number 1, express your answer as a number and a simplified fraction. Try to be strategic in how you solve the problems .... Think about your plan before you begin, try to find the simplest, most straightforward way to simplify the expression. The first exercise is done for you, in two different ways.

52. $5 - \frac{2}{3} + \frac{1}{3}$.

53. $\frac{2}{3} - \frac{3}{3}$ bananas minus 3 bananas is a debt of 1 banana.

\[
\frac{2}{3} - \frac{3}{3} = \frac{2}{3} - 1 = \frac{2}{3} - \frac{3}{3} = \frac{2}{3} \cdot \frac{5}{5} - \frac{3}{3} = \frac{2}{3} \cdot \frac{5}{5} - \frac{3}{3} = \frac{10}{15} - \frac{9}{15} = \frac{1}{15}
\]

54. $-\frac{2}{3} + \frac{3}{3}$

55. $-\frac{2}{3} - \frac{3}{3}$

56. $-\frac{2}{3} - \frac{3}{5}$

\[
-\frac{2}{3} \cdot \frac{5}{5} + \frac{3}{5} \cdot \frac{3}{3} = \left(\frac{-10}{15}\right) + \left(\frac{9}{15}\right) = -\frac{1}{15}
\]

57. $\frac{2}{3} - 1$

Ask yourself: what’s the difference between $\frac{2}{3}$ and 1? How do you get from 1 to $\frac{2}{3}$? You go “backwards” $\frac{1}{3}$. So the answer is $-\frac{1}{3}$.

You could do this by finding a common denominator and simplifying.

58. $-\frac{2}{3} - 2$
Ask yourself: what’s the difference between $-\frac{2}{3}$ and 2? How do you get from 2 to $-\frac{2}{3}$? You go “backwards” 2 and $\frac{2}{3}$. So the answer is $-\frac{2}{3}$.

You could do this by finding a common denominator and simplifying:

$$-\frac{2}{3} - 2 \cdot \frac{3}{3} = -\frac{2}{3} - \frac{6}{3} = -\frac{8}{3} = -\frac{6 + 2}{3} = -\frac{2}{3}$$

As always, there are many paths to a solution.

Before you decide on a path, think about the options: which path is easiest? The “easiest” path will be different for different people.

59. $\frac{2}{-3} - 2$

See previous question 😊 Are they equivalent? Yes!

60. $-2 - \frac{-2}{5}$

$$-2 - \frac{-2}{5} = -2 + \frac{2}{5}$$

$$-2 + \frac{2}{5} = (-2) \frac{5}{5} + \frac{2}{5}$$

$$-\frac{10}{5} + \frac{2}{5} = -\frac{8}{5} = \frac{-5 + 3}{5} = -1 + \frac{-3}{5} = -(1 \frac{3}{5})$$

61. $-2 - (\frac{2}{5} \cdot \frac{3}{4})$

$$-2 - (\frac{2}{5} \cdot \frac{3}{4}) = -2 - (\frac{3}{5} \cdot \frac{2}{4})$$

(it doesn’t matter what order you multiply numbers)

$$-2 - (\frac{3}{5} \cdot \frac{1}{2}) = -2 - \frac{3}{10} = -2 + \frac{-3}{10} = -2 \frac{3}{10}$$

(as always, this is one path to the solution... you might find your own path)

62. $-\left(\frac{2}{5} \cdot \frac{3}{4}\right) - \left(\frac{2}{5} \cdot \frac{3}{4}\right)$ (consider several ways to simplify this BEFORE you begin)

$$-\left(\frac{2}{5} \cdot \frac{3}{4}\right) + \left(-\frac{2}{5} \cdot \frac{3}{4}\right) = -2 \left(\frac{2}{5} \cdot \frac{3}{4}\right)$$

(-1 banana) + (-1 banana) = -2 bananas. In this case, the banana is $\left(\frac{2}{5} \cdot \frac{3}{4}\right)$

$$-2 \left(\frac{2}{5} \cdot \frac{3}{4}\right) = -2 \left(\frac{2}{5} \cdot \frac{3}{4}\right) = \left(-\frac{2 \cdot 2}{5} \cdot \frac{3}{4}\right)$$

$$\left(\frac{-2 \cdot 2}{5} \cdot \frac{3}{4}\right) = \left(-\frac{-2 \cdot 2}{4} \cdot \frac{3}{5}\right) = (-1) \left(\frac{3}{5}\right) = -\frac{3}{5}$$

This is a great example of how you can save yourself some time of you spend MORE time at the beginning deciding on your path to the solution, and the path to the solution might take LESS time)
63. \( \frac{3}{10} - \left( \frac{2}{5} \cdot \frac{3}{4} \right) \)

\[
\frac{3}{10} - \left( \frac{2}{5} \cdot \frac{3}{4} \right) = \left( \frac{3}{10} \right) - \left( \frac{2}{5} \cdot \frac{3}{4} \right) = \left( \frac{3}{10} \right) - \left( \frac{1}{2} \cdot \frac{3}{5} \right)
\]

\[
\frac{3}{10} - \left( \frac{1}{2} \cdot \frac{3}{5} \right) = \left( \frac{3}{10} \right) - \left( \frac{3}{10} \right) = 0
\]

64. \( \frac{2}{x} - \frac{3}{x} \) (write this as an expression with only one fraction bar, meaning as one fraction)

\[
\frac{2}{x} - \frac{3}{x} = -\frac{1}{x}
\]

65. \( \frac{2}{x} - \frac{3}{5} \) (write this as one fraction)

\[
\frac{2}{x} - \frac{3}{5} = \frac{2}{x} \cdot \frac{5}{5} - \frac{3}{5}
\]

\[
\frac{2}{x} \cdot \frac{5}{5} - \frac{3}{5} = \frac{10 - 3x}{5x}
\]

66. \( \frac{2}{x} - \frac{3x}{5} \) (write this as one fraction)

\[
\frac{2}{x} - \frac{3x}{5} = \frac{2}{x} \cdot \frac{5}{5} - \frac{3x}{5}
\]

\[
\frac{2}{x} \cdot \frac{5}{5} - \frac{3x}{5} = \frac{10 - 3x}{5x}
\]

\[
x \cdot x \text{ is also written as } x^2, \text{ so the answer can be written like this as well:}
\]

\[
\frac{10 - 3x^2}{5x}
\]

67. \( \frac{2}{x} - \frac{3}{5x} \) (write this as one fraction)

\[
\frac{2}{x} - \frac{3}{5x} = \left( \frac{2}{x} \right) \left( \frac{5}{5} \right) - \frac{3}{5x}
\]

\[
\left( \frac{2}{x} \right) \left( \frac{5}{5} \right) - \frac{3}{5x} = \left( \frac{10}{5x} \right) - \left( \frac{3}{5x} \right) = \frac{7}{5x}
\]

68. \( 2 - \frac{3}{5x} \) (write this as one fraction)

\[
2 - \frac{3}{5x} = \frac{2 \cdot 5x}{5x} - \frac{3}{5x}
\]

\[
\frac{2 \cdot 5x}{5x} - \frac{3}{5x} = \frac{10x - 3}{5x}
\]

69. \( x - \frac{3}{5} \) (write this as one fraction)

\[
x - \frac{3}{5} = x \cdot \frac{5}{5} - \frac{3}{5}
\]

\[
\frac{5x}{5} - \frac{3}{5} = \frac{5x - 3}{5}
\]
70. \(2x - \frac{3x}{5}\) (write this as one fraction)

\[
2x - \frac{3x}{5} = 2\cdot \frac{5}{5} - \frac{3x}{5}
\]

\[
\frac{10x}{5} - \frac{3x}{5} = \frac{7x}{5}
\]

71. \(-\frac{2}{x} - \frac{-3}{5x}\) (write this as one fraction)

\[-\frac{2}{x} - \frac{-3}{5x} = -\frac{2}{x} \cdot \frac{5}{5} - \frac{-3}{5x}
\]

\[-\frac{2 \cdot 5}{x \cdot 5} + \frac{-3}{5x} = -\frac{10}{5x} - \frac{-3}{5x}
\]

\[-\frac{10}{5x} - \frac{-3}{5x} = -\frac{10}{5x} + \frac{3}{5x} = -\frac{7}{5x}
\]
Division of fractions exercises:

1. Show why \( \frac{2}{3} \) and \( 4 \cdot \frac{1}{9} \cdot \frac{1}{5} \cdot \frac{1}{7} \) are equivalent expressions. Your answer should reflect your thinking, feel free to refer back to the text for help.

2. How many halves fit into 1? In other words, how many half-cookies are there in 1 cookie?
   2

3. How many thirds fit into 1? In other words, how many third-pizzas fit into a pizza?
   3

4. How many halves fit into 5? In other words, how many half-cookies are there in 5 cookies?
   10

5. Simplify each of the following based on your answers to the questions above:
   a. \( \frac{1}{2} \cdot \frac{1}{2} = 2 \)
   b. \( \frac{1}{3} \cdot 3 = 3 \)
   c. \( \frac{5}{2} \cdot \frac{1}{3} = 10 \)

6. Based on what your wrote above, what is a “rule” you could use for dividing a number by a unit fraction? In other words, what is a rule you could use to simplify the following expression (as long as B≠0)
   \[
   \frac{A}{\frac{1}{B}} = (A)(B)
   \]

7. Explain why \( \frac{2}{3} \cdot \frac{9}{9} \) is equivalent to \( \frac{4}{5} \cdot \frac{1}{9} \). Because \( \frac{4}{9} = 4 \cdot \frac{9}{9} \), because it doesn’t matter what order I multiply, so I can “move” the numbers so that it is \( 4 \cdot \frac{9}{9} = 4 \cdot 1 \)

8. Explain why \( \frac{4}{7} \cdot \frac{1}{9} \) simplifies to \( \frac{4}{7} \cdot \frac{1}{9} \). Because \( \frac{5}{7} \cdot 7 = 5 \), so the expression in bold is equivalent to 5:

9. \( \frac{\frac{3}{5}}{\frac{4}{9}} = \frac{28}{45} \)
10. \( \frac{7}{2} \div 2 = \frac{7}{4} \)

11. Explain why \( \frac{21}{4} \) is equivalent to 5 and a quarter. 4 fits into 21 five times, with 1 “leftover”, and that “leftover” has to get divided by 4 as well.

12. How many servings (including fractions of a serving) are there in 3 and a half cups of yoghurt, if a serving is 2 thirds of a cup? *For the solution to this, please read the text prior to this exercise.*

13. What is 2 thirds of 3 and a half cups of yoghurt? \( \left( \frac{2}{3} \right) \left( 3 + \frac{1}{2} \right) = 2 + \frac{1}{3} \)

14. If I have 3 and a half cups of yoghurt, and you have 2 thirds of a cup, what’s the ratio of what I have to what you have? \( \frac{3+\frac{1}{2}}{\frac{2}{3}} = \frac{(3+\frac{1}{2}) \cdot 2}{\frac{2}{3} \cdot 2} \)

\[ \frac{7}{\left( \frac{3}{2} \right)^3} = \frac{7\cdot3}{\left( \frac{3}{2} \right)^3} = \frac{21}{4}. \text{ The ratio is 21 to 4.} \]

15. If I have 3 and a half cups of yoghurt, and you have 2 thirds of a cup, what do we have combined? \( 3 + \frac{1}{2} + \frac{2}{3} = 3 + \frac{1\cdot3}{2\cdot3} + \frac{2\cdot2}{3\cdot2} \)

\[ 3 + \frac{\frac{1}{3}}{2\cdot3} + \frac{\frac{2}{2}}{3\cdot2} = 3 + \frac{\frac{3}{3} + \frac{4}{6}}{6} \]

\[ 3 + \frac{\frac{3+4}{6}}{6} = 3 + 1 + \frac{1}{6}. \text{ We have 4 cups and 1 sixths of a cup, altogether.} \]

16. If I have 3 and a half cups of yoghurt, and you have 2 thirds of a cup, what’s the difference between what I have and what you have? \( 3 + \frac{1}{2} - \frac{2}{3} = 3 + \frac{1\cdot3}{2\cdot3} - \frac{2\cdot2}{3\cdot2} \)

\[ 3 + \frac{\frac{1}{3}}{2\cdot3} - \frac{\frac{2}{2}}{3\cdot2} = 3 + \frac{\frac{3}{3} - \frac{4}{6}}{6} \]

\[ 3 + \frac{\frac{3-4}{6}}{6} = 3 + \frac{-1}{6} \]

\[ 3 + \frac{-\frac{1}{6}}{2+\frac{5}{6}}. \text{ The difference between what I have and what you have is 2 cups and 5 sixths of a cup.} \]

17. If I have 3 and a half cups of yoghurt, and you have 2 thirds of a cup, how much do you need to get so that you would have same amount as I do? Same answer as previous question; 2 cups and 5 sixth of a cup.

18. What is 3 and a half times more than 2 thirds of a cup? \( \left( \frac{2}{3} \right) \left( 3 + \frac{1}{2} \right) = 2 + \frac{1}{3} \)

19. If I have 3 and a half cups of yoghurt, and you have 2 thirds of a cup, what’s the ratio of what you have to what I have? *See the answer to question #14. If the ratio of what I have to what you have is 21 to 4, then the ratio of what you have to what I have is 4 to 21.*

20. Fill in the blank: 7 divided by 2 is the same as 7 times \( \frac{1}{2} \)

21. Fill in the blank: The number of thirds that fit into 9 is the same as 9 times 3

22. Fill in the blank: The number of threes that fit into 9 is the same as 9 times \( \frac{1}{3} \)
23. What is “Nine divided into thirds”? Hmmmm…. This is tricky. It’s the kind of language that makes problem-solving confusing. I think it means “9 divided by 3”, but I would admit that the language is confusing. Avoid this kind of phrasing, and if someone asks you a question like this, be sure you are clear what they are asking.

Nine divided by 3 is \( \frac{9}{3} = 3 \)

Nine divided by a third is \( \frac{9}{\frac{1}{3}} = 27 \)

24. My share of Great Aunt Gertrude’s estate is a third of the total value of the estate. My uncle’s share of the estate is 2 thirds of the total value of the estate. My uncle’s share is how many times bigger than mine? (how many of one share fit into another share?)

\[
\frac{1}{3} x = \text{my share}, x = \text{total value of estate}
\]

\[
\frac{2}{3} x = \text{uncle's share}, x = \text{total value of estate}
\]

\[
\frac{\text{uncle's share}}{\text{my share}} = \frac{\frac{2}{3} x}{\frac{1}{3} x} = 2
\]

My uncle’s share is twice as much as mine.

25. My share of Great Aunt Sophie’s estate is \( \frac{2}{7} \) of the total value of the estate. My uncle’s share of the estate is \( \frac{3}{5} \) of the total. My uncle’s share is how many times bigger than mine? (how many of one share fit into another share?)

\[
\frac{2}{7} x = \text{my share, } x = \text{total value of estate}
\]

\[
\frac{3}{5} x = \text{uncle's share, } x = \text{total value of estate}
\]

\[
\frac{\text{uncle's share}}{\text{my share}} = \frac{\frac{3}{5} x}{\frac{2}{7} x} = \frac{3}{5} \cdot \frac{1}{2} \cdot \frac{1}{7}
\]

\[
\frac{3}{5} \cdot \frac{1}{2} \cdot \frac{1}{7} = \frac{3}{5} \cdot \frac{1}{2} \cdot \frac{1}{7} = \frac{3 \cdot 1}{5 \cdot 2 \cdot 7} = \frac{3}{10} \cdot \frac{21}{10} = \frac{21 + 10}{10} = \frac{31}{10} = 3 + \frac{1}{10} = 3 \frac{1}{10}
\]

Uncle’s share is 2 and a tenth bigger than mine.
26. What part of, or how many, $\frac{2}{3}$ fits into $\frac{3}{7}$? Before you do any calculations, ask yourself if the answer should be “greater than one $\frac{2}{3}$ fits into $\frac{3}{7}$” or “only a part of $\frac{2}{3}$ will fit into $\frac{3}{7}$”. In other words, can you figure out which is bigger, $\frac{2}{3}$ or $\frac{3}{7}$, without doing any calculations? (hint: you should be able to).

$\frac{2}{3}$ is bigger than a half (2 is more than half of 3)

$\frac{3}{7}$ is less than half (because 3 is less than half of 7)

So, only part of $\frac{2}{3}$ will fit into $\frac{3}{7}$. So, the answer should be less than 1. Now that I’ve thought about that, I can find the exact answer. If my answer ends up being bigger than 1, I’ll know I made a mistake.

$$\frac{\frac{2}{3}}{\frac{3}{7}} = \frac{\frac{2}{3} \cdot \frac{7}{3}}{\frac{3}{7} \cdot \frac{7}{3}} = \frac{2}{3} \cdot \frac{7}{3} \cdot \frac{7}{3} = \frac{2 \cdot 7}{3 \cdot 7} = \frac{14}{21} = \frac{2}{3} \cdot \frac{7}{3}$$

So, only part of $\frac{2}{3}$ will fit into $\frac{3}{7}$. So, the answer should be less than 1. Now that I’ve thought about that, I can find the exact answer. If my answer ends up being bigger than 1, I’ll know I made a mistake.

$\frac{\frac{2}{3}}{\frac{3}{7}} = \frac{\frac{2}{3} \cdot \frac{7}{3}}{\frac{3}{7} \cdot \frac{7}{3}} = \frac{2}{3} \cdot \frac{7}{3} \cdot \frac{7}{3} = \frac{2 \cdot 7}{3 \cdot 7} = \frac{14}{21} = \frac{2}{3} \cdot \frac{7}{3}$

27. What is $\frac{2x}{3y}$ divided by $\frac{3}{7}$?

$$\frac{\frac{2x}{3y}}{\frac{3}{7}} = \frac{2x \cdot \frac{7}{3}}{3 \cdot \frac{7}{3}} = \frac{2x \cdot \frac{7}{3}}{3 \cdot \frac{7}{3}} = \frac{2x}{3y} \cdot \frac{7}{3}$$

$$\frac{2x}{3y} \cdot \frac{7}{3} = \frac{14x}{9y}$$

28. Summarize how you will remember, for the rest of your life, how to figure out how to divide fractions if you forget the procedure.

You’ll have to write your own solution here.

29.

a. Is memorizing a procedure different from working out solutions that make sense? You’ll have to write your own solution here.

b. What are the advantages of memorizing, what are the drawbacks? To me, memorizing allows you to answer quickly, and that’s sometimes a good thing. But, ideas that are memorized but not fit into other ideas like puzzle pieces are easier to forget.
30. What is half of two thirds? *One third.*

31. What is two thirds divided by 2? *One third.*

32. Simplify: \[
\frac{3+\frac{1}{3}}{2} = \frac{21}{4} = 5 + \frac{1}{4}
\]

33. Putting together all you know about multiplication, division, fractions, adding, and subtracting, answer the following questions that refer to the unknown numbers marked on this number line. Remember, negative 1,000 is a smaller number than 5, simply because it is negative.

\[a \quad b \quad c \quad d \quad e \quad f \quad g \quad h\]

-2 -1 0 1 2 3

a. Which is bigger, \(f\) or \((f)(e)\)? What is your reasoning, how do you know?

\(e\) is less than 1, so \((f)(e)\) means “a piece of \(f\)”, which is smaller than \(f\). So, \(f\) is bigger than \((f)(e)\).

b. Which is bigger, \(g\) or \((g)(h)\)? What is your reasoning, how do you know?

\(g\) is and \(h\) are both bigger than 1. So, I’m repeating \(g\), \(h\) times. Or, I’m repeating \(h\), \(g\) times. Either way, \((g)(h)\) has to be bigger than \(g\).

c. Which is bigger, \(f\) or \((f)(g)\)? What is your reasoning, how do you know? You can think of \((f)(g)\) as \((g)\) repetitions of \((f)\). \((g)\) is bigger than 1, so \((g)\) repetitions of \((f)\) has got to be bigger than \((f)\). So, \((f)(g)\) is bigger.

d. Which is bigger, \(g\) or \((g)(d)\)? What is your reasoning, how do you know? \((g)\) is a positive number. \((d)\) is a negative number. \((g)(d)\) is a negative number, so \((g)\) is bigger.

e. Which is bigger, \(g\) or \((g)(b)\)? What is your reasoning, how do you know? Same reasoning as previous question. \((g)\) is bigger because it is a positive number.

f. Which is bigger, \((a)\) or \((a)(b)\)? What is your reasoning, how do you know? Both \((a)\) and \((b)\) are negative, so \((a)(b)\) is positive. \((a)(b)\) is bigger.

g. What is bigger, \((a)(b)\) or \((f)\)? What is your reasoning, how do you know? \((b)\) looks pretty close to -1, so \((a)(b)\) would be pretty close to \(-(a)\). \(-(a)\) is almost equal to 2 (maybe 1.75-ish), so \(-(a)\) is bigger than \((f)\). [Note: \(-(a)\) is a positive number!]
h. What is bigger, \((b)(c)\) or \((b)(d)\)? What is your reasoning, how do you know? (b)(c) is pretty close to \(-c\), which would be a little more than 0.5. (b)(d) is pretty close to \(-(d)\), which looks like it’s around -.25 or -.3. (b)(c) is bigger.

i. What is bigger, \((b)(e)\) or \((b)(f)\)? What is your reasoning, how do you know? (b)(e) is around -0.3, (b)(f) is around -0.6. (b)(e) is bigger [the absolute value is smaller, but -0.3 is bigger than -0.6].

j. What is bigger, \(-b\) or \((c)(d)\)? What is your reasoning, how do you know? \(-b\) is close to 1. (c)(d) will be a small positive number (it’s a piece of a piece, or, a fraction of a fraction). So, \(-b\) is bigger.

k. What is bigger, \(\frac{f}{e}\), or \(\frac{g}{h}\)? What is your reasoning, how do you know? \(\frac{f}{e}\) is “how many e fit into f?” It looks like about 3 or 4 e fit into f. \(\frac{g}{h}\) is “what part of h fits into g?” and the answer to that is less than 1. So, \(\frac{f}{e}\) is bigger.

l. What is bigger, \(\frac{c}{d}\), or \(b\)? What is your reasoning, how do you know? \(\frac{c}{d}\) means “how many d fit into c?” and the answer looks like between 2 and 3. B is very close to negative 1, so it’s smaller because it’s a negative number.

m. What is bigger, \(\frac{c}{d}\), or \(-b\)? What is your reasoning, how do you know? \(\frac{c}{d}\) means “how many d fit into c?” It looks like between 2 and 3. \(-b\) is close to 1, so \(\frac{c}{d}\) is bigger.

n. Of all the labeled numbers, which one is closest to \((a)(b)\)? What is your reasoning, how do you know? \((a)(b)\) is close to \(-a\), which looks like it’s about almost 2. The closest is g.
Exercises:

Find the solutions to the following equations without doing any formal algebra... consider what the equation means and then find the solution.

1. \((x)(x)(x)=1\) (in words: what number, when multiplied to itself 3 times, is 1)
   
   \(x\) must be equal to 1.

2. \((x)(x)=1\) (in words: what number, when multiplied to itself twice, is 1)
   
   \(x\) could be 1 or \(-1\)

3. \(10-x=8\) (If you remove this number from 10, you’d get 8. Or, the difference between 10 and this number is 8)
   
   \(x\) must be 2

4. \(2x=10\) (in words: twice this number is 10, or, if you double this number, you’d get 10).
   
   \(x\) must be 5

Exercises

In the following examples, explain what mathematical operation is done to the first equation to get to the second equation. (you are NOT necessarily solving for \(x\), you’ll get to do that later...). Please check your answers as you go because it’s important you don’t re-enforce wrong ideas!

5. \(3x + 3 = 6\) \(\rightarrow\) \(3x = 3\)

   Subtract 3 from both sides of the first equation to get the second equation

6. \(3x + 3 = 6\) \(\rightarrow\) \(x + 1 = 2\)

   Take a third of both sides of the first equation to get the second equation. Divide both sides of the first equation by 3 to get the second equation. Multiply both sides of the first equation by \(\frac{1}{3}\) to get the second equation.

7. \(\frac{3}{x} = 9\) \(\rightarrow\) \(3 = 9x\)
Multiply both sides of the first equation by \(x\).

8. \(x - 3 = 5\) \(x = 8\)

Add 3 to both sides of the first equation

9. \(x - 3 = 5\) \(x - 8 = 0\)

Subtract 5 from both sides of the first equation to get the second equation

10. \(3 - x = 5\) \(3 = 5 + x\)

Add \(x\) to both sides of the first equation

11. \(x_1 + \Delta x = x_2\) \(x_1 = x_2 - \Delta x\)

(These two equations show the relationship between addition and “removing what was added”)

Subtract \(\Delta x\) from both sides of the first equation to get the second equation

12. \(x_1 + \Delta x = x_2\) \(\Delta x = x_2 - x_1\)

(These two equations show the relationship between addition and “finding the difference”)

Subtract \(x_1\) from both sides of the first equation to get the second equation

13. \(\frac{4}{5} = x\) \(4 = 5x\)

Multiply both sides of the first equation by 5 to get the second equation.

14. \(\frac{T}{B} = x\) \(\frac{T}{B} - x = 0\)

Subtract \(x\) from both sides of the first equation

15. \(\frac{3}{4x} = 2\) \(\frac{3}{x} = 8\)
Multiply both sides of the first equation by 4 to get the second equation

Exercises: Solve for $x$ in the following equations. Some might take multiple steps. Keep “doing the same thing” to both sides of resulting equations until you have isolated $x$.

16. $x + 3 = 6$ \quad \Rightarrow \quad x = 3$  

17. $\frac{3}{x} = 9$ \quad \Rightarrow \quad \frac{1}{3}$  

18. $x - 3 = -5$ \quad \Rightarrow \quad x = -2$  

19. $x + 2 = -7$ \quad \Rightarrow \quad x = -9$  

20. $3 - x = 5$ \quad \Rightarrow \quad x = -2$  

21. $x_1 + \Delta x = x_2$  
   \quad (solve for $x_1$) \quad x_1 = x_2 - \Delta x$  

22. $x_1 + \Delta x = x_2$  
   \quad (solve for $\Delta x$) \quad \Delta x = x_2 - x_1$  

23. $\frac{4}{x} = 5$  
   \quad \Rightarrow \quad \frac{4}{5} = x$  
   \quad (Please don’t freak out because the $x$ is isolated on the right side of the equal sign. It doesn’t matter)  

24. $\frac{x}{B} = A$ \quad \Rightarrow \quad x = AB$  

25. $\frac{3}{4x} = 2$  
   \quad \Rightarrow \quad \frac{3}{8} = x$  

26. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not
a. Lucy has twice as much money as Carl  
\[ L = 2C \]

b. Carl has half as much money as Lucy  
\[ C = \left(\frac{1}{2}\right)L \]

These are equivalent. You can divide both sides of the first equation by 2 and get the second equation.

27. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. Roxane weighs 40 pounds more than Fred  
\[ R = F + 40 \]

b. If you add 40 to Roxane’s weight, you’ll get Fred’s weight  
\[ 40 + R = F \]

Nope, these are not equivalent. You can’t get from one equation to the other equation.

28. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. If you divide 2 into x equal pieces, each piece will be exactly 7  
\[ \frac{2}{x} = 7 \]

b. If you repeat x 7 times, you get 2  
\[ 7x = 2 \]

Yep, these are equivalent

29. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. Half of Roxane’s money is 3 times more than Melinda’s money  
\[ \frac{1}{2}R = 3M \]

b. Melinda’s money is exactly a sixth of Roxane’s money  
\[ M = \frac{1}{6}R \]

Yep, if you divide both sides of the first equation by 3, you get the second equation

30. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. Half of Roxane’s money is 3 times more than Melinda’s money  
\[ \frac{1}{2}R = 3M \]

b. Roxane’s money is exactly 1 and a half times more than Melinda’s money  
\[ R = 1 \frac{1}{2}M \text{ or, } R = \frac{3}{2}M \]
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Nope, you can’t get from one to the other…. You get $R=6M$, not one and a half $M$.

31. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. Half of Roxane’s money is 3 times more than Melinda’s money
\[ \frac{1}{2} R = 3M \]

b. Roxane’s money is exactly 6 times more than Melinda’s money
\[ R=6M \]

Yep, these are equivalent statements.

32. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. I started out with $30 under my mattress, and each week I added $10, so after 1 week I had $40, after 2 weeks I had $50, after 3 weeks I had $60, etc….
\[ 30 + 10 \cdot \text{Weeks} = \text{Total money under mattress} \]

b. If you take what I have in my mattress and subtract $30, and then divide by $10, you’ll know how many weeks I’ve been putting money in the mattress.
\[ \frac{\text{Total under mattress} - 30}{10} = \text{weeks} \]

yes, these are equivalent

33. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

a. Pirio’s car gets 30 miles on a gallon of gas. Her car uses one gallon of gas to to 30 miles.
\[ \frac{\text{gallons}}{\text{miles}} = \frac{1}{30} \quad \text{or} \quad \frac{\text{miles}}{\text{gallon}} = \frac{30}{1} \]

b. Pirio’s car uses 4 gallons of gas to drive 120 miles
\[ \frac{4}{120} = \frac{\text{gallons}}{\text{miles}} \]

\[ \frac{4}{120} = \frac{1}{30} = \frac{\text{gallons}}{\text{miles}} \]

Yes, these are equivalent
34. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not
   a. 10 divided by x is exactly the same as y divided by 2  \( \frac{10}{x} = \frac{y}{2} \)
   
   b. y divided by x is exactly 20  \( \frac{y}{x} = 20 \)

   no, you can’t get from one of these equations to another. They do not describe the same relationship

35. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not
   a. 10 divided by x is exactly the same as y divided by 2  \( \frac{10}{x} = \frac{y}{2} \)
   
   b. x divided by y is exactly 20  \( \frac{x}{y} = 20 \)

   nope, these are not equivalent

36. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not
   a. 10 divided by x is exactly the same as y divided by 2  \( \frac{10}{x} = \frac{y}{2} \)
   
   b. x divided by y is exactly 1 twentieth  \( \frac{x}{y} = \frac{1}{20} \)

   nope, these are not equivalent

37. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not
   a. 10 divided by x is exactly the same as y divided by 2  \( \frac{10}{x} = \frac{y}{2} \)
   
   b. x times y is exactly 20  \( xy = 20 \)

   yes, these are equivalent. If you multiply both sides of the equal sign by 2x, you get the second equation. These two equations describe the same relationship (as long as x does not equal zero)
38. Write equations that correspond to the following scenarios, and then determine if the two equations are equivalent or not

   a. 10 divided by x is exactly the same as y divided by 2 \[ \frac{10}{x} = \frac{y}{2} \]

   b. x times y is exactly 1 twentieth

     nope, these are not equivalent. See solution to previous question.
**Problems/exercises/questions on fractions: putting it all together.** In all of the following, simplify your answer as much as possible. All fractions in your answer should be less than 1, if not, write the number differently (e.g. $\frac{5}{2}$ is more than 1, so rewrite it like $2\frac{1}{2}$ or $2 + \frac{1}{2}$).

1. What is the size of each serving if you divide 17 cups of ice-cream into 9 servings?

   $\frac{17}{9} = \frac{9+8}{9} = 1 + \frac{8}{9} = 1 \frac{8}{9}

   Each serving is $1 \frac{8}{9}$ of a cup, just a little under two cups. This makes sense because 17 is just a little less than (2)(9).

2. What is 17 divided by 9?

   $\frac{17}{9} = \frac{9+8}{9} = 1 + \frac{8}{9} = 1 \frac{8}{9}$

3. What part of 17 fits into 9? (Compare this question with the previous one… are they the same?)

   $\frac{9}{17} \quad No, they are not the same.

4. I have $900 to spend, but I want to buy a boat that costs $1700. What fraction of the total cost do I have?

   $\text{Fraction} = \frac{\text{part}}{\text{whole}} = \frac{900}{1700} = \frac{9}{17}

   I have only 9 seventeenths of the total to spend. I have a little more than half.

5. How much flour do I have if I have two thirds of one and a half cups?

   $\frac{2}{3} (1 + \frac{1}{2})

   This is the same as $\frac{2}{3}$ of 1, combined with $\frac{2}{3}$ of $\frac{1}{2}$.

   \[
   \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{3} + \frac{1}{3} \cdot \frac{2}{2} = \frac{2}{3} + \frac{1}{3} = \frac{3}{3} = 1
   \]

   I have exactly 1 cup of flour.

   Did you find a different path to the solution? Great!
6. How much flour do I have if I combine two thirds of a cup with one and a half cups?

\[
\frac{2}{3} + \left(1 + \frac{1}{2}\right) =
\]

\[
= \frac{2 \cdot 2}{3} + 1 \cdot \frac{6}{6} + \frac{1 \cdot 3}{3} = \]

\[
= \frac{4}{6} + \frac{6}{6} + \frac{3}{6} = \frac{13}{6} =
\]

\[
\frac{6+6+1}{6} = 1 + \frac{1}{6} = 2 + \frac{1}{6}
\]

I have exactly 2 and a sixth cups of flour.

7. In the flour bin, there are 9 tenths of a cup of flour. The recipe for cookies requires 2 thirds of a cup to make 10 cookies. How many cookies can I make with 9 tenths of a cup of flour?

There are lots of paths to this solution. Here is one of them. You need to find your own path to the solution, as well as follow along with this one:

The number of cookies and the amount of flour are proportional.

\[
\frac{\text{Cookies}}{\text{Flour}} = \text{the constant ratio} \quad \frac{10}{\left(\frac{2}{3}\right)}
\]

That ratio is messy, so I am going to find an equivalent one that is much prettier by simplifying \(\frac{10}{\left(\frac{2}{3}\right)}\).

\[
\frac{10}{\left(\frac{2}{3}\right)} = \frac{10}{(1)} \cdot \frac{3}{(2)} = 15
\]

The ratio of cookies to flour is 15.

\[
\frac{\text{Cookies}}{\text{Flour}} = 15 = \left(\frac{x}{\left(\frac{9}{10}\right)}\right),
\]

\(x\) in this equation stands for the number of cookies I could make with \(\frac{9}{10}\) cups of flour.

To solve for \(x\), one path I could take is to multiply both sides by \(\frac{9}{10}\) and simplify:

\[
15 \cdot \frac{9}{10} = \left(\frac{x}{\left(\frac{9}{10}\right)}\right) \cdot \frac{9}{10}
\]

\[
5 \cdot 3 \cdot \frac{9}{52} = x \cdot \left(\frac{9}{9}\right) = x
\]
I can make 13 \( \frac{1}{2} \) cookies

8. If triangles have the same angles, then the ratios of the lengths of sides of triangles are always the same. So, for example, the ratios of the lengths of sides for any triangle with angles 90°, 30°, and 60°, will always be the same, no matter how big or small the triangle is. The ratios of the lengths of sides of any triangle with angles equal to 90°, 10°, 80° will always be the same, no matter how big or small the triangle is.

You can visualize how the sides of triangles with the same angles have proportional sides by imagining projecting a triangle on a screen, and then zooming in and out: the size of the triangle changes, but the angles stay the same.

a) Suppose I have a triangle with the following angles: \( P°, Q°, R° \). The length of the longest side is 5, the length of the shortest side is 3, and the length of the middle side is 4.

If I have another triangle with those same angles, and the longest side 8, what are the lengths of the other sides?

To answer this question, I create the following true equations (you don’t need all of them, just any three that show the relationship between the sides, Long, Middle, and Short. But I don’t know which are the easiest to use yet, so I write them all down):

\[
\frac{\text{long}}{\text{short}} = \frac{5}{3} = \frac{8}{\text{short}} \\
\frac{\text{long}}{\text{middle}} = \frac{5}{4} = \frac{8}{\text{middle}} \\
\frac{\text{short}}{\text{middle}} = \frac{3}{4} \\
\frac{\text{middle}}{\text{short}} = \frac{4}{3} \\
\frac{\text{middle}}{\text{long}} = \frac{4}{5} = \frac{\text{middle}}{8}
\]
\[
\begin{align*}
\text{short} & = 3 \cdot \frac{\text{short}}{5} = 8 \\
\text{long} & = \frac{3}{5} = \frac{\text{short}}{8}
\end{align*}
\]

I could use the first two equations to solve for “short” and “middle” respectively. But, it looks like the algebra will be a little easier if I use the last two equations to solve for “short” and “middle”. I’ll leave it up to you to solve the easier way; I’ll write down the more tedious way here:

\[
\frac{5}{3} = \frac{8}{\text{short}}
\]

I need to get “short” by itself. First, I need to multiply both sides of the equal sign by “short”.

\[
\frac{5}{3} \cdot \text{short} = \frac{8}{\text{short}} \cdot \text{short}
\]

\[
\frac{5}{3} \cdot \text{short} = 8
\]

Now, I multiply both sides by 3 and divide by 5.

Or, I divide both sides by 5 and multiply by 3.

Or, I divide both sides by \( \frac{5}{3} \). Any of these options will get me to:

\[
\text{short} = 8 \cdot \frac{3}{5}
\]

\[
8 \cdot \frac{3}{5} = \frac{24}{5}
\]

\[
\frac{24}{5} = 20 + \frac{4}{5}
\]

\[
\frac{20 + 4}{5} = \frac{24}{5} = 4 + \frac{4}{5}
\]

Short side has length of \( 4 \frac{4}{5} \)

I’m going to use this equation to find out what the length of the middle side is:

\[
\frac{5}{4} = \frac{8}{\text{middle}}
\]

Here’s what needs to happen: multiply both sides of the equation by “middle”, multiply both sides of the resulting equation by 4, and divide both sides by 5.

Or,

Multiply both sides by “middle” and divide both sides by \( \frac{5}{4} \).

You’ll end up with:

\[
\frac{8 \cdot 4}{5} = \text{middle}
\]
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\[
\begin{align*}
8 \cdot 4 &= \frac{32}{5} \\
\frac{30 + 2}{5} &= 6 + \frac{2}{5} \\
6 + \frac{2}{5} &= 6 + \frac{2}{5}
\end{align*}
\]

The length of the middle side is \( \frac{2}{5} \).

b) Suppose I have a triangle with the following angles: A˚, B˚, C˚. The length of the longest side is 10, the length of the shortest side is 5, and the length of the middle side is approximately \( \frac{8}{3} \) (remember that \( \frac{8}{3} \) is the same as \( \frac{8}{3} \)).

If I have another triangle with those same angles, but the shortest side is 4, what are the lengths of the other sides?

\[
\begin{align*}
\frac{long}{short} &= \frac{10}{5} = \frac{long}{4} \\
\frac{long}{middle} &= \frac{10}{\frac{8}{3}} \\
\frac{short}{middle} &= \frac{5}{\frac{8}{3}} = \frac{4}{middle} \\
\frac{middle}{short} &= \frac{\frac{8}{3}}{5} = \frac{middle}{4} \\
\frac{middle}{long} &= \frac{\frac{8}{3}}{10} \\
\frac{short}{long} &= \frac{5}{10}
\end{align*}
\]

Somewhat randomly, I’m going to choose an equation to start to answer the questions

\[
\frac{10}{5} \cdot 4 = \frac{long}{4} \cdot 4
\]
So, the long side is \( \frac{40}{5} = 8 \). You might have noticed that \( \frac{10}{5} = 2 \), and simplified \( \frac{10}{5} \cdot 4 = 2 \cdot 4 = 8 \). It never matters what order you multiply!

I can use that answer in the second to last equation to get:

\[
\frac{\text{middle}}{\text{long}} = \frac{\frac{8}{3}}{10} = \frac{\text{middle}}{8}
\]

I’ll multiply both sides of the equal sign by 8 to get

\[
\frac{8\frac{2}{3}}{10} \cdot 8 = \frac{\text{middle}}{8} \cdot 8
\]

\[
\frac{8}{3} + \frac{2}{10} \cdot 8 = \frac{\text{middle}}{8} \cdot 8
\]

\[
\frac{8}{3} + \frac{2}{10} \cdot 8 = \frac{24}{3} + \frac{2}{3} \cdot 8
\]

\[
\frac{26}{3} \cdot 8 = \frac{26}{3} \cdot 1 \cdot 8
\]

I need to factor out the weird forms of number 1:

\[
\frac{26}{3} \cdot \frac{1}{10} \cdot 8 = \frac{2 \cdot 13}{3} \cdot \frac{1}{2 \cdot 5} \cdot 2 \cdot 4 =
\]

\[
\frac{2 \cdot 13}{3} \cdot \frac{1}{2 \cdot 5} \cdot 2 \cdot 4 = \frac{2}{3} \cdot \frac{1}{3} \cdot 2 \cdot 4 \cdot 13 =
\]

\[
\frac{2}{3} \cdot \frac{1}{3} \cdot 2 \cdot 4 \cdot 13 = \frac{1}{3} \cdot 2 \cdot 4 \cdot 13 =
\]

\[
\frac{104}{15}
\]

To simplify that, I need to find out how many “fifteens” fit into 104. I happen to have memorized that \( 3 \cdot 15 = 45 \), so I’m going to work off that fact:

\[
\frac{104}{15} = \frac{3 \cdot 15 + 3 \cdot 15 + 10 + 4}{15}
\]

\[
\frac{3 \cdot 15 + 3 \cdot 15 + 10 + 4}{15} = \frac{3 + 3 + \frac{14}{15}}{15}
\]

\[
= \frac{6 + \frac{14}{15}}{15} = \frac{6\frac{14}{15}}{15}
\]
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The length of the middle side is $6 \frac{14}{15}$

9. What is $\frac{1}{9}$, repeated 17 times?

$$\frac{1}{9} \cdot 17 = \frac{17}{9} = \frac{9 + 8}{9} = 1 + \frac{8}{9}$$

$\frac{1}{9}$, repeated 17 times, is $1 \frac{8}{9}$, also known as seventeen nineths. Seventeen nineths is nine ninths plus eight nineths. Get it?

10. How many 1 inch by 1 inch squares (also known as square inches) are there in a space that is 2 and a half inches long and 3 and a half inches wide? (you might want to draw a picture).

$$\left(2 + \frac{1}{2}\right) \cdot \left(3 + \frac{1}{2}\right) =$$

**DRAW A PICTURE!!!**

$$\left(2 + \frac{1}{2}\right) \cdot \left(3 + \frac{1}{2}\right) = \left(\frac{5}{2}\right) \cdot \left(\frac{7}{2}\right) = \left(\frac{35}{4}\right)$$

$$\left(\frac{35}{4}\right) = \left(\frac{32 + 3}{4}\right)$$

$$\left(\frac{32 + 3}{4}\right) = 8 + \frac{3}{4}$$

The area is 8 and three quarters square inches.

Here’s another way to solve this, more visually:

<table>
<thead>
<tr>
<th>Length</th>
<th>Area</th>
<th>Area</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>1</td>
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<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
</tr>
</tbody>
</table>
In this rectangle, one side is 2 and half units long, the other side is 3 and a half units long. The total area is the sum of the little areas: there are 6 unit squares, 5 half-unit squares, and 1 quarter-unit squares.

Adding them all up will give you a total area of \(6 + \frac{5}{2} + \frac{1}{4}\)

I’ll leave it up to you to get from there to the same as the answer above: 8 and three quarters.

11. How many unit cubes would fit in a box that is 5 and a half units tall, 2 and a half units wide, and half a unit deep? (you can cut up the unit cubes so they fill up the box completely with no room leftover. In other words, what is the volume?)

\[(5 + \frac{1}{2})(2 + \frac{1}{2})(\frac{1}{2}) = \]

Here’s a visual solution:

One flat side of this box looks like this:

```
   +---+---+---+
   |   |   |   |
   +---+---+---+
   |   |   |   |
   +---+---+---+
```

The area is a total of 10 unit areas, 7 half-areas, and 1 quarter area.

So, the total area is \(10 + \frac{7}{2} + \frac{1}{4}\)

\[10 + \frac{7}{2} + \frac{1}{4} = 10 + \frac{14}{4} + \frac{1}{4}\]

\[10 + \frac{15}{4} = 10 + 3 + \frac{3}{4}\]

The area of this side is 13 and \(\frac{3}{4}\) unit squares.

It’s only half a unit deep, so the total volume is \((\frac{1}{2})(13 and \frac{3}{4})\)

Half of 13 is 6 and a half.

Half of \(\frac{3}{4}\) is \(\frac{3}{8}\)

So, the volume is \(6 + \frac{1}{2} + \frac{3}{8}\)
\[ 6 + \frac{1}{2} + \frac{3}{8} = 6 + \frac{4}{8} + \frac{3}{8} \]
\[ 6 + \frac{4}{8} + \frac{3}{8} = 6 + \frac{7}{8} \]

*Volume is 6 and 7 eighths unit cubes.*

12. You have 6 donuts and you want to give \( \frac{2}{3} \) of them to a friend and keep \( \frac{1}{3} \) for yourself. How many donuts would your friend get?

\( \frac{2}{3} \) of 6 means: take two pieces of a third of 6. Instead of writing this out arithmetically, I’m going to just think about it and write down my thoughts. A third of 6 is 2. Two thirds of 6 is therefore two twos, or 4.

So, I get 2 donuts, and my friend gets 4 donuts.

13. You will inherit \( \frac{5}{6} \) of your great-grandparents’ estate. If the estate is worth twice as much as your own estate, and your estate is worth $300, how much will you inherit?

*I inherit \( \frac{5}{6} \) of \( x \), where \( x \) stands for the value of the estate.*

\[ x = 2 \text{(my estate)} = 2 \times 300 = 600 \]

So, I inherit \( \frac{5}{6} \) of \( x \).

\[ \frac{5}{6} \times 600 = 5 \times 100 = 500. \]

I will inherit $500

14. Wendy's hair was originally 10 inches long. She asked her hairdresser to cut 3 inches off. What fraction of her hair did she cut off? What fraction of her original length was she left with?

\[ 10-3=\text{new length}=7 \]

7 out of 10 is \( \frac{7}{10} \)

She is left with \( \frac{7}{10} \) of her original length. She cut off \( \frac{3}{10} \) of the original length.

15. It takes two-thirds of a box of nails to build a birdhouse. If you wanted to build six birdhouses, how many boxes would you need?

*You need to repeat two-thirds of a box of nails 6 times:*

\( \left( \frac{2}{3} \right)(6) = 2 	imes 2 = 4. \)

You need 4 boxes of nails for 6 birdhouses.
16. A bakery used eight and a third cups of flour to make a full size cake. If they wanted to make a cake that was one-quarter the size, how many cups of flour would they need?

One quarter of eight and a third is One quarter of eight, plus one quarter of a third.

One quarter of 8 is 2. One quarter of a third is a twelfth.

So, the baker would need 2 and a twelfth cups of flour.

You could also write it out:

\[
\left(\frac{1}{4}\right) \left(8 + \frac{1}{3}\right) = \left(\frac{1}{4}\right) \left(\frac{24}{3} + \frac{1}{3}\right) = \left(\frac{1}{4}\right) \left(\frac{25}{3}\right) = \frac{25}{12} = \frac{24+1}{12} = 2 + \frac{1}{12}
\]

17. A fast food restaurant had 9 and a half pounds of flour. If they split the flour evenly among 4 batches of chicken, how much flour would each batch use? Between what two whole numbers does your answer lie?

You need to divide 9 and a half into 4 equal pieces.

9 divided into 4 equal pieces is \(\frac{9}{4} = 2 + \frac{1}{4}\)

A half divided into 4 equal pieces is \(\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}\)

So, in total the restaurant needs \(2 + \frac{1}{4} + \frac{1}{8}\) for each batch of chicken.

\[2 + \frac{1}{4} + \frac{1}{8} = 2 + \frac{2}{8} + \frac{1}{8} = 2 + \frac{3}{8} = 2 \frac{3}{8}\]

Each batch of chicken needs \(2 \frac{3}{8}\) cups of flour

18. The serving size for the granola that Roxane likes to eat for breakfast is \(\frac{1}{2}\) cup. How many servings are there in a box that holds 5 \(\frac{3}{4}\) cups?

How many servings fit into the box?\(\frac{\text{size of box}}{\text{size of serving}}\)

\[
\frac{5 + \frac{3}{4}}{\frac{1}{2}} = \frac{\left(5 + \frac{3}{4}\right)}{\frac{1}{2}} = \left(5 + \frac{3}{4}\right) \cdot \frac{1}{\left(\frac{1}{2}\right)} = \left(5 + \frac{3}{4}\right) \cdot 2
\]

\[
\left(5 + \frac{3}{4}\right) \cdot 2 = \left(\frac{20}{4} + \frac{3}{4}\right) \cdot 2
\]

\[
\left(\frac{23}{4}\right) \cdot 2 = (23)\left(\frac{2}{4}\right)
\]

\[
(23)\left(\frac{2}{4}\right) = (23)\left(\frac{1}{2}\right)
\]

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(23) \(\frac{1}{2} = \frac{20 + 3}{2} = \frac{20 + 2 + 1}{2}\)

There are 10 plus 1 plus \(\frac{1}{2}\) servings in the box.

The box has 11 and a half servings.

This is a very important type of problem. You have to understand the meaning of division, and you have to be able to divide fractions.

Hooray for all the learning you are doing!

19. A restaurant had 7 days to sell 102 gallons of ice cream before it expired. How much should they sell each day? Which two whole numbers does your answer lie between?

102 gallons, divided by 7 equal days

\[
\frac{102}{7} = \text{how many sevens fit into 102}
\]

\[
\frac{102}{7} = \frac{70 + 28 + 4}{7}
\]

\[
\frac{70 + 28 + 4}{7} = \frac{10 + 4 + \frac{4}{7}}{7}
\]

\[
10 + 4 + \frac{4}{7} = 14 \frac{4}{7}
\]

They should try to sell between 14 and 15 gallons every day.

20. A pan of brownies was left out on the counter and \(\frac{1}{4}\) of the pan had already been eaten. Then John came along and ate \(\frac{2}{3}\) of what was left.

a) What fraction of the total pan of brownies was on the counter before John came along?

\(\frac{1}{4}\) of the pan had already been eaten. \(\frac{3}{4}\) of the pan was sitting there for John to eat.

b) What fraction of a whole pan of brownies did John eat?

John ate \(\frac{2}{3}\) of what was left. \(\frac{3}{4}\) of the pan was sitting there for John to eat.

\[
\frac{2}{3} \text{ of } \frac{3}{4} \text{ is }
\]

\[
\frac{2}{3} \cdot \frac{3}{4} = 2 \cdot \left\{ \frac{1}{3} \right\} \cdot \frac{1}{4} =
\]
\[
2 \cdot \frac{1}{3} \cdot \frac{1}{4} = 2 \cdot \frac{1}{3} \cdot \frac{1}{4} \\
2 \cdot \frac{1}{3} \cdot \frac{1}{4} = 6 \cdot \frac{1}{12} \\
\frac{6}{12} = \frac{1}{2}
\]

*John ate half a pan of brownies! He’d better hit the gym.*

c) What fraction of a whole pan was left after John finished eating?

*Before John came along, there was \( \frac{3}{4} \) of the pan sitting there. He ate half a pan of brownies. In other words, John removed half from three quarters, leaving one quarter.*

\[
\frac{3}{4} - \frac{1}{2} = \frac{3}{4} - \frac{2}{4} = \frac{1}{4}
\]

d) If a whole pan of brownies is 5,000 calories, how many calories did John eat?

*John ate 2,500 calories, or half of 5,000.*

21. Write this as one fraction \( \frac{A}{B} - \frac{C}{D} \) (I won’t say “simplify” because it’s not particularly simpler to express this as one fraction, but it is equivalent)

\[
\frac{A}{B} - \frac{C}{D} = \left( \frac{A \cdot D}{B \cdot D} \right) - \left( \frac{C \cdot B}{D \cdot B} \right) \\
\frac{A \cdot D}{B \cdot D} - \frac{C \cdot B}{D \cdot B} = \frac{AD - CB}{BD}
\]

22. I have X dollars, and you have M times what I have.

e) How many dollars do you have, in terms of X and M?

*You have \((X)(M)\)*

f) If M is less than 1, do you have more or less money than I have?

*You have less money than I have.*

g) If M is more than 1, do you have more or less money than I have?

*You have more money if \(M\) is more than 1.*

h) If \((X)(M)\) is greater than \(X\), what do I know about \(M\)?

*\(M\) must be greater than 1.*

i) If \((X)(M)\) is less than \(X\), what do I know about \(M\)?

*\(M\) must be less than 1.*
23. How many halves fit into 9? Write a mathematical expression that shows this.

\[
\frac{9}{\frac{1}{2}} = 18
\]

24. A baker is making cakes for a big party. She uses \( \frac{1}{4} \) cup of oil for each cake. How many cakes can she make if she has a bottle of oil that has \( 7\frac{1}{3} \) cups in it?

**How many \( \frac{1}{4} \) cups fit into \( 7\frac{1}{3} \) cups?**

\[
\frac{7 + \frac{1}{3}}{\frac{1}{4}} = \frac{21 + \frac{1}{3}}{\frac{1}{4}}
\]

\[
\frac{21 + \frac{1}{3}}{\frac{1}{4}} = \frac{22}{3} \cdot \frac{1}{\frac{1}{4}}
\]

\[
\frac{22}{3} \cdot \frac{1}{\frac{1}{4}} = \frac{22}{3} \cdot 4
\]

\[
\frac{22}{3} \cdot 4 = \frac{88}{3}
\]

\[
\frac{66 + 9 + 6 + 6 + 1}{3} = 22 + 3 + 2 + 2 + \frac{1}{3} = 29 \text{ and } \frac{1}{3}
\]

**OR **

There are 28 quarter cups in 7 cups, and \( \frac{1}{3} \cdot 4 = \frac{4}{3} = 1 + \frac{1}{3} \)

So, \( 28 + 1 + \frac{1}{3} = 29 \text{ and } \frac{1}{3} \)

**You decide which path was easiest to get to this answer.**

25. To make this equation true, \( \frac{4}{x} = \frac{3}{2} + 2 \) what number must \( x \) equal? If \( x \) is not a whole number, between what two whole numbers is it?

Considering the right hand side of this equation, I can simplify the right-hand side of this equation to:
So I get this new equation:

\[ \frac{4}{x} = \frac{7}{2} \]

Multiply both sides by \( x \), divide both sides by 4, multiply both sides by 2, divide both sides by 7 (in any order you like) to get:

\[ x = \frac{8}{7} = 1 + \frac{1}{7} \]

26. These next questions are probably the most important questions so far in this book. They require that you put together ideas from all the previous chapters. Pay attention, work slowly and carefully and make sure you understand before moving on to the next problem.

An empty box weighs 7 and a half pounds. After 5 doo-dads are put into the box, it and the doo-dads weigh 8 and two thirds pounds. How much does each doo-dad weigh?

\[ 7 + \frac{1}{2} + 5x = 8 + \frac{2}{3} \] ... I've gotta solve for \( x \), which stands for the weight of each doo-dad.

OR

\[ 7 + \frac{1}{2} + y = 8 + \frac{2}{3} \] ... I've gotta solve for \( y \), which stands for the weight of all 5 doo-dads, and then figure out the weight of each one by dividing eight of each one by dividing \( y \) by 5.

Hmm, I'll work with the second equation. There are other equivalent equations you could set up, too.

If I subtract 7 and subtract \( \frac{1}{2} \) from both sides of the equal sign, I end up with:

\[ y = 1 + \frac{2}{3} - \frac{1}{2} \]

\[ y = 1 + \frac{2}{3} \cdot \frac{2}{2} - \frac{1}{3} \cdot \frac{3}{3} \] remember the weird form of number 1 in order to subtract or combine fractions?

\[ y = 1 + \frac{4}{6} - \frac{3}{6} \]

\[ y = 1 + \frac{1}{6} \]

All 5 doo-dads, put together, weigh \( 1 + \frac{1}{6} \) pounds. So, each of them weigh \( \frac{1 + \frac{1}{6}}{5} \).
Each doo-dad weighs $\frac{7}{30}$ of a pound!

If you arrived at this answer another way, that’s great!

27. An empty box weighs B pounds. After N widgets are put into the box, the box and the widgets together weigh T pounds. How much does each widget weigh (in terms of B, N and T?)

$$B + Nx = T \quad \text{... you could start with this equation. Do same thing to both sides of equal sign until you get. I'm not writing the steps down because there are many paths to this solution.}$$

$$x = \frac{T - B}{N}$$

28. An empty box weighs 9 and two thirds pounds. It is filled with some cookies, each cookie weighs 2 fifths of a pound. If the box and cookies altogether weigh 12 and a half pounds, how many cookies (including fractions of cookies) are in the box?

$$9 + \frac{2}{3} + \frac{2}{5}x = 12 + \frac{1}{2}$$

You want to solve for $x$. Here’s one way: Subtract 9 from both sides of the equal sign. Subtract $\frac{2}{3}$ from both sides of the equal sign. Then, multiply both sides of the resulting equation by $\frac{5}{2}$.

You get that there were 7 and $\frac{1}{12}$ cookies in the box.

There are lots of ways to solve for $x$ in this equation. If you arrived at the answer through a different path, that’s great!

29. An empty box weights B pounds. It is filled with some cookies, each cookie weighs C pounds. If the box and the cookies altogether weigh T pounds, how many cookies are in the box, in terms if B, C and T?

$$B + Cx = T \quad \text{I want to isolate x. I can do this by first subtracting B from both sides of the equal sign, and then dividing by C:}$$

$$x = \frac{T - B}{C}$$
30. What is an equation that describes the relationship between flour to sugar in a recipe that needs 8 cups flour and 5 cups sugar? \( \frac{8}{5} = \frac{F}{S} \) or any equivalent equation, such as \( 8S = 5F \), or \( F = \frac{8}{5}S \).

31. What is an equation that describes the relationship between flour to sugar in a recipe that needs 8 cups flour and 2 cups sugar? \( 4 = \frac{F}{S} \) or any equivalent equation.

32. What is an equation that describes the relationship between flour to sugar in a recipe that needs 1 cup flour and a third cup sugar? \( \frac{1}{3} = \frac{F}{S} \) or any equivalent equation.

33. What is an equation that describes the relationship between flour to sugar in a recipe that needs a third cup flour and 1 cup sugar? \( \frac{1}{3} = \frac{F}{S} \) or any equivalent equation.

34. What is an equation that describes the relationship between cost and pounds of apples if it costs $15 to buy 3 pounds? \( \frac{5}{3} = \frac{\text{dollars}}{\text{pound}} \). Or equivalent equation.

35. What is an equation that describes the relationship between cost per ounce of something if 15 ounces cost $20? \( \frac{4}{3} = \frac{\text{cost}}{\text{ounces}} \) or equivalent equation.

36. If 16 Canadian dollars are worth about 12 US dollars, what is an equation that describes the relationship of Canadian dollars to US dollars? \( \frac{4}{3} = \frac{\text{USD}}{\text{CAD}} \).

37. There are almost exactly 20 miles per 32 km. What is an equation that describes the relationship between miles to km? Express your answer as a simplified fraction. \( \frac{5}{8} = \frac{\text{miles}}{\text{km}} \).

38. My car used 10 gallons of gas going 300 miles. What is an equation that describes the relationship between gallons I use and miles I drive? Express the answer as a simplified fraction. \( \frac{30}{1} = \frac{\text{miles}}{\text{gal}} \).

39. I walked 8 miles in 2 hours. What is an equation that describes the relationship between the distance I walked and the time I spent walking? \( 4 = \frac{\text{Distance in miles}}{\text{time in hours}} \) or equivalent equation.

40. I earned $500 working 20 hours. What is an equation that describes the relationship between the time I work and the money I make? (assume that I get paid at a constant rate) \( 25 = \frac{\text{money}}{\text{hour}} \).

41.

a. If you drive for 2 hours at 50 miles per hour, how far have you driven? 100 miles

b. If you drive for 3 hours at 50 miles per hour, how far have you driven? 150 miles

c. If you drive for 10 hours at 50 miles per hour, how far have you driven? 500 miles
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d. If you dive for 2 hours at 15 miles per hour, how far have you dived? 30 miles

e. If you drive for \( x \) hours at \( y \) miles per hour, how far have you driven, in terms of \( x \) and \( y \)? you’ve driven \( x \cdot y \) miles

f. If you drive for \( H \) hours at \( S \) speed for a total of \( D \) miles, what is an equation that describes the relationship between \( H \), \( S \) and \( D \)? What are all the equations?

\[
S = \frac{D}{H} \text{ (as long as } H \text{ is not zero), or } D = SH. \text{ Or } H = \frac{D}{S} \text{ only if } S \text{ is not zero}
\]

42. If you ride your bike with average speed of 18 miles per hour to the store, and it takes you \( H \) hours to get there, and you ride your bike at an average speed of 15 miles per hour home, and it takes you \( T \) hours to get home, what is a true equation that describes the relationship between \( H \) and \( T \) (you take the exact same route to the store as home, so the distance you go is the same).

\[18H = 15T, \text{ because the distance to the store = distance home from the store.}\]

43. This refers to the previous question: if it took you 40 minutes (that’s \( \frac{2}{3} \) of an hour) to get to the store, how long did it take you to get home?

\[18 \cdot \frac{2}{3} = \text{distance to store}\]

\[15T = \text{distance back from store, which is equal to the distance to the store.}\]

\[18 \cdot \frac{2}{3} = 15T\]

\[2 \cdot \frac{18}{3} = 2 \cdot 6\]

\[12 = 15T\]

\[\frac{12}{15} = T\]

\[\frac{4}{5} = T\]

It takes you \( \frac{4}{5} \) of an hour to get home, which is 48 minutes (4 fifths of 60 is 48).

44.

a. You have one job that pays $20 per hour, and another job that pays $25 per hour. If you work \( H \) hours on the first job, and \( T \) hours on the second job, and you make a total of \( M \) dollars, what is an equation that describes the relationship between the numbers \( H \), \( T \) and \( M \)?
\[20H + 25T = M\]

b. This question refers to the previous question: If you made a total of $17.50 and worked for 30 minutes at the second job, how long did you work at the first job?

\[17.5 = (20) \cdot H + 25 \left( \frac{1}{2} \right)\]

the goal is to solve for \(H\).

\[17.5 = 20H + 12.5\]

\[5 = 20H\]

\[\frac{5}{20} = H\]

\[H = \frac{1}{4}\]

You work for a quarter of an hour, or 15 minutes, at the first job.

c. Suppose you need to make $510, and you work 7 hours per week on the first job, and, since you are a student, you want to work a total of 15 hours every week. How many weeks (including fractions of weeks) till you can make $510? (just assume you get paid cash).

45.

a. Suppose you start painting a fence and you paint one post every 10 minutes. How many posts do you paint in 20 minutes?

\[\frac{1 \text{ post}}{10 \text{ minutes}} \cdot 20 \text{ minutes} = 2 \text{ posts}\]

b. How many posts do you paint in 60 minutes?

\[\frac{1 \text{ post}}{10 \text{ minutes}} \cdot 60 \text{ minutes} = 6 \text{ posts}\]

c. How many posts do you paint in \(M\) minutes? (in terms of \(M\))

\[\frac{1 \text{ post}}{10 \text{ minutes}} \cdot M \text{ minutes} = \text{posts}\]

d. Suppose you paint a total of \(P\) posts, and it takes you \(M\) minutes. What is an equation that describes the relationship between the number of posts you paint, \(P\), and the minutes you’ve spent painting, \(M\)?

\[\frac{1 \text{ post}}{10 \text{ minutes}} \cdot M \text{ minutes} = P \text{ posts}\]

or, if you don’t like all those words:

\[\frac{1}{10} \cdot M = P\]

I find the words helpful, however. They help me keep track of what the letter stand for.
Any equivalent equation would also describe this relationship. Like:
\[
\frac{P}{M} = \frac{1}{10}
\]
e. Suppose Jesse paints one post every 5 minutes. What is an equation that describes the relationship between the number of posts Jesse paints, J, and the time they spend painting, T?
\[
\frac{1 \text{ post}}{5 \text{ minutes}} = \frac{J}{T}
\]
Or, any equivalent equation, like 5J = T

f. If you and Jesse both paint posts, what is an equation that describes the relationship between the total posts painted, Y, and the time you’ve spent painting, M, and the time Jesse has spent painting, T?

The posts I paint = \frac{1}{10} \cdot M

g. Suppose you and Jesse both paint posts for the exact same amount of time. How long will it take to get 78 posts painted?

The number of posts I paint in M minutes is \frac{1}{10}M

The number of posts Jesse paints in M minutes is \frac{1}{5}M

So, if we add those amounts, it will be the total number of posts painted, which is 78

\[
\frac{1}{10}M + \frac{1}{5}M = 78
\]

A tenth of M plus a fifth of M is three tenths (review adding fractions if this is not clear)

\[
\frac{3}{10}M = 78
\]

Solve for M (divide by 3 and multiply by 10 is one way) to get

M = 260

So, it takes us 260 minutes.

h. Suppose you paint for exactly twice as long as Jesse. How long will each of you work to get exactly 156 posts painted?

The number of posts I paint is \frac{1}{10} \cdot 2T, where T is the time Jesse spent painting

The number of posts Jesse paints in T minutes is \frac{1}{5}T
Altogether, we paint \(\frac{1}{10} \cdot 2T + \frac{1}{5} \cdot T = 156\)

Solve for \(T\) to get 390 minutes. I worked for twice that, or 780 minutes.

Let’s check this: I work for 780 minutes, Jesse works for 156 minutes. How many posts do we paint?

\(\frac{1}{10} \cdot 780 + \frac{1}{5} \cdot 390 = 156\), yep! Looks right.

There is not “a way” to do math, other than making sense of the situation, keeping track of the meaning of all the symbols you use. The key is to understand the power of using an equation, and to do that you need to know the meanings of addition, subtraction, multiplication, division and the equal sign.

Rely less on memorizing procedures, and more on understanding meaning.
Parentheses, Distributing, Factoring:

Adding, Subtracting, Multiplying and Dividing in the Company of Parentheses, and the Distributive Property.

1. Do parentheses matter when you are adding numbers? For example is the expression \(a + (b+c)\) the same as \((a+b)+c\), and is this the same as \(a+b+c\)? Explain your answer.

   No, it doesn’t matter. If I add items to a bag, I end up with the same combination of items no matter what order I put them in there.

2. Do the parentheses matter in this expression: \(9-(2+4)\) ?

   Yep, they sure do. If I remove 2 and 4 from 9, I get 3. If I write the expression without the parentheses, it would be \(9-2+4\), which would be removing 2 from 9 to get 6, then adding that to 4, which would get me 10.

3. Compare the situation where I am combining items in my purse with items on the table, to the situation where I combine items in my purse, and then throw my purse out the window. How can you tell when the parentheses matter? Explain why the parentheses are important and necessary in one situation, and not relevant in the other situation.

   If I combine items in a parentheses, and then combine that parentheses to something else, it doesn’t matter what order I combined items. It’s just one big happy combination of stuff.

   If I combine items in a parentheses, and then subtract that combination from something, the initial combination matters, because it means I am subtracting EVERYTHING in the parentheses (like throwing all the things away that are in the purse).

   Find equivalent expressions that don’t have any parentheses (Do NOT do the arithmetic, just write an equivalent expression with the same amount of numbers.) If the original parentheses mattered, explain what changes you had to make when re-writing the expression. The first one is done for you:

4. \(9-(2+4) = 9-2-4\). I had to subtract every item in the parentheses.

5. \(10-(2+3) = 10-2-3\)

6. \(x-(a+b) = x-a-b\)
7. \(-x+(a-b) = -x+a-b\)

Exercises: Write equivalent expressions without any parentheses. Don’t do any arithmetic, just write equivalent expressions:

8. \((x+y)-a = x+y-a\)

9. \(X-(a+b) = X-a-b\)

10. \(10 -(2 \cdot 3) = 10 -2 \cdot 3\)

11. \(X+(3)(y) = X + 3y\)

12. \(X-(3)(y) = X -3y\)

13. \(Y-2(x) = Y -2x\)

Exercises: Write equivalent expressions that do not have any parentheses.

14. \(2-(x+y) = 2-x-y\)

15. \(3(x+y) = 3x+3y\)

16. \(X(2+5) = 2X+5X = 7x\)

17. \(x+(y+z) = x+y+z \quad \text{or} \quad x+y+z\)

18. \(x+2(a+y) = x+2a+2y\)
19. \(x-2(x+a) = x-2x-2a = -x-2a\)

20. \(-2(x+y) = -2x - 2y\)

21. \(3-2(x) = 3-2x\)

22. \(5+1(x) = 5+x\ \text{or} \ 5+1x\)

Question:

23. Do parentheses matter if you are only multiplying? Write down three examples that demonstrate your answer.

*Your answers will vary. But, basically, something like half of 8 is 4*

\[
\frac{1}{2} \cdot 8 = 4
\]

*And,*

\[
\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4.
\]

*one half, repeated 8 times, is also equal to 4*

\[
(8 \cdot \frac{1}{2}) = 4
\]

*This is ONE example. You should write three examples.*
24. Suppose you have a coupon that lets you pay only 80% of the listed price of a new purse, and another coupon that lets you pay only 75%. AND, you are allowed to combine these coupons. (note, 80% of something means \( \frac{80}{100} \) of it, and 75% of something means \( \frac{75}{100} \) of it. \( \frac{80}{100} \) can be simplified to \( \frac{4}{5} \), and \( \frac{75}{100} \) can be simplified to \( \frac{3}{4} \). So 80% of x is the same as \( \frac{4}{5} \) \( \cdot \) \( x \).)

a. Does it matter which coupon you apply first to determine what you pay? Try it both ways and then explain your answer.

**Suppose the price-tag is \( X \) dollars.**

If I pay 80%, that turns out to be \( \frac{4}{5} \) \( \cdot \) \( X \)

If I then pay 75% of that, then I pay \( \frac{3}{4} \) \( \cdot \) \( \frac{4}{5} \) \( \cdot \) \( X \) = \( \frac{3}{5} \) \( \cdot \) \( X \)

I end up paying \( \frac{3}{5} \) \( \cdot \) \( X \).

If I apply the 75% coupon first, then I will pay

\[ \frac{4}{5} \cdot \frac{3}{4} \cdot X = \frac{4}{5} \cdot \frac{3}{4} \cdot X = \frac{3}{5} \cdot X. \]

Basically, it doesn’t matter what order you multiply, therefore, it doesn’t matter what order you apply discounts.

b. What percent of the list price do you end up paying?

\[ \frac{3}{5} \cdot X = \frac{?}{100} \cdot X \]

If I multiply \( \frac{3}{5} \) by the weird form of the number \( 1 \frac{20}{20} \), I will get \( \frac{3}{5} \cdot \frac{20}{20} = \frac{60}{100} \).

Applying a coupon that lets me pay only 75% with another coupon that lets me pay only 80%, means that I end up paying only 60%.

25. Without “plugging in” any numbers, decide whether \((3)(xz)\) is always, sometimes, or never, the same as \((3x)(3z)\), and justify your answer. (if you don’t justify your answer, the whole point of this question is lost, so you’d better do so, and your justification had better make sense).

The parentheses in \((3)(xz)\) do not matter, because the only mathematical operation is multiplication, so \((3)(xz) = 3xz\)

The parentheses in \((3x)(3z)\) also do not matter, because the only mathematical operation is multiplication, so \(3x3z = 9xz\).

\(3xz\) and \(9xz\) aren’t the same, because how could 3 of something be the same as 9 of something... wait! Unless \( x \) or \( z \) equal 0, these expression cannot be equal.

So, the answer is “sometimes”, because \((3)(xz)\) is the same as \((3x)(3z)\) if \( x \) OR \( z \) happen to be 0, but in general, they are not the same.
26. Without “plugging in” any numbers, decide whether \((3)(xz)\) is always, sometimes, or never, the same as \((xz)(3)\), and justify your answer. (if you don’t justify your answer, the whole point of this question is lost, so you’d better do so, and your justification had better make sense).

The parentheses in \((3)(xz)\) do not matter, because the only mathematical operation is multiplication, so \((3)(xz) = 3xz\)

The parentheses in \((xz)(3)\) also do not matter, because the only mathematical operation is multiplication, so \((xz)(3) = 3xz\).

So, the answer is “always”.

Exercises: For the following, write in words what the mathematical expression is showing. The first one is done for you. Don’t do the arithmetic, just write if it is addition or multiplication

27. \((3\frac{1}{2})\) means 3 plus \(\frac{1}{2}\)

28. \(\frac{2x}{y}\) means 2 times \(\frac{x}{y}\)

29. \((2)(\frac{1}{3})\) means 2 times \(\frac{1}{3}\)

30. \(x\frac{1}{3}\) means x times \(\frac{1}{3}\)

31. \(7\frac{1}{3}\) means 7 PLUS \(\frac{1}{3}\). COMPARE THIS TO THE PREVIOUS QUESTION.

32. \((7\frac{1}{3})\) means 7 TIMES \(\frac{1}{3}\). COMPARE THIS TO THE PREVIOUS QUESTION

33. \((7\frac{1}{3})\) means 7 TIMES \(\frac{1}{3}\). COMPARE THIS TO THE PREVIOUS QUESTION

34. \(7\frac{1}{3}\) means 7 TIMES \(\frac{1}{3}\). COMPARE THIS TO THE PREVIOUS QUESTION

35. \(\frac{1}{3}(7)\) means 7 TIMES \(\frac{1}{3}\). COMPARE THIS TO THE PREVIOUS QUESTION

Exercises: Simplify the following expressions

36. \(\frac{10}{8+2} = 1\)

37. \(\frac{10x}{8+2} = x\)

38. \(\frac{x}{(8+2)} = x\)
39. \( (10) \frac{x}{8+2} = x \)

40. \( (10) \frac{8+2}{5} = 20 \)

41. \( \frac{x+2}{x} \) write this as the sum of two fractions

\[
\frac{x+2}{x} = \frac{x}{x} + \frac{2}{x} \quad \text{Or} \quad \frac{x+2}{x} = \frac{x}{x} + \frac{2}{x} = 1 + \frac{2}{x}
\]

This would be ok, too: \( \frac{x+2}{x} = \frac{x}{x} + \frac{2}{x} = 1 + \frac{2}{x} \)

42. \( \frac{x^2}{x} \) (simplify this) = 2

43. Compare the previous two examples. The distinction between them is the first example has addition, so there are some of those “invisible parentheses”. The latter exercise has multiplication, so thinking about parentheses is not helpful.

44. Show how \( \frac{x+2+y}{x+2} \) is equivalent to \( 1 + \frac{y}{x+2} \)

The key to seeing this path to the answer is to group some terms together so you get a weird form of number 1.

\[
\frac{x+2+y}{x+2} = \frac{(x+2)+y}{(x+2)} \quad \text{I put these parentheses there to help me “group” those terms together.}
\]

\[
\frac{(x+2)+y}{(x+2)} = \frac{(x+2)}{(x+2)} + \frac{y}{(x+2)}
\]

\[
\frac{(x+2)}{(x+2)} + \frac{y}{(x+2)} = 1 + \frac{y}{(x+2)}
\]

Another way to show how \( \frac{x+2+y}{x+2} \) is equivalent to \( 1 + \frac{y}{x+2} \).

\[
1 + \frac{y}{x+2} = \frac{x+2}{x+2} + \frac{y}{x+2}
\]

\[
\frac{x+2}{x+2} + \frac{y}{x+2} = \frac{x+2+y}{x+2}
\]

45. Show how \( \frac{a}{b} \) is equivalent to \( \frac{b}{c} \)

I’m going to show how you can get to the first expression from the second expression. You should be able to go the other way, too: from the first to the second:
\[
\frac{a}{b} = \frac{a}{1} \cdot \frac{1}{b} \\
\frac{a}{c} \cdot \frac{1}{b} = \frac{a}{c} \cdot \frac{1}{b} \\
\frac{a}{c} \cdot \frac{1}{b} = \frac{a}{c} \cdot b \\
\frac{a}{c} \cdot b = \left(\frac{ab}{c}\right) \\
\frac{ab}{c} = a \left(\frac{b}{c}\right) \\
a \left(\frac{b}{c}\right) = a \frac{b}{c}
\]

46. Show how \(a \frac{b}{c}\) is equivalent to \((b) \left(\frac{a}{c}\right)\)

I’ll go from \(a \frac{b}{c}\) to \((b) \left(\frac{a}{c}\right)\).

\[
a \frac{b}{c} = \frac{ab}{c} \\
a \frac{b}{c} = b \frac{a}{c} \\
b \frac{a}{c} = (b) \left(\frac{a}{c}\right)
\]

47. Show how two thirds of 5 is equivalent to a third of 10.

Two thirds of 5 is \(\left(\frac{2}{3}\right) (5) = \frac{10}{3}\)

One third of 10 is \(\left(\frac{1}{3}\right) (10) = \frac{10}{3}\)

48. Show how \(a \frac{b}{c}\) is equivalent to \((ab) \left(\frac{1}{c}\right)\)

\[
a \frac{b}{c} = (a) \cdot (b) \cdot \left(\frac{1}{c}\right) \\
(a) \cdot (b) \cdot \left(\frac{1}{c}\right) = (ab) \cdot \left(\frac{1}{c}\right)
\]

49. Is it true that, in general, \(a \frac{b+x}{c}\) is equivalent to \(\frac{ab}{c} + \frac{ax}{c}\)? Explain why or why not. (Assume \(c\) is not equal to zero, because in that case, you’d be dividing by zero, and that doesn’t make sense)

The best way I think to approach this is to write those parentheses that are currently invisible.

\[
a \frac{b+x}{c} = a \cdot \frac{(b+x)}{c} \\
a \frac{(b+x)}{c} = a \cdot \frac{(b+x)}{c}
\]
\[
\frac{a(b+x)}{c} = \frac{(ab+ax)}{c} \\
\frac{(ab+ax)}{c} = \frac{ab}{c} + \frac{ax}{c}
\]

Yes, those two expression are equivalent.

50. Is it true that, in general, \( a \frac{b+x}{c+x} \) is equivalent to \( (a) \left( \frac{b}{c+x} \right) + \left( \frac{x}{c+x} \right) \)? Explain why or why not. (assume \( c + x \) is not equal to 0)

\[
a \frac{b+x}{c+x} = a \frac{(b+x)}{c+x} \\
a \frac{(b+x)}{c+x} = a \left( \frac{b}{c+x} + \frac{x}{c+x} \right) \quad \text{This is NOT equivalent to} \quad (a) \left( \frac{b}{c+x} \right) + \left( \frac{x}{c+x} \right).
\]

\( (a) \left( \frac{b}{c+x} \right) + \left( \frac{x}{c+x} \right) \) does not indicate that that \( (a) \) gets multiplied to \( \frac{x}{c+x} \).

\( (a) \left( \frac{b}{c+x} \right) + \left( \frac{x}{c+x} \right) \) could be written like \( a \cdot \frac{b}{c+x} + \frac{x}{c+x} \), which is equal to \( \frac{ab+x}{c+x} \).

51. Is it true that \( a \frac{b+x}{c+x} \) is equivalent to \( (a) \cdot \left( \frac{b}{c+x} \right) + \left( \frac{x}{c+x} \right) \)? Explain why or why not. (assume \( c + x \) is not equal to 0). Those curly brackets are just fancy parentheses.

Yes, those are equivalent. Check out the solution to the previous question.

52. Is it true that \( a \frac{b+x}{c+x} \) is equivalent to \( \left( \frac{ab}{c+x} \right) + \left( \frac{ax}{c+x} \right) \)? Explain why or why not. (assume \( c + x \) is not equal to 0)

\[
a \frac{b+x}{c+x} = a \frac{(b+x)}{c+x} \\
a \frac{(b+x)}{c+x} = a \frac{(b+x)}{c+x} \\
a \frac{(b+x)}{c+x} = \frac{ab+ax}{c+x} \\
a \frac{b+x}{c+x} = \frac{ab}{c+x} + \frac{ax}{c+x}
\]

Yes, they are equivalent.
Exercises: Expand (meaning multiply and write an equivalent expression for) the following expressions. You will have to figure out how to expand expressions with negative numbers. To do this, remember that x-y is the same as x+(-y), and that you can have a negative length (just think of a negative length as an arrow with a direction opposite to the direction of a positive number). So the length x-3 can be thought of as a combination of the length x with the length -3.

53. (4+x)(x-2+y)

   \[ (4+x)(x-2+y) = 4x - 8 + 4y + (x)(x) - 2x + xy. \]
   \[ \text{This can be simplified, because } 4x-2x=2x. \]

54. (x-y)(a+b-c)

   \[ (x-y)(a+b-c) = ax + bx - cx - ay - by + cy \]

55. (-2x)(3y)

   \[ You \ gotta \ think... \ do \ those \ parentheses \ matter? \]
   \[ (-2x)(3y) = -2x3y = -6xy \]

56. (2x-3)(3x)

   \[ (2x-3)(3x)=6(x)(x)-9x \]
   \[ \text{NOTE: you can’t simplify this. } (x)(x) \text{ and } (x) \text{ are NOT like terms.} \]

57. (2x+1)(2x-1)

   \[ Drawing \ a \ box \ will \ REALLY \ help \ here: \]
   \[ (2x+1)(2x-1)= 4(x)(x) - 1 \]
58. \((-2+x)(3y)\)
\((-2 + x)(3y) = -6y + 3xy\)

59. \((-2+x)(3+y)\)
\((-2 + x)(3 + y) = -6 - 2y + xy + 3x\)

60. \((-2x)(3+y)\)
\((-2x)(3 + y) = -6x - 2xy\)

61. \(\left(\frac{x}{2} + 3\right)(-3 + \frac{2}{x})\)

If I were you, I’d definitely draw a box for this one...

\[
\left(\frac{x}{2} + 3\right)(-3 + \frac{2}{x}) = \frac{-3x}{2} - 8 + \frac{6}{x}
\]

STOP. Check all your answers … make note of what you did wrong, especially pay attention to patterns in your mistakes. As usual, always celebrate your mistakes and misunderstandings. Think: “oooh… what did I get wrong and what do I get to learn now? How exciting!” Don’t even think about skipping this part of your learning. Reviewing and attending to your mistakes is the ONLY way to learn this material…

62. Factor out an x from the expression \((1+2x)\) (assume x does not equal 0)

\((1+2x) = x\left(\frac{1}{x} + 2\right)\)

63. Factor out a 3 from the expression \((3+3x)\)

\(3 + 3x = 3(1 + x)\)

64. Factor out a 2 from the expression \((2x+4y)\)

\(2x + 4y = 2(x + 2y)\)

65. Simplify the following expression by factoring the top and bottom (numerator and denominator) of the fraction and then simplifying further if possible.
Chapter 13
Solutions, Parentheses, Distributing, Factoring

p. 23

\[
\frac{(2x+8)}{(16x+8)} = \frac{2(x+4)}{8(2x+1)}
\]

\[
\frac{2(x+4)}{8(2x+1)} = \frac{1(x+4)}{4(2x+1)}
\]

66. Simplify the following expression by factoring the top and bottom (numerator and denominator) of the fraction and then simplifying by finding “weird forms of the number 1” (assume that \(x \neq -4\) and \(x \neq -\frac{1}{2}\), because if \(x\) equaled either of those values, you’d be dividing by 0)

\[
\frac{(2x+8)}{(16x+8)(x+4)} = \frac{2(x+4)}{8(2x+1)(x+4)}
\]

\[
\frac{2(x+4)}{8(2x+1)(x+4)} = \frac{2}{8(2x+1)} \cdot \frac{(x+4)}{(x+4)}
\]
67. For each expression in the left column, find ALL of the equivalent expressions in the right column. If there are more than 1 matches, **WRITE ALL OF THE MATCHES**. Write the corresponding letter next to the expression in the left column.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Corresponding Letters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{x+2y}{x} ) This is the same as G, and only G.</td>
<td>A. 2y + 1</td>
</tr>
<tr>
<td>( \frac{(x)(2y)}{x} ) This is the same as E, and only E</td>
<td>B. x + 2y</td>
</tr>
<tr>
<td>( \frac{x^y}{1} \left(\frac{1}{x}\right)(2) ) This is the same as E, and only E</td>
<td>C. x(3z)</td>
</tr>
<tr>
<td>( \frac{(2y)}{x} )</td>
<td>D.</td>
</tr>
<tr>
<td>( (2y) )</td>
<td>E.</td>
</tr>
<tr>
<td>( 1 + 2y )</td>
<td>F.</td>
</tr>
<tr>
<td>( 1 + \frac{2y}{x} )</td>
<td>G.</td>
</tr>
</tbody>
</table>
68. For each expression in the left column, find ALL of the equivalent expressions in the right column. If there are more than 1 matches, **WRITE ALL OF THE MATCHES.** Write the corresponding letter next to the expression in the left column.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Right Column</th>
</tr>
</thead>
<tbody>
<tr>
<td>3(xz)</td>
<td>A. (3x)(3z)</td>
</tr>
<tr>
<td>This is the same as B and C (note that B and C are, therefore, the same)</td>
<td></td>
</tr>
<tr>
<td>3(x + z)</td>
<td>B. (3x)(z)</td>
</tr>
<tr>
<td>This is the same as D, and only D</td>
<td></td>
</tr>
<tr>
<td>3xz</td>
<td>C. 3xz</td>
</tr>
<tr>
<td>D. 3x + 3z</td>
<td>E. 3x + z</td>
</tr>
<tr>
<td>F. (3x + z)(3)</td>
<td></td>
</tr>
</tbody>
</table>

69. The length of one side of a room is 4 feet less than 2x (in other words, the length is 2x-4). The length of the other side of the room is c less than the sum a+b (in other words, the length is a+b-c). Write an expression for the area of the room that does not have any parentheses.

Length one side = 2x-4  
Length of other side = a+b-c  
Total Area = (2x-4)(a+b-c)  
\[2ax + 2bx - 2cx - 4a - 4b + 4c\]
Draw a box, one side has three lengths: $a$, $b$ and $-c$. The other side has two lengths: $2x$ and $-4$.

The total area written without parentheses is:

$$2ax + 2bx - 2cx - 4a - 4b + 4c$$
Decimal Notation and Exponents

Exercises:

Note: these exercises lead you down a path to understanding. They MUST be done in the order they are written or you won’t build the understanding they are designed to create. If you get stuck, don’t go on. Stop and figure it out, or you’ll go down the wrong path!

The exercises in BOLD represent BIG IDEAS! The other exercises lead you to those BIG IDEAS. PAY ATTENTION TO WHAT IS IN BOLD.

1. Write \((x)(x)(x)(x)(x)\) in exponential notation.

\(x^5\) note this is NOT \(x^4\), or \(4x\). The \(4\) MUST be a little superscript to the right of the \(x\).

2. Write \(x^3 \cdot x^5\) in expanded form (as \(x\)’s multiplied to each other)

\[(x)(x)(x)(x)(x)(x)(x)(x)\]

3. Write \{\((x)(x)(x)\)\} \cdot \{(x)(x)(x)(x)(x)\} in exponential notation

\(x^8\)

4. Based on your answers from the previous two questions, what do you think is a general rule for simplifying expressions like \(x^A \cdot x^B\)? (express your rule as an equation)

\[x^A \cdot x^B = x^{A+B}\]

STOP if you do not completely get this. Play with this idea with examples you make up yourself. Are you convinced that you can explain how \(x^A \cdot x^B\) is equivalent to \(x^{A+B}\)?

5. Write \(\frac{x^5}{x^3}\) in expanded form, and simplify.

\[\frac{x^5}{x^3} = \frac{(x)(x)(x)(x)}{(x)(x)(x)}\]

\[\frac{(x)(x)(x)(x)}{(x)(x)(x)} = \frac{(x)(x)(x)}{(x)(x)(x)}(x)(x)\]

\[\frac{(x)(x)(x)}{(x)(x)(x)}(x)(x) = 1(x)(x) = x^2\]
6. Write \( \frac{x^6}{x^3} \) in expanded form, and simplify.

\[
\frac{x^6}{x^3} = \frac{(x)(x)(x)(x)(x)(x)}{(x)(x)(x)(x)}
\]

\[
\frac{(x)(x)(x)(x)(x)(x)}{(x)(x)(x)(x)} \cdot \frac{1}{(x)(x)(x)(x)} = 1 \cdot \frac{1}{(x)(x)(x)(x)}
\]

\[
\frac{1}{(x)(x)(x)(x)} = \frac{1}{x^3}
\]

7. Based on your answers from the previous two questions, what do you think is the general rule for simplifying expressions like \( \frac{x^A}{x^B} \)? (express your rule as an equation)

\[
\frac{x^A}{x^B} = x^{A-B}
\]

STOP if you do not completely get this. Play with this idea with examples you make up yourself. Are you convinced that you can explain how \( \frac{x^4}{x^2} \) is equivalent to \( x^{A-B} \)?

8. Write \( \frac{x^3}{x^6} \) in expanded form, and simplify.

\[
\frac{x^3}{x^6} = \frac{(x)(x)(x)}{(x)(x)(x)(x)(x)(x)}
\]

\[
\frac{(x)(x)(x)(x)(x)(x)(x)}{(x)(x)(x)(x)(x)(x)(x)} \cdot \frac{1}{(x)(x)(x)(x)(x)(x)} = 1 \cdot \frac{1}{(x)(x)(x)(x)(x)(x)}
\]

\[
\frac{1}{(x)(x)(x)(x)(x)(x)} = \frac{1}{x^3}
\]

9. Write \( \frac{x^2}{x^5} \) in expanded form, and simplify.

\[
\frac{x^2}{x^5} = \frac{(x)(x)}{(x)(x)(x)(x)(x)}
\]
10. In exercise 7, you came up with a general rule for simplifying expressions like \( \frac{x^A}{x^B} \). Use that rule to simplify the expressions \( \frac{x^3}{x^6} \) and \( \frac{x^2}{x^5} \).

\[
\frac{x^3}{x^6} = x^{3-6} = x^{-3}
\]

\[
\frac{x^2}{x^5} = x^{2-5} = x^{-3}
\]

compare your answer when you used the rule, to your answers for the previous two questions.

When I expanded, I got \( \frac{1}{x^3} \). When I used the rule, I got \( x^{-3} \).

11. Based on your answer above, what is another way to write expressions with negative exponents, as in the expression \( x^{-y} \)?

\[
x^{-y} = \frac{1}{x^y}
\]

The key here is to recognize that \( x^{-y} \) does not mean \( (x)(-y) \). It is not a repetition of negative fives. It is 1 divided by \( x^y \).

STOP! Make sure you can explain how on earth \( x^{-A} \) is equivalent to \( \frac{1}{x^A} \).

Make up some of your own examples, where you start out dividing \( \frac{x_A}{x_B} \) for an example in which \( B \) is BIGGER than \( A \).
12. Write $\frac{x^5}{x^5}$ in expanded form, and simplify.

$$\frac{x^5}{x^5} = \frac{(x)(x)(x)(x)(x)}{(x)(x)(x)(x)(x)}$$

$$= 1$$

13. Write $\frac{x^3}{x^3}$ in expanded form, and simplify.

$$\frac{x^3}{x^3} = \frac{(x)(x)(x)}{(x)(x)(x)} = 1$$

14. **Use the general rule you came up with** for simplifying expressions like $\frac{x^A}{x^B}$ to simplify $\frac{x^5}{x^5}$ and $\frac{x^3}{x^3}$ look back and compare your answers to the previous two questions.

$$\frac{x^5}{x^5} = x^{5-5}$$

$$= x^0$$

$$\frac{x^3}{x^3} = x^{3-3}$$

$$= x^0$$

15. Based on your answer to the previous question, what do you think is another way to write an expression zero exponent, as in $x^0$?

I know that $\frac{x^3}{x^3} = 1$, and $\frac{x^5}{x^5} = 1$

If I apply the rule, I get that $\frac{x^5}{x^5} = x^0$ and $\frac{x^3}{x^3} = x^0$

So it must be true that $x^0 = 1$.

STOP! $x^0$ DOES NOT MEAN $(0)(x)$, OR $(x)(0)$. 
$x^0$ means that you divided $x$ to some number by $x$ to that same number and applied a rule of simplifying exponents. Anything divided by itself is 1. When you have a zero exponent, it’s the same thing as the number 1.

Make sure you wrap your head around this idea… it’s important.

16. Write $(x^3)^4$ in expanded form, and then write the result in the form $x$ to a power

$$(x^3)^4 = x^3 \cdot x^3 \cdot x^3$$

$x^3 \cdot x^3 \cdot x^3 = xxxxxxxx$

$xxxxxxx = x^{12}$

17. Write $(x^4)^3$ in expanded form, and then write the result in the form $x$ to a power

$$(x^4)^3 = x^4 \cdot x^4 \cdot x^4$$

$x^4 \cdot x^4 \cdot x^4 = xxxxxxxx$

$xxxxxxx = x^{12}$

18. Write $(x^2)^3$ in expanded form, and then write the result in the form $x$ to a power

$$(x^2)^3 = x^2 \cdot x^2 \cdot x^2$$

$x^2 \cdot x^2 \cdot x^2 = xxxxx$

$xxxxx = x^6$

19. Based on your answers to the previous questions, what do you think is a general rule for simplifying expressions like $(x^A)^B$? (write your rule as an equation)

$$(x^A)^B = (x)^{A \cdot B}$$

20. Write $(xy)^3$ in expanded form, and then write the result in the form $(x$ to a power) times $(y$ to a power). [hint: $xy$ means $x$ times $y$, and it does not matter what order you multiply numbers]

$$(xy)^3 = (xy)(xy)(xy)$$

$(xy)(xy)(xy) = xyxyxy$
\[ xyxyxyy = xxyyyy \]
\[ xxyyyy = (x)^3(y)^3 \]

21. Write \((xy)^5\) in expanded form, and then write the result in the form \((x\text{ to a power})\) times \((y\text{ to a power})\)

\[ (xy)^5 = (xy)(xy)(xy)(xy)(xy) \]
\[ (xy)(xy)(xy)(xy)(xy) = xxyxyxxxyyxy \]
\[ xxyxyxxxyyxy = xyyyyyyyyy \]
\[ xyyyyyyyyy = (x)^5(y)^5 \]

22. Based on your answers to the previous questions, what do you think is a general rule for “distributing” exponents in expressions like \((xy)^A\)? (write your rule as an equation)

\[ (xy)^A = (x)^A(y)^A \]

23. Write \(\frac{x^5}{x^4}\) in expanded form, then simplify.

\[ \frac{x^5}{x^4} = \frac{xxxxx}{xxxx} \]
\[ \frac{xxxxx}{xxxx} = \frac{xxxx}{xxxx} x \]
\[ \frac{xxxx}{xxxx} x = 1x \]
\[ 1x = x \]

24. Use the general rule you came up with for simplifying expressions like \(\frac{x^A}{x^B}\) to the expressions \(\frac{x^5}{x^4}\) and \(\frac{x^7}{x^6}\) and then explain what \(x^1\) means.

\[ \frac{x^5}{x^4} = x^{5-4} \]
\[ x^{5-4} = x^1 \]
\[ x^1 = x \]

STOP! Don’t get this confused with \(x^0 = 1\). Are you sure you got this? If not, go back and think slowly about what you are doing.

25. Write \((a + x)^2\) in expanded form, then expand by multiplying (feel free to use the box model to help).
\[(a + x)^2 = (a + x)(a + x)\]

I could draw a box to figure this out. Each side of the box is \(a + x\), which gives me 4 little boxes inside the big box. STOP!!! Go ahead a draw the box in your workbook…

The total area is: \((a)(a) + (x)(x) + (a)(x) + (a)(x)\).

Combining like terms, and using exponential notation, I get:

\[(a + x)^2 = a^2 + x^2 + 2ax\]

26. Write \((ax)^2\) in expanded form, then simplify.

\[(ax)^2 = (ax)(ax)\]

\[(ax)^2 = a^2(x)^2\]

27. Explain what is different between the two previous exercises that allows you to distribute the exponent in one of them but not the other.

There are several different ways to explain this, one is that if you draw boxes to help wrap your head around these two questions, the sides of the box in exercise 25 is a SUM of TWO different lengths, \(a\) and \(x\).

But, if you draw a box for the problem in exercise 26, each side is just ONE length, \(ax\).

So, for exercise 26, there aren’t any “little boxes” to add up, but there are 4 “little boxes” to add up for exercise 25.

28. Write an equation that corresponds to this sentence: there is a number \(x\), such that if you square it (meaning raise it to the power 2), you’ll get the number \(y\).

\[(x)^2 = y\]

29. In the equation \(x^2 = 4\), will both \(x = 2\) and \(x = -2\) make the equation true? In other words, can \(x = 2\) or \(x = -2\)? Are both of these numbers solutions to this equation?

Yes, both \(x = 2\) and \(x = -2\) are solutions to the equation \(x^2 = 4\).

30. How many solutions are there to the equation \(x^2 = 4\)?

There are two solutions, see the answer to the previous question.
31. How many solutions are there to the equation $x^3 = 27$? [hint: $3 \cdot 3 \cdot 3 = 27$]

*Only one solution. $x = 3$ is the only solution to the equation $x^3 = 27$*

32. How many solutions are there to the equation $x^4 = 16$? [hint: $2 \cdot 2 \cdot 2 \cdot 2 = 16$]

*There are two solutions, a positive solution and a negative solution.*

$x = 2$ and $x = -2$ are both solutions to the equation $x^4 = 16$

33. **What is the rule for determining how many solutions there are to equations of the form $x^a = B$?** [describe your rule in words, not an equation]

*If the power is even, there might be two solutions, because if you multiply a negative to itself an even number of times, you’ll get a positive number. So, if $B$ is a positive number, $x^a = B$ will have two solutions, an $x$ that is positive, and an $x$ that is negative.*

*If $a$ is an odd number, then there is only one solution.*

34. If you “do the same thing to both sides of an equation, you might get another equivalent equation (but don’t divide by zero).” If you wanted to solve for $x$ in the equation $x^2 = 9$, you could raise both sides to the $\frac{1}{2}$ power. **You will get**

$(x^2)^{\frac{1}{2}} = 9^{\frac{1}{2}}$. **Simplify the left-hand side of that equation to solve for $x$.**

*Remember the rule $(x^A)^B = (x)^{A\cdot B}$? Using that rule, I get that $(x^2)^{\frac{1}{2}} = (x)^{\frac{2}{2}}$*

$(x)^{\frac{2}{2}} = (x)^{1}$

*Therefore, $(x^2)^{\frac{1}{2}} = x$*

*A solution to the equation $x^2 = 9$ is, therefore, $9^{\frac{1}{2}}$*

**STOP! Did you follow what I just wrote? I started with the equation $x^2 = 9$**

*I did something to both sides, and I got $x = 9^{\frac{1}{2}}$. Next, we’ll figure out what that heck $9^{\frac{1}{2}}$ means.*

35. How many solutions are there to the equation $x^2 = 9$? What are they?
There are two solutions because 2 is an even number.

The two solutions are \( x = 3 \) and \( x = -3 \). I didn’t use any algebra to figure this out. I remember that 3 times 3 is 9.

Remember, finding \( x \) in the equation \( x^2 = 9 \) means to find the numbers that, if you square them, will equal 9.

36. Explain how the solution to \( x^2 = 9 \) is \( 9^{\frac{1}{2}} \) and \( -9^{\frac{1}{2}} \), and that, therefore, \( 9^{\frac{1}{2}} \) must be another way to write the number 3, and \( -9^{\frac{1}{2}} \) must be another way to write the number -3.

In the question 34, I found that \( x = 9^{\frac{1}{2}} \) is a solution to the equation \( x^2 = 9 \). Since I know that there must be two solutions, one negative and one positive, the other solution is \( x = -9^{\frac{1}{2}} \).

In exercise 35, I found that the solutions to \( x^2 = 9 \) are 3 and -3.

So, \(-9^{\frac{1}{2}}\) and \(9^{\frac{1}{2}}\) must just be really weird ways to write -3 and 3.

37. \( x^3 = 27 \). Solve for \( x \) by raising both sides to the \( \frac{1}{3} \) power and expressing the number 27 as \( 3^3 \) (This procedure lets you figure out what number gives you 27 if you multiply it to itself 3 times!)

\( x^3 = 27 \) is the same equation as \( x^3 = 3^3 \), because \( 3^3 \) is exactly 27.

If I raise both sides of this equation: \( x^3 = 3^3 \) to the \( \frac{1}{3} \) power, I get:

\[
\begin{align*}
x^3 & = 3^3 \\
\left( x^3 \right)^{\frac{1}{3}} & = \left( 3^3 \right)^{\frac{1}{3}} \\
x^1 & = 3^1 \\
x & = 3
\end{align*}
\]

38. \( x^2 = 4 \). Find a solution to this equation by raising both sides to the \( \frac{1}{2} \) power, and expressing 4 as \( (\pm 2)^2 \) and figure out what number gives you 4 if you multiply it to itself twice.

You already know the answer to this, just by knowing that if you square 2 or \(-2\) you’ll get 4.
But, what if the equation has a more difficult number that you can’t figure out in your head?

First, consider that $x^2 = 4$ is the exact same equation as $x^2 = (±2)^2$

Next, raise both sides of this equation to the $\frac{1}{2}$ power.

$$x^{2 \cdot \frac{1}{2}} = (±2)^{2 \cdot \frac{1}{2}}$$

And you get that $x = ±2$

Exponents that are also a fractions

The square-root symbol: $\sqrt{}$.

The square-root of $x$ is written like this $\sqrt{x}$ “The square-root of $x$.”

It means “find all the numbers that, if you square them, will be equal to $x$.”

So, $\sqrt{9} = +3$ and $-3$, because if you square -3 you get 9, and if you square 3, you also get 9:

$(±3)^2 = 9$, therefore $±3 = \sqrt{9}$. In other words, $\sqrt{9}$ is the same as 3 or -3.

What would happen if I took the equation $(±3)^2 = 9$ and raised both sides to the $\frac{1}{2}$ power?

$(±3^2)^{\frac{1}{2}} = 9^{\frac{1}{2}}$

$±3 = 9^{\frac{1}{2}}$.

9$^{\frac{1}{2}}$ is equivalent to $\sqrt{9}$

Exercises, continued:

39. Based on what you just read and thought about, how is the expression $x^{\frac{1}{2}}$ related to the expression $\sqrt{x}$? Going back and reviewing is probably a good idea as you answer this question.

   It means the same thing.

40. Write the expression using exponential notation: $\sqrt{x}$

   $\sqrt{x}$ is equivalent to $x^{(\frac{1}{2})}$
This is not the same as \( x^{\frac{1}{7}} \). It’s really really really important that you see that the fraction is in a superscript, and when you write, it’s really really really important you write it as a superscript.

41. Write the expression using exponential notation: \( (\sqrt[8]{x})^7 \)

\[
(\sqrt[8]{x})^7 = x^{\left(\frac{7}{8}\right)}
\]

This is not the same as \( x^{\left(\frac{8}{7}\right)} \). It’s really really really important that you see that the fraction is in a superscript, and when you write, it’s really really really important you write it as a superscript.

42. Write the expression using exponential notation: \( 8\sqrt[8]{x} \)

\[
8\sqrt[8]{x} = 8 \cdot x^{\left(\frac{1}{8}\right)}
\]

43. Write the expression using exponential notation: \( 7\sqrt[8]{x} \)

\[
7\sqrt[8]{x} = 7 \cdot x^{\left(\frac{1}{8}\right)}
\]

44. Write the expression using exponential notation: \( (\sqrt[8]{x})^7 \), and then simplify

\[
(\sqrt[8]{x})^7 = x^{\left(\frac{7}{8}\right)}
\]

\[
x^{\left(\frac{7}{8}\right)} = x
\]

45. Simplify (without a calculator) \( \sqrt[8]{4^2} \)

As a strategy, it’s usually good to first factor everything that can be factored. Here... I can factor the 4:

\[
\sqrt[8]{4^2} = \sqrt[8]{(\pm 2^2)^2}
\]

\[
\sqrt[8]{(\pm 2^2)^2} = (\pm 2^2)^{\left(\frac{1}{2}\right)}
\]

\[
(\pm 2^2)^{\left(\frac{1}{2}\right)} = (\pm 2)^{2 \cdot \left(\frac{1}{2}\right)}
\]

\[
(\pm 2)^{2 \cdot \left(\frac{1}{2}\right)} = \pm 2^{\left(\frac{4}{2}\right)}
\]
\[ \pm 2^{\left(\frac{1}{2}\right)} = \pm 2^{(2)} \]
\[ \pm 2^{(2)} = 4 \]

46. Simplify as much as possible (without a calculator), and write your answer using exponential notation:

\[ \sqrt[2]{9^{\left(\frac{1}{2}\right)}} \]

\[ \sqrt[2]{9^{\left(\frac{1}{2}\right)}} = 9^{\left(\frac{1}{2}\right)} \]

*I need to factor 9 to see if this can simplify any further:*

\[ 9^{\left(\frac{1}{2}\right)} = (\pm 3)^{\left(\frac{1}{2}\right)} \]

*Finally, I get that the simplification is:*

\[ (\pm 3)^{\left(\frac{1}{2}\right)} = (\pm 3)^{\left(\frac{1}{2}\right)} \]

*Because there are no real numbers that have a negative square-root, writing the simplification of \[ \sqrt[2]{9^{\left(\frac{1}{2}\right)}} \] as \( (3)^{\left(\frac{1}{2}\right)} \) is fine for this math class.*

*STOP: do you understand that the answer could also be written as \( \sqrt[3]{3} \)?*

*But, I asked you to write the answer using exponential notation.*

47. Simplify (without a calculator):

\[ \sqrt[3]{x^2} \]

\[ \sqrt[3]{x^2} = x^{\frac{2}{3}} \]

48. Simplify (without a calculator), and write your answer with exponential notation:

\[ \sqrt[2]{9^{(4)}} \]

\[ \sqrt[2]{9^{(4)}} = 9^{\left(\frac{4}{2}\right)} \]

\[ 9^{\left(\frac{4}{2}\right)} = 9^{(2)} = 81 \]

49. Explain whether or not this equation is true for any positive values of \( x \):

\[ \sqrt[4]{(x^B)} = (\sqrt[4]{x})^B \]  In other words, are the expressions on either side of the equal sign equivalent? (hint, re-write both sides using exponential notation, and then simplify)
I’ll start with the left side of the equation and re-write it using exponential notation:

\[ A\sqrt[\frac{1}{x}]{B} = \left( x^{B} \right)^{\frac{1}{A}} = x^{\frac{B}{A}} \]

Now, here’s the right side of the equation, re-written using exponential notation:

\[ \left( \sqrt[\frac{1}{A}]{x} \right)^{B} = x^{\frac{B}{A}} \]

Yep, those sure are the same thing!

So, the equation \[ A\sqrt[\frac{1}{x}]{B} = \left( \sqrt[\frac{1}{A}]{x} \right)^{B} \] is true for any positive value of \( x \). This is another way of saying that \( A\sqrt[\frac{1}{x}]{B} \) is equivalent to \( \left( \sqrt[\frac{1}{A}]{x} \right)^{B} \)

50. To solve for \( x \) in the equation \( x^{\frac{1}{7}} = A \), George raises both sides of the equation to the \( \frac{1}{7} \) power, and then gets the equation \( \left( x^{\frac{1}{7}} \right)^{\frac{1}{7}} = A^{\frac{1}{7}} \), and because George remembers the rule that \( (x^{m})^{n} = x^{m\cdot n} \), he knows that \( \left( x^{\frac{1}{7}} \right)^{\frac{1}{7}} = x^{1} = x \). So, finally, George says that if \( x^{\frac{1}{7}} = A \), then \( x = A^{\frac{1}{7}} \).

Lisa looks at the equation \( x^{\frac{1}{7}} = A \) and solves for \( x \) by taking the seventh root of both sides of the equation: \( \sqrt[7]{x^{\frac{1}{7}}} = \sqrt[7]{A} \) and says that if \( x^{\frac{1}{7}} = A \) then \( x = \sqrt[7]{A} \).

George and Lisa get into an argument over who is correct. How can you help settle their argument?

They are both correct! It’s important they (and you) are able to explain why they are both correct.

51. \( x^{3} = (2)^{-3} \) (how many solutions are there and why?)

Raise both sides of the equal sign to the \( \frac{1}{3} \) power, and get this equation:

\[ x^{1} = (2)^{-1} \]

\[ x = \frac{1}{2} \]

Could there be a positive answer as well? No, because if you raise a negative number to an odd power (in this case 3), you’d get a negative number, and if you raise a positive number to an odd power, you get a positive number. So the only answer is \( x \) equals one half.

52. \( x^{4} = 3^{-4} \) (how many solutions are there and why?)
Raise both sides of the equal sign to the \( \frac{1}{4} \) power, and get this equation:

\[
x^1 = 3^{-1}
\]

\[
x = \frac{1}{3}
\]

Could there be a negative answer as well? YES, because if you raise a negative number to an even power (in this case 4), you’d get a positive number. So there could be two answers:

\[
x = \frac{1}{3} \text{ or } x = -\frac{1}{3}
\]
Exercises: Solve the following equations for $x$, without using a calculator. Of course, some of these exercises require the skills and thinking from the chapters.

1. $x \cdot 10^3 = 2 \cdot 10^{-3}$

   Divide both sides by $10^3$, OR, multiply both sides by $10^{-3}$ (you know that dividing by $10^3$ is the same as multiplying by $10^{-3}$, right? If not, review what a negative exponent means), to isolate $x$.

   $x = (2 \cdot 10^{-3})/(10^{-3})$

   $x = (2 \cdot 10^{-6})$

2. $10^4 = 500x \cdot 10^{-4}$ (how many solutions are there and why?)

   There is only 1 solution. $x$ is simply multiplied to 500 and to $10^{-4}$, so if I divide both sides of the equal sign by 500 and $10^{-4}$, I'd isolate $x$.

   \[
   \frac{10^4}{500 \cdot 10^{-4}} = \frac{500x \cdot 10^{-4}}{500 \cdot 10^{-4}}
   \]

   \[
   \frac{10^4}{500 \cdot 10^{-4}} = x
   \]

   \[
   \frac{10^4}{5 \cdot 10^2 \cdot 10^{-4}} = x
   \]

   \[
   \frac{1}{5} \cdot 10^{4-2-(-4)} = x
   \]

   \[
   \frac{1}{5} \cdot 10^6 = x
   \]

   Since you are not using a calculator, you could leave your answer as that… but you might also know that $\frac{1}{5}$ is 0.2, so the answer could be written as:

   $0.2 \cdot 10^6 = x$

   $2 \cdot 10^{(-1)} \cdot 10^6 = x$

   $2 \cdot 10^5 = x$
The answer is $200,000 = x$

There are several different ways to write 200,000, but there is only one answer. All the different ways to write the answer are the same number.

3. What would you rather have, 500 thousand dollars of half a million dollars? Explain your answer.

500 thousand is 500,000

Half a million is $0.5 \cdot 10^6 = 5 \cdot 10^5 = 500,000$

It’s the same amount of money, so I’d be happy with it, no matter what you wanna call it.

4. What would you rather have, 250 thousand or a quarter million dollars? Explain your answer

250 thousand is 250,000.

A quarter million is $(0.25) \cdot 10^6 = 25 \cdot 10^{-2} \cdot 10^6 = 25 \cdot 10^4 = 250,000$

It’s the same amount of money.

5. What would you rather have, 18 hundred dollars or 0.18 million dollars? Explain your answer

18 hundred is 1800

0.18 million is $(0.18) \cdot 10^6 = 18 \cdot 10^{-2} \cdot 10^6 = 18 \cdot 10^4 = 180,000$

Give me the 0.18 million!

6. Which is more money, $10^9$ pennies or a million dollars?

$10^9$ pennies is $10^7$ dollars. (there are 100 pennies in a dollar).

$10^7 = 10,000,000$

This is quite a bit more than a million dollars!

But a pile of 1,000,000,000 pennies might be hard to deal with. But I would still take it and figure how to deposit the money into my bank account somehow. I wonder how many pennies are actually in circulation now?
7. The total weight of a container of widgets is 20,000 pounds. Each widget weighs 0.00004 pounds.

   a. Are there a lot of widgets or just a few widgets in the container? (just think about it)

      *It’s a really heavy container, and each widget is really light. So there are a whole lot of widgets.*

   b. How many widgets are there in the container? Express your answer two different ways: as a whole number times a power of 10 AND as a number in standard form.

      *I see two ways of approaching this problem. The first way is to think: “how many of or part of one thing fit into another is “division”, so I’ll set up and simplify this division expression:*

      \[
      \frac{20,000}{0.00004} = 20,000,000 \\
      \]

      *Another way to approach this is to set up an equation with a variable and then solve. The variable is “the number of widgets in the container”*

      *I know that if each widget weighs 0.00004 pounds, and if I repeat that weight over and over, I need to get to 20,000 pounds.*

      \[(0.00004)x = 20,000.\]

      *To solve for x, divide both sides by 0.00004.*

      \[
      \frac{20,000}{0.00004} = \frac{2 \cdot 10^4}{4 \cdot 10^{-5}} \\
      \]

      \[
      \frac{2 \cdot 10^4}{4 \cdot 10^{-5}} = \frac{1}{2} \cdot 10^{4-(-5)} \\
      \]

      \[
      \frac{1}{2} \cdot 10^{4(-5)} = \frac{1}{2} \cdot 10^9 \\
      \]

      *That’s half a billion.*

      *Or, you could write your answer like this:*

      \[
      \frac{1}{2} \cdot 10^9 = (0.5) \cdot 10^9 \\
      (0.5) \cdot 10^9 = (5) \cdot 10^{-1} \cdot 10^9 \\
      (5) \cdot 10^{-1} \cdot 10^9 = (5) \cdot 10^8 \\
      \]

      *Now, go back and read the exercise. It requires that I write the answer as a whole number times a power of ten:*

      \[(5) \cdot 10^8\]
AND as number in standard form:
500,000,000

8. Simplify the expression \((2^{16})(\frac{1}{2})\). Write your answer with no exponents or root symbols.

\[
(2^{16})(\frac{1}{2}) = (2)^{16}(\frac{1}{2})
\]

\[
(2)^{16}(\frac{1}{2}) = (2)^{-16}
\]

\[
(2)^{-16} = (2)^{-2}
\]

\[
(2)^{-2} = \frac{1}{2^2} = \frac{1}{4}
\]

9. Simplify the expression \((4 \cdot 10^8)(\frac{1}{2})\). Write your answer with no exponents or root symbols. (Don’t use a calculator!)

\[
(4 \cdot 10^8)(\frac{1}{2}) = (4)(\frac{1}{2})(10^8)(\frac{1}{2})
\]

\[
(4)(\frac{1}{2})(10^8)(\frac{1}{2}) = \left(\frac{1}{4}\right)(\frac{1}{2})(10)^{-4}
\]

\[
\left(\frac{1}{4}\right)(\frac{1}{2})(10)^{-4} = \left(\frac{1}{2^2}\right)(10)^{-4}
\]

\[
\left(\frac{1}{2^2}\right)(10)^{-4} = \left(\frac{1}{2}\right)(10)^{-4}
\]

\[
\left(\frac{1}{2}\right)(10)^{-4} = \frac{1}{2}(10)^{-4}
\]

\[
\frac{1}{2}(10)^{-4} = (0.5)(10)^{-4}
\]

\[
(0.5)(10)^{-4} = 0.00005
\]

Make sure you count all the decimal places!!!
10. Simplify and write your answer in standard form: \(10,000(10^{-3} + 10^{-3})\)

   *Lots of ways to look at this. You could distribute the 10,000. But, I’m going to add what is in the parentheses first, because I notice that it is the same number, so pretty easy to add:*

   \[
   10,000(10^{-3} + 10^{-3}) = 10,000(2 \cdot 10^{-3})
   \]

   \[
   10,000(2 \cdot 10^{-3}) = 10^4(2 \cdot 10^{-3}) \text{ Now, I have three numbers multiplied to each other. It never matters what order you multiply numbers, so I'm going to multiply (10^4) and (10^{-3}) to get 10.}
   \]

   \[
   10^4(2 \cdot 10^{-3}) = 2 \cdot 10
   \]

   *The expression 10,000(10^{-3} + 10^{-3}) is, simply, 20.*

11. \(10,000(10^{-3})(10^{-3}) = (10^4)(10^{-3})(10^{-3})\)

   \[
   (10^4)(10^{-3})(10^{-3}) = 10^{-2}
   \]

   \[
   (10^{-2}) = \frac{1}{100}
   \]

12. Simplify and write your answer in standard form: \((2^{-3} + 2^{-3})\)

   \[
   (2^{-3} + 2^{-3}) = (2)(2^{-3})
   \]

   \[
   (2)(2^{-3}) = (2^1) \cdot (2^{-3})
   \]

   \[
   (2^1) \cdot (2^{-3}) = 2^{-2}
   \]

   \[
   2^{-2} = \frac{1}{2^2}
   \]

   \[
   \frac{1}{2^2} = \frac{1}{4}
   \]

   *If you got this answer another way, that's great! You could have started by changing 2^{-3} to be \(\frac{1}{8}\) and then gone from there.*

13. Simplify and write your answer in standard form: \((2^{-3} \cdot 2^{-3})\)

   Here’s one way to simplify this: \((2^{-3} \cdot 2^{-3}) = (2^{-3})^2\)

   \[
   (2^{-3})^2 = (2)^{-6}
   \]

   \[
   (2)^{-6} = \frac{1}{2^6}
   \]

   \[
   \frac{1}{2^6} = \frac{1}{64}
   \]

14. Simplify and write your answer in standard form: \(16(2^{-3} + 2^{-3})\)
This simplifies to 4, there are many ways to get there. Noticing that $2^{-3} = \frac{1}{8}$ is helpful.

15. Simplify and write your answer in standard form: $16(2^{-3} \cdot 2^{-3})$

   *This simplifies to $\frac{1}{4}$*

16. Simplify and write your answer in standard form: $(2)(-3)(2)(-3) = 36$