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# The World Before Calculus: Historical Approaches to the Tangent Line Problem

Lindsay Skinner
*Western Washington University*

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# THE WORLD BEFORE CALCULUS

## HISTORICAL APPROACHES TO THE TANGENT LINE PROBLEM

LINDSAY SKINNER

## Introduction

Pierre de Fermat and René Descartes were two brilliant 17th century mathematicians who have had lasting impacts on modern mathematics. Descartes laid the groundwork for the Cartesian coordinate system that is frequently employed in modern mathematics and Fermat's last theorem vexed the mathematics community until Wiles' proof was published in 1995. Amidst their many ground-breaking accomplishments these two men produced solutions for another mathematical problem - developing a general method to find the tangent line to a curve.

In spite of their apparent genius, neither man's method had the lasting impact of their other works. Descartes' and Fermat's methods were quickly superseded by the development of calculus thirty years later. In many ways these methods contributed to the development of calculus, yet in others they drastically deviated from it. This deviation is the driving force behind this investigation: why were their methods so different? And what does this reveal about the development of mathematics?

## History of the Tangent Line Problem

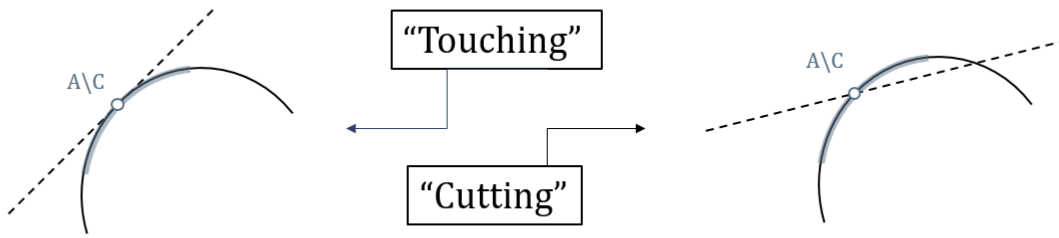
In order to fully grasp the significance of Descartes' and Fermat's tangent line methods one should begin by comparing our modern understanding of tangent lines to their historical treatment leading up to the time of Descartes and Fermat.

Modern mathematicians use calculus to find the tangent line to some curve, say the graph of a function $f(X)$, at some point, $(a, f(a))$. The tangent line is thus defined as $Y = f(a)+f'(a)(X-a)$, assuming the derivative $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists. Notice that this definition is dependent upon limits and the existence of arbitrarily small positive numbers, two concepts that were absent from the earliest definitions of tangent lines.

The earliest known consideration of tangent lines can be found in Euclid's *Elements*, Book Three, in which he defines a line tangent to a circle. Regarding this line Euclid states, "A straight line is said to touch a circle which meeting the circle and, being produced, does not cut the circle." [9] This distinction between lines that "touch" and "cut" was the generally accepted understanding of tangent lines for hundreds of years after Euclid. In fact it is from this definition that the term "tangent" arises, derived from the Latin *tangere* which means "to touch".

Using modern notation we can understand this to mean the following:

> For some curve $A$ composed of the points $(X, f(X))$ a line, $T(X)$, is *tangent* to the curve at the point $C = (X_0, f(X_0))$ if the line goes through the point $C$ and all points on the curve $A$ sufficiently close to $C$ lie on one side of the line. If the line goes through the point $C$ but every collection of points on the curve $A$ around $C$ contains points on both sides of the line, then the line is said to cut the curve $A$ at the point $C$. [1]




> Notice that for the line that touches the curve the points on the curve near the point C lie entirely on one side of the line. However, for the line that cuts the curve, any collection of points around the point C contains points on both sides of the line.

Clearly this definition is not equivalent to the modern one. Consider, for example, the inflection point of $Y = X^3$. The tangent line at this point exists but, due to the fact that the function changes concavity at this point, the line will cut the curve and, according to Euclid's definition, cannot be tangent. However, when one considers the types of problems dealt with at this time, Euclid's definition is sufficient and does accurately describe all possible tangent lines for those specific curves, namely conic sections. Euclid's definition of tangency prevailed for hundreds of years, though it was adjusted and expanded upon in order to consider a larger class of curves.

Archimedes[2] was the first known mathematician after Euclid to consider tangents to a curve other than a circle. He applied the notion of tangents to his Archimedean spiral in order to find the arch length for a given segment of his spiral.[10]

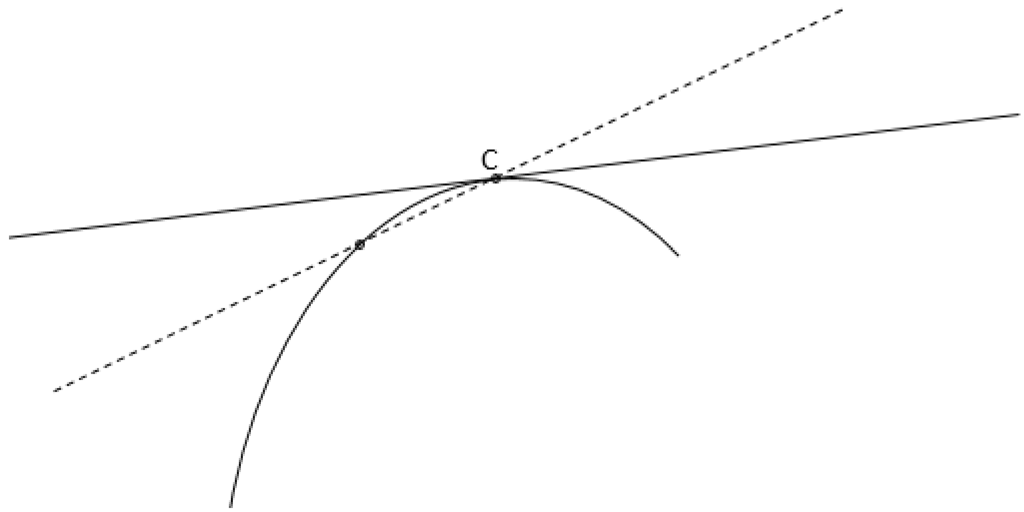
Additionally, Apollonius of Perga[3] built upon Euclid's definition in order to consider tangent lines to conics. Regarding these tangent lines Apollonius states, "If a straight line be drawn through the extremity of the diameter of any conic parallel to the ordinates to that diameter, the straight line will touch the conic, and no other straight line can fall between it

[1] Euclid did not need to restrict the locality of this behavior because he only considered tangents to circles. Other mathematicians that followed him expanded this notion to curves in general. However, because they did not consider a line to be infinite but rather a line segment, Euclid's successors were able to use this definition for curves without restricting the domain of their curve. Instead, they could simply limit the length of the line segment being employed. The idea, then, is that if no such line segment exists for a given direction, then this is not the direction of the tangent line.

[2] Archimedes is considered one of the greatest scientists in classical antiquity. He lived from around 287 BCE to 212 BCE and studied mathematics, physics, engineering and astronomy. Around a dozen of his works survive today.

[3] Apollonius of Perga was a Greek Geometer and Astronomer who lived from 262BCE to 190BCE. He paid particular attention to conic sections in his geometry, which is the context in which he explored the tangent line problem.

and the conic."[3] That is, if we were to start a line segment between the given tangent line and curve on one side of the point in question that then intersects the curve at that point and continues beyond for some distance, then that line segment will cut rather than touch our curve.




Here is one example illustrating this property. The solid line is tangent to the curve at a point, C, and the dotted line segment (which begins between the tangent line and the curve on the left hand side of the image and crosses through the point C at which the solid line is tangent to the curve) cuts the curve at the point C (and at another point).

Notice that Apollonius's understanding expanded Euclid's work by proposing the uniqueness of tangent lines.[4]

Generally speaking, this is the way in which tangent lines were thought about for over a thousand years. However this understanding is only applicable to a very specific set of curves. As a result, more sophisticated means of understanding tangent lines had to develop. In the seventeenth century drastically different definitions, that more closely resemble our own modern definition of the tangent line, began to gain popularity.

Gilles Personne de Roberval[5] made one of the most significant contributions to these changes. Unlike Euclid's purely geometric definition, Roberval explained the tangent line via a dynamic system. He claims that the tangent line can be understood via the movement of a point traveling along the curve under investigation. "By means of the specific properties of the curved line, examine the various movements made by the point which describes it at the location where you wish to draw the tangent: from all these movements compose a single one; draw the line of direction of the composed movement, and you will have the tangent of the curved line."[13] In other words, he makes an explicit connection between the tangent line and the instantaneous velocity of a point moving along the curve under investigation.

[4]Though it would not have been considered so at the time, as lines were considered line segments, thus one could conceive of numerous different sized segments that meet the above requirement.

[5]Gilles Personne de Roberval lived from 1602 to 1675, making him a contemporary of Descartes and Fermat. He was a French mathematician particularly interested in many of the problems that were eventually solved with the use of calculus, including the tangent line problem, and finding the quadrature and cubature of surfaces and volumes.

Roberval's definition eliminated many of the problems present with Euclid's and allowed mathematicians to consider tangent lines to a variety of curves that had previously been ignored.[6] Roberval is noteworthy for another reason as well. He was a contemporary of Descartes and Fermat, having corresponded with the latter on several occasions.[13]

Descartes' and Fermat's tangent line solutions arose in a century of rapid development after more than a thousand years of subdued progress regarding the problem. Their solutions were published just prior to the development of calculus, the most popular modern method for solving these problems. Both men, extremely intelligent and well-equipped to deal with this problem, were on the cusp of a major mathematical development yet they were only on the cusp. Each man's argument relied on geometric methods[7] and, while their explanations utilized some dynamic notions, in practice each man's method only considers a non-moving geometric system. In this way both men gave some consideration to a limit-like notion (both considered behavior as one point neared another) but neither directly applied this to his method, nor do they conceptualize limits the way a modern mathematician would.[8] As a result their methods were only useful when dealing with a limited number of curves; they lacked the universality of calculus.

## Descartes' Method

René Descartes was a famous French philosopher and one of the most confident - some might even say "egotistical" - mathematicians of his day.[9] He dealt with a large variety of subjects, having written twelve major treatises on material including philosophy, theology, human anatomy, mathematics, music and more. Amidst those treatises is *La Geometrie*, an appendix to his *Discourse de la Methode*. This treatise is best known as the text in which Descartes proposed his analytic geometry, a means of reducing geometric figures to algebraic equations and the inspiration for the modern Cartesian coordinate system. However, this work dealt with many other topics, among them Descartes' method for solving the tangent line problem.

The method is proposed in Book Two of *La Geometrie*, in which Descartes seeks to discuss the nature of curves, which were extremely important in the seventeenth century due to mathematicians' interest in optics. Seeking to investigate the effects of lenses and the properties of reflection and refraction, the study of optics is closely tied to an investigation of intersecting curves and, subsequently, their tangent lines. Descartes' text begins with a discussion of ancient mathematics, the rules ancient mathematicians followed, and the limitations they faced. He then delves into a discussion of notable geometric problems and provides a solution for Pappus' five line problem by employing his analytic geometry.

[6] For example, this new way of understanding the tangent line would lead to the correct tangent line on a cubic graph at the inflection point, which was impossible with the previous definition.

[7] At this time geometry was the only widely accepted method of proof.

[8] Limits are by nature dependent upon the existence of arbitrarily small positive numbers. The epsilon-delta definition of a limit ($\lim_{x\to a} f(x) = L$ iff $\forall\epsilon > 0, \exists\delta$ s.t. $|x-a| < \delta \implies |f(x) - L| < \epsilon$) requires that $\epsilon$ be able to become arbitrarily small. This understanding of limits, as well as the modern definition of continuity, were developed by Cauchy in the nineteenth century, well over a hundred years after Descartes and Fermat developed their methods to solve the tangent line problem.

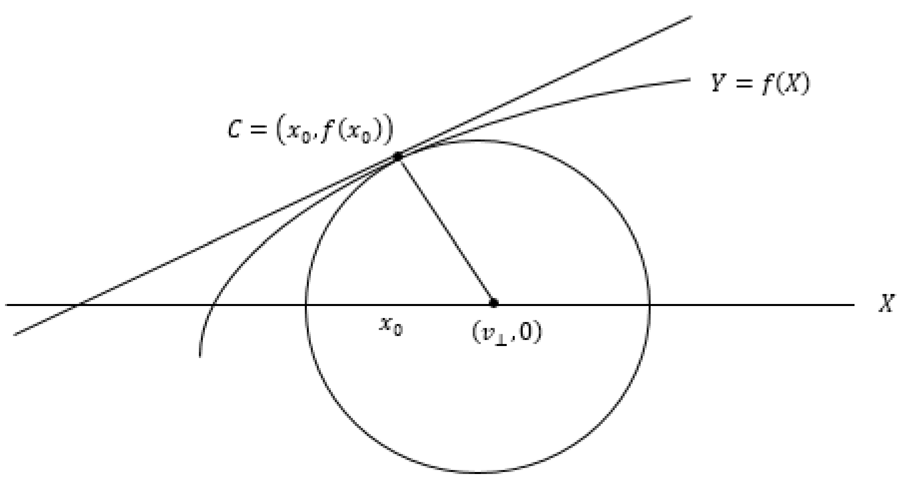
[9] Descartes has been known to refer to his contemporaries as "two or three flies," "less than a rational animal," "a little dog," and "extremely contemptible," among other, less flattering (i.e. scatological) descriptions. [8]

Following this solution Descartes gives a general statement regarding problems that deal with curves, in which he claims:

> "All other properties of curves depend only on the angles which these curves make with other lines. But the angle formed by two intersecting curves can be as easily measured as the angle between two straight lines, provided that a straight line can be drawn making right angles with one of these curves at its point of intersection with the other. This is my reason for believing that I shall have given here a sufficient introduction to the study of curves when I have given a general method of drawing a straight line making right angles with a curve at an arbitrarily chosen point upon it. And I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know."[4]

It's worth exploring, then, how exactly Descartes went about solving "the most useful and most general problem in geometry."

**Employing the Method.** Say we are trying to find the tangent line to the graph of a function $f(X)$ at the point $C = (x_0, f(x_0))$.[10] Descartes assumes the existence of a tangent line at this point and, subsequently, the existence of a normal line to the tangent. Provided this normal line is not horizontal[11] and our point does not lie on the $X$-axis[12] it will intersect the $X$-axis at some point, which we shall label $(v_\perp, 0)$.




[10]Descartes's method was applicable to more general curves, not just those that that can be represented by graphs of functions. However, for ease of notation, we will proceed under the assumption that the curve we are investigating is the graph of a function.

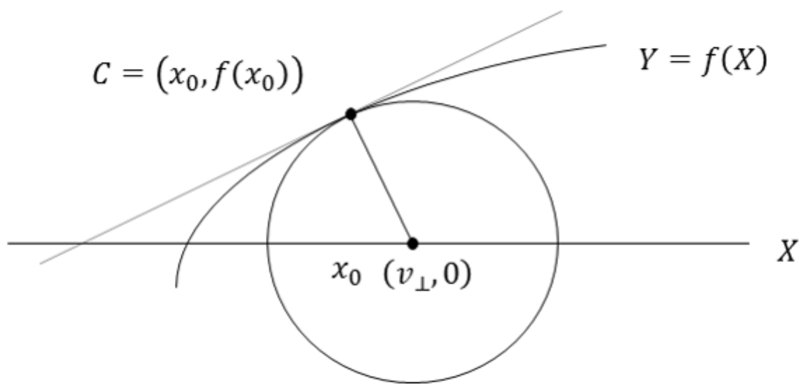
[11]Presumably, vertical tangent lines need not be considered, as they result from an uninteresting or obvious case. If the normal line were horizontal then the tangent line would be vertical, which would have been "obvious" if one were to examine the geometric depiction of this curve.

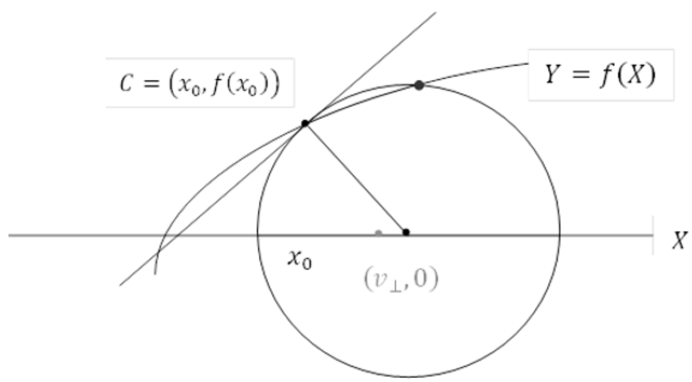
[12]At the time coordinate systems were not used in the way that they are today. Often mathematicians considered simple free-standing geometric figures with labeled lengths rather than orient themselves using a fixed set of coordinate axes. As a result, the choice of the $X$-axis in this recreation is arbitrary, meaning if our point of interest lies on the $X$-axis then all we need to do is shift our axis to a different location.

Now, following Descartes' method, we form a circle centered around $(v_\perp, 0)$ and touching the graph of $f(X)$ at the point $C$.

Descartes remarked that this circle is the only circle of this type (contacting $C$ with its center on the $X$-axis) which will "touch" but not "cut" the curve. [13]

> "The circle about $[v_\perp]$ as center and passing though the point $C$ will touch but not cut [the graph of $Y = f(X)$]; but if this point $[v_\perp]$ be so little nearer to or farther from $[x_0]$ than it should be, this circle must cut the curve not only at $C$ but also in another point."[14] [4]




> The top image shows the circle centered at $(v_\perp, 0)$ which touches the graph of $f(X)$ at $C$. In contrast, the bottom figure portrays an improperly centered circle, which cuts the graph of $f(X)$ at two separate points.

Descartes further remarked that the closer the circle's center is to this correct point $(v_\perp, 0)$ the closer the two points at which the circle cuts the graph of $f(X)$ will be. If the circle is centered at $(v_\perp, 0)$ then we will be left with only the point $C$.[15] So, assuming the existence

[13] Recall Euclid's distinction between "to touch" and "to cut".

[14] The numerous changes made to this quote are due to the fact that modern notation differs drastically from that which Descartes used.

[15] This observation is extremely significant as it demonstrates that Descartes essentially relies the notion of limits in order to justify his method. However, he does not do so explicitly and his final conclusion does not invoke this notion of the two points "falling together". The man's accidental invocation of limit-like

of this circle, one is able to represent this circle via the expression $r^2-(v_\perp-X)^2-f(X)^2$. It should be clear that this circle does indeed "touch" the graph of $f(X)$ (i.e. share a tangent line with the graph of $f(X)$ at the point $C$) since the line segment normal to the graph of $f(X)$ between $C$ and $(v_\perp,0)$ is the radius of the circle and thus tangent to the circle.

Since the circle "touches" the curve at the point $C$, meaning the single point of intersection between the curve and the circle is a double root, we can write the above expression in the form $(X-x_0)^2H(X)$, where $H(X)$ is some expression that we determine based on the behavior of $f(X)$. This is perhaps best demonstrated through an example.

Given the curve $f(X)=X^2$ suppose we want to discover the tangent line to the curve at the point $(1,1)$. Then we begin with the equation $r^2-(v_\perp-X)^2-f(X)^2=(X-x_0)^2H(X)$ which, plugging in $f(X)=X^2$ and $C=(1,1)$, yields $r^2-(v_\perp-X)^2-X^4=(X-1)^2H(X)$. We can see from this equation that we have an $X^4$ term on the left, meaning we must also have an $X^4$ term on the right. This then means that $H(X)$ must be a quadratic, so we let $H(X)=a_0+a_1X+a_2X^2$, for some constants $a_0,a_1,a_2$. Making this substitution Descartes obtains:

$$r^2-(v_\perp-X)^2-(X^2)^2=(X-1)^2H(X)$$
$$r^2-(v_\perp-X)^2-(X^2)^2=(X-1)^2(a_0+a_1X+a_2X^2)$$
$$r^2-v_\perp^2+2v_\perp X-X^2-X^4=a_0+(-2a_0+a_1)X+(a_0-2a_1+a_2)X^2+(a_1-2a_2)X^3+a_2X^4$$

We then match terms of the same degree in the equation in order to determine the unknown constants of $H(X)$ and thus solve for $v_\perp$.

$$\begin{aligned} -X^4 &= a_2X^4 && \implies a_2=-1\\ (0)X^3 &= (a_1-2a_2)X^3, a_2=-1 && \implies a_1=-2\\ -X^2 &= (a_0-2a_1+a_2)X^2, a_2=-1, a_1=-2 && \implies a_0=-4\\ 2v_\perp X &= (-2a_0+a_1)X, a_1=-2, a_0=-4 && \implies v_\perp=3 \end{aligned}$$

We can then use calculus to confirm that this is the correct value for $v_\perp$.

If $f(X)=X^2$ then we know that $f(1)=1, f'(X)=2X$ and $f'(1)=2$. This means that the normal line at $(1,1)$, $N(X)$, must be of the form $N(X)=(-1/f'(X))X+b=\frac{-1}{2}X+b$. We can then use the fact that this line passes through the point $(1,1)$ to solve for $b$:

$$1=\frac{-1}{2}(1)+b$$
$$\implies b=\frac{3}{2}.$$

Thus the equation of the normal line to $f(X)$ at the point $(1,1)$ is $N(X)=\frac{-1}{2}X+\frac{3}{2}$. Since $(v_\perp,0)$ lies on this line we can use $N(X)$ to solve for $v_\perp$,

$$N(v_\perp)=0=\frac{-1}{2}v_\perp+\frac{3}{2}$$
$$\implies v_\perp=3$$

which is the same value that we determined using Descartes's method.

Of course, this method is not perfect; two major issues with Descartes' method quickly become apparent. First and foremost is the method's dependency on H(X). No method

notions is of particular interest, and will be investigated much more thoroughly in the second half of this paper.

is given for determining H(X); as he does in other places throughout the text, Descartes assumes the choice will be obvious to the reader.[16] However this choice is not obvious when the functions under investigation become more complicated.[17] Of course, given the types of curves he was dealing with, Descartes would not have considered these more complicated functions. [18]

The second major issue with this method is a practical one, which plagued Descartes's contemporaries as well as the modern reader. The potential complexity of this method is cause for concern. With just a slight increase in the complexity of $f(X)$, the method becomes much more complicated.

To illustrate, when working with a quadratic function this method requires one to solve a system of four equations (since the $f(X)^2$ term would yield a fourth degree polynomial and one would be required to match terms for $X, X^2, X^3$, and $X^4$ in order to determine $v_\perp$). However, increasing the complexity of the curve slightly, say to a quartic, would result in a drastic increase in the number of equations one would have to solve; in this case it would double the number of required equations. This may not be particularly burdensome for today's mathematicians but, if one considers the lack of technology and common notation used during Descartes' time,[19] it becomes apparent that this increased complexity would very quickly reach unmanageable levels in the seventeenth century. Several of Descartes' contemporaries shared this concern with complexity, and voiced their concerns to the philosopher.

## Fermat's Method

Among those contemporaries was Pierre de Fermat who, in his letter to Mersenne commented upon the complexity of Descartes' method, critiqued the fact that Descartes did not explicitly address cases involving maxima and minima, and suggested that Mersenne share with Descartes Fermat's own, far simpler, method for finding tangent lines.[8] Descartes, who at the time was a far more accomplished mathematician than Fermat, gave an extremely

[16] "I shall not give the constructions for the required tangents and normals in connection with the method just explained, since it is always easy to find them, although it often requires some ingenuity to get short and simple methods of construction."[4]

[17] While we may not always be able to find $H(X)$, we can say something about its behavior. Due to the fact that the equation derived from Descartes's method is of the form $r^2 - v_\perp^2 + 2v_\perp X - f(X)^2 = (X-1)^2 H(X)$ we can see that the only coefficients of $H(X)$ that truly matter when determining $v_\perp$ are the constant and degree one coefficients, because when multiplied by $(X - x_0)^2$ these are the only coefficients that will appear in front of the $X$ term of $(X-1)^2 H(X)$. This implies that, no matter how high the degree of the $X$ terms in $H(X)$, we really only care about the linear term. (Of course, using this method the higher degree coefficients are necessary to find the lower ones. In that sense, all of the coefficients are important.) This makes sense; we are using the equation to determine first derivative information. The higher order terms tell as about the curvature of the figure, whereas its linear behavior is dependent on the linear terms.

[18] In *La Geometrie* Descartes only dealt with conics and polynomials.

[19] Due to the fact that coordinate axes were not a well accepted tool until well after Descartes' time, all of these calculations were carried out with constants from labeled geometric figures. For example, when considering a square-root function Descartes' method yields the following expression $zz\frac{+2bcddz--2bcdez--2cddvz--2bdevz--bddss+bddvv--cdss+cddvv}{bdd+cee+eev--ddv}$ to represent the function $r^2-(v_\perp-X)^2-(\sqrt{z})^2$

negative response to the man's advice, beginning one of the most famous feuds in the history of mathematics.[20]

In the face of this criticism Fermat defended his method.

> "I maintain that my methods are just as certain as the construction of the first proposition of the Elements. Perhaps having them put forward naked and without demonstration, they were not understood or they appeared too simple to M. Descartes, who has made so much headway and has taken such a difficult path for these tangents in his Geometry" [8]

For the most part Fermat was justified in his defense. His method was far simpler and, when considering its practicality one can't help but note that Fermat's approach is far more similar to our current limit definition of the derivative than Descartes'.

**Employing the Method.** As before, suppose we are trying to find the tangent line to some curve, represented by the graph of a function $f(X)$, at some point, $C = (x_0, f(x_0))$.[21] Fermat assumes the existence of the tangent line at this point and, provided it is not horizontal[22] and does not lie on the X-axis [23] this line crosses the x-axis at some point, call it $(v, 0)$. We shall define this line by the function $T(X)$.[24] Fermat then denotes another point on this line near $(x_0, f(x_0))$,call it $(x_0 + e, T(x_0 + e))$.

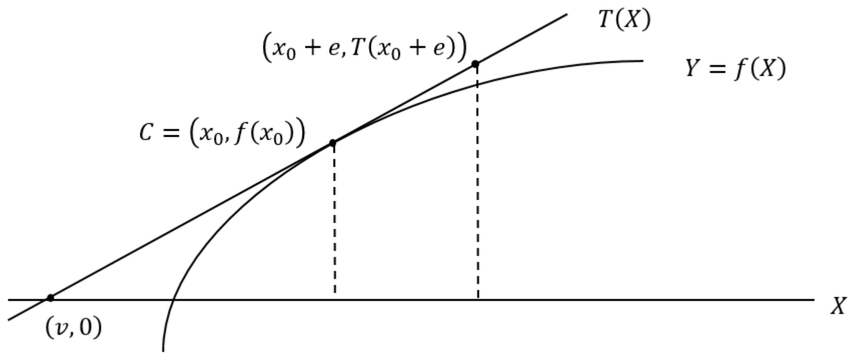
[20]In response, Descartes wrote that "[Fermat's rule] is nothing more than a false position, founded on a means of demonstration that reduces to the impossible, and that is the least valued and the least clever of all of that are used in mathematics. Whereas mine is drawn from a familiarity with the nature of equations that were never explained, but in the third book of my Geometry ... and it follows from the noblest means of demonstrating what could be, to know that which is called a priori." [8] This correspondence was the beginning of a longlasting feud between the two men. While it may seem that their interactions eventually calmed over time due to the lack of correspondences in their later years, one cannot help but draw attention to Fermat's final jab at his rival, delivered in a funeral speech honoring Descartes. "The conclusions that can be taken from the fundamental proposition of M. Descartes' Dioptrique are so beautiful and ought naturally to produce such lovely results throughout every part of the study of refraction that one would wish not for the glory of our deceased friend, but more for the argumentation and embellishment of the sciences that this proposition were genuine and legitimately demonstrated, and all the more as it is from these [conclusions] that one is able to say that multa sunt falsa probabiliora veris (often, falsehoods are more acceptable than truth" [8]

[21]As before, Fermat's method was applicable for many types of curves, not just those represented by the graphs of functions. However, for notation's sake we will proceed assuming the curve is the graph of a function.

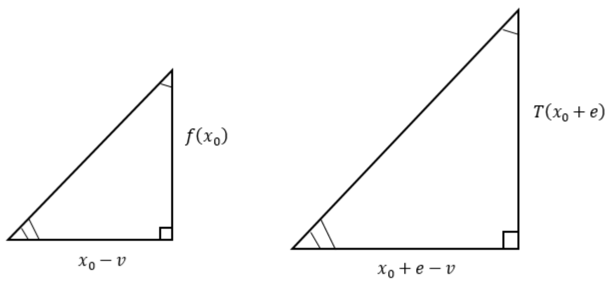
[22]Fermat had already dealt with the case of horizontal tangent lines before the introduction of this problem in his *Maxima et Minima*.

[23]For the same reason as before, we are able to shift the X-axis in order to prevent this issue from arising

[24]The only instance in which this would not be a function is if the line were vertical. However, as was the case for Descartes, vertical lines would have been uninteresting to Fermat and thus need not be considered.




Notice, then, that similar triangles can be formed involving these points.




Fermat leverages the proportionality of similar triangles in order to come up with the equation $\frac{f(x_0)}{x_0-v} = \frac{T(x_0+e)}{x_0+e-v}$. He then uses the notion of "adequality". Essentially, this involves employing an approximate equality in the solution in order to make simplification possible, only to replace this approximation with the true equality once simplified. This allows Fermat to claim that $f(x_0+e) \approx T(x_0+e)$, and thus write

$$\frac{f(x_0)}{x_0-v} \approx \frac{f(x_0+e)}{x_0+e-v} \tag{1}$$

From this point he is able to solve for $v$, which is perhaps best demonstrated by an example.

Using the same example as in Descartes's method, $f(X) = X^2$ and $C = (1,1)$, we can use the above general equation and write $\frac{1^2}{1-v} = \frac{(1+e)^2}{1+e-v}$ which then allows us to solve for $v$ by first cross-multiplying, which yields $1 - v + 2e - 2ev + e^2 - ve^2 = 1 + e - v$. Simplifying, we obtain $e - 2ev + e^2 - ve^2 = 0$. Notice that the remaining terms of the equation are all multiples of $e$. Since we are attempting to solve for $v$ we want to simplify the equation as much as possible, which requires us to divide by the highest possible power of $e$ common to all terms, resulting in $1 - 2v + e - ev = 0$.[25] Fermat then points out that, since $f(x+e)$ and $T(x+e)$ are only equivalent if $e = 0$, we must allow $e = 0$ in order to have a true equality. Thus we eliminate all of the remaining terms containing $e$ which yields $1 - 2v = 0$ and allows us to determine $v = \frac{1}{2}$.

We can then use calculus to confirm that this is the correct value of $v$. From our earlier work we know that $f'(1) = 2$ which means that $T(X) = 2X + b$. Since the point $(1,1)$ must

[25] Notice that Fermat had no concerns about dividing by zero

lie on the graph of $T(X)$ we know

$$\begin{aligned} T(1) = 1 = 2(1) + b \\ \implies b = -1. \end{aligned}$$

We also know that $(v, 0)$ must lie on the graph of $T(X)$, thus

$$\begin{aligned} T(v) = 0 = 2v - 1 \\ \implies v = \frac{1}{2} \end{aligned}$$

which is the same value that was determined for $v$ using Fermat's method.

This method, in practice, is extremely similar to the modern definition of the derivative. In fact, if one were to rearrange the above equation (1) and apply a limit then we could see that $v = \lim_{e\to\infty} x_0 + \frac{e}{f(x_0)-f(x_0+e)} f(x_0) = x_0 - \frac{f(x_0)}{f'(x_0)}$ which is the correct answer for $v$ in the general case. Once again, this may be confirmed using calculus.

We know that the tangent line to the graph of $f(x)$ at the point $C = (x_0, f(x_0))$ is of the form $Y = f'(x_0)X + b$. Since this lines passes through the point $(x_0, f(x_0))$ we can determine $b$ since

$$\begin{aligned} f(x_0) = f'(x_0)x_0 + b \\ \implies b = f(x_0) - f'(x_0)x_0. \end{aligned}$$

This means that the equation of the tangent line is $Y = f'(x_0)X + f(x_0) - f'(x_0)x_0$. Since we know the tangent line crosses the X-axis at $(v, 0)$ we can now determine $v$. From the equation $0 = f'(x_0)v + f(x_0) - f'(x_0)x_0$ we can determine that

$$\begin{aligned} v &= \frac{f'(x_0)x_0 - f(x_0)}{f'(x_0)} \\ &= x_0 - \frac{f(x_0)}{f'(x_0)} \end{aligned}$$

which is the answer we get if we apply limits to Fermat's method.

However, Fermat did not utilize the concept of limits which, as a result, restricted the applicability of his method. Without limits these concepts are not always able to be simplified, as $f(x_0)$ and $f(x_0 + e)$ do not always yield terms that will cancel out.[26] Despite the similarities between this method and our current one, Fermat's method is still only applicable to a limited group of curves.[27]

Both methods share this limitation, as they are predicated upon a stagnant geometric understanding of tangents rather than a dynamic one, as calculus is. To clarify, both of the above methods assume the figures are fixed in place. Either the two points on Descartes's circle lie on top of each other or they do not, there is no "sliding together" (which would imply the existence of limits). A similar observation may be made regarding Fermat's method. Rather than use the language of $(x + e, T(x + e))$ approaching $(x, f(x))$, Fermat simply says that they are equal.

[26] Consider functions like $e^X, \sin X$, etc.

[27] Of course this wouldn't have been concerning at the time, as the curves to which both of these methods are limited were the only types of curves being considered in the seventeenth century. Functions like $e^X, \sin X, \log X$, etc. would not have been explored.

The limit notion is at the heart of what makes modern calculus so powerful. It allows us to solve more complicated problems which could not have been solved using previous methods. In doing so, it opened the door to a whole new class of curves to be taken under consideration. But the question remains, why did Descartes and Fermat fail to take this approach, especially when both men invoked a limit-like notion in their observations?

Descartes looked at the behavior of tangents to his circle in relation to the tangent to his curve as he changed the location of the circle's center. He observed that if the center is closer to $v_{\perp}$ then the points of intersection between the circle and curve will be closer together. This observation, which is almost the same as allowing one point to approach another, is strongly tied to limits. However, Descartes did not use this notion of points *approaching* one another beyond this brief description; he simply states that the center of his circle must exist at his proscribed point.

Fermat's method, while much more closely related to the approach taken in calculus to find a tangent line, is not nearly as reliant on this limit notion as Descartes'. Fermat's method is composed of steps that the modern mathematician justifies by employing limits; a notion that Fermat does not once allude to. Fermat has the opportunity to invoke limits when he equates $T(x+e)$ and $f(x+e)$ but he does not do so. Rather than consider the behavior as $(x+e, T(x+e))$ approaches $(x, f(x))$, Fermat simply states that $(x+e, T(x+e))$ must equal $(x, f(x))$ if his equation is to hold.

This begs the question, why do these two methods, one which invokes a limit-like observation and the other which is closely related to our modern definition (which is predicated on the notion of limits), fail to consider limits?

## History of Infinity, Limits and Continuity

The seventeenth century marks a distinct moment of change in a long history of confusion and general wariness when dealing with issues surrounding infinity, including limits and continuity  the two concepts central to calculus that are lacking in Descartes' and Fermat's methods.

**The Ancients.** This history of confusion begins, most famously, with Zeno's paradoxes.[28] The most significant of the paradoxes for this investigation is often referred to as the "dichotomy" paradox. This paradox is predicated upon the notion "That which is in locomotion must arrive at the half-way stage before it arrives at the goal."[6] That is, if one attempts to traverse a finite distance $D$, he must traverse half the distance $\frac{D}{2}$ before he is able to traverse the entire distance. However, this effect is compounding, meaning that at that halfway point one still has a distance of $\frac{D}{2}$ left to traverse. Before traversing this distance he must traverse half this distance, $\frac{D}{4}$, landing him at a distance of $\frac{3D}{4}$ away from his original starting point. This pattern continues on, the traveler always having a distance of $\frac{D}{2^n}$ left to travel at step $n$. Zeno then explains that this implies that one must traverse and infinite number of distances in order to traverse this finite distance $D$, which he believed impossible to do in a finite time. However, this is ludicrous as people are clearly capable of traversing finite

[28]Zeno's paradoxes are a famous set of philosophical problems attributed to Zeno of Elea, who lived between 490 and 430 BCE. These paradoxes are preserved in Plato's *Parmenides* and Aristotle's *Physics*.

distances in a finite span of time. Thus a paradox arises.[29] This paradox was the cause of much controversy when it was proposed resulting in the development of two major schools of thought: the atomists and the continuists.

In order to resolve the conflict the atomists attacked the notion of an infinitely divisible distance. They argued that a distance cannot be halved *ad infinitum* as Zeno had claimed. Rather, they developed the notion of an indivisible distance.[30] The existence of this indivisible implies that a finite distance cannot be halved an infinite number of times. Thus, according to the atomists, a body in motion does not traverse an infinite number of halves, as Zeno claimed, but rather a finite number. Thus no paradox arises.

In opposition to the atomist views, continuist thinkers emerged. The most famous among these was Aristotle,[31] whose work constituted the main body of scientific and philosophical authority through the Middle Ages. The continuist perspective regarding this issue is preserved in Book III of Aristotle's *Physics*, which explicitly addresses this paradox. Contrary to the atomists, Aristotle argues that a continuum can be divided *ad infinitum*, but only potentially. Throughout Aristotle's work he strives to maintain this qualification, distinguishing between the "potential infinite" and the "actual infinite" wherever possible.

> "It is evident from what has been said, then, that no infinite body exists in actuality ... Accordingly, we are left with the alternative that the infinite exists potentially. However, the potential existence of the infinite must not be taken to be like that of a statue; for what is potentially a statue may come to be actually a statue, but this is not so for what is potentially infinite. But since to be has many senses, the infinite exists in the sense in which the day exists or times exist, namely, by always coming into being one after another ..."[1]

For Aristotle mankind is only able to consider infinity when imagining an ongoing process. Having undertaken repetitive processes, man is able to conceptualize the idea that this process could continue on forever, however this process will never actually continue on forever. Everything with physical and temporal extension is finite, according to Aristotle. Thus, while we can imagine something like a line continuing on forever by just extending its length every time we approach an end. However because that line can only actually exist in the real world, which limits it to a finite existence, it will never actually extend infinitely.

According to Aristotle this continuum can be divided in half an infinite number of times, but doing so causes it to lose the quality of being continuous.

> "A continuous motion is of something continuous and in that which is continuous an infinite number of halves do exist, but potentially and not in actuality. And if he were to make these [halves] actual, he would not be making something continuous but would be stopping (something which evidently happens to one who is counting the halves)." [1]

[29]Of course, we know that Zeno's assumption that it is impossible to traverse an infinite number of distances in a finite time is false. Implicit in this paradox is Zeno's assumption that $\Sigma_{n=1}^{\infty}(\frac{1}{2})^n = \infty$, which we know to be false. This is a geometric summation, meaning $\Sigma_{n=1}^{\infty}(\frac{1}{2})^n = 1$ which is finite.

[30]The Greek term *atom*, from which the school's name arises, means "indivisible".

[31]Aristotle, a student of Plato, was one of the most influential Greek philosophers. He lived between 384 and 322 BCE, though his authority lasted for thousands of years. He wrote on numerous subjects including physics, metaphysics, biology, ethics, psychology and more.

As a result one must qualify Zeno's claims that a moving body traverses an infinite number of halves.

> "If these things be considered as existing actually, it is not possible, but if potentially, then it is possible; for he who is in motion continuously traverses the infinite not in an unqualified way but accidentally, in view of the fact that the line is an infinity of halves in an accidental way while its substance or its being is something else." [1]

The idea presented is that if one begins with a finite continuum then it can be halved a potentially infinite number of times without changing its nature. However actually halving the continuum results in a non-continuous collection of lengths. Thus an actually infinite collection of halves cannot be of the same nature as the collection derived from this continuum. No infinite collection is actually traversed, according to Aristotle, because no infinite collection actually exists in the physical world. But a potentially infinite collection does exist, thus refuting atomists' claims regarding the existence of an indivisible. This distinction between "actual" and "potential" infinity carried on through the Middle Ages and the seventeenth century.

**The Middle Ages.** One of the most influential medieval figures to transmit and expand upon Aristotle's ideas was St. Thomas Aquinas.[32] Aquinas was extremely interested in Aristotle's works, having written a commentary on his *Physics* and frequently citing "the philosopher" as an authority in his *Summa Theologica*.

Aquinas perpetuated Aristotle's notion of continuity in his discussion of the motion of angels. When discussing the question "Whether an angel passes through intermediate space?"[33] Aquinas argued that "the local motion of an angel can be continuous, and non-continuous."[34] He then distinguishes between continuous and non-continuous motion, frequently citing Aristotle and invoking his notion of continuity. "Accordingly, since magnitude is infinitely divisible and the points in every magnitude are likewise infinite in potentiality, it follows that between every two places there are infinite intermediate places."[15] Aquinas also maintained Aristotle's distinction between actual and potential infinity throughout the *Summa*.[35]

However, Aquinas's works do reveal a major shift in the conceptual history of infinity that is notably distinct from Aristotle's arguments. Aquinas admitted the existence of an actual infinity. Specifically he believed that the Christian God was actually infinite. ". . . it is clear

[32] Thomas Aquinas was a 13th century Dominican friar and one of the most influential medieval theologians. He is the father of Thomism and much of the development of modern philosophy can be viewed as a reaction against the ideas that he perpetuated.

[33] The *Summa* is broken into a series of questions, which function like separate books or chapters within the larger work. This question deals with the issue of whether angels simply appear and disappear when changing location or actually pass through the space in between different locations.

[34] To clarify, angels can move by both sudden appearance and by moving through the space in-between different locationis.[15]

[35] " . . . every kind of multitude must belong to a species of multitude. Now the species of multitude are to be reckoned by the species of numbers. But no species of number is infinite; for every number is multitude measured by one. Hence it is impossible for there to be an actually infinite multitude, either absolute or accidental . . . But a potentially infinite multitude is possible; because the increase of multitude follows upon the division of magnitude; since the more a thing is divided, the greater number of things result. Hence, as the infinite is to be found potentially in the division of the continuous, because we thus approach matter, as was shown in the preceding article, by the same rule, the infinite can be also found potentially in the addition of multitude." [15]

that God Himself is infinite and perfect."[15] These claims reflect the beliefs of the larger populace throughout the Middle Ages.

Gregory of Nyssa was the first notable Christian theologian to outline a detailed argument for God's infinitude, having done so between 381 and 383 CE. His arguments were based upon the works of earlier Cappadocian fathers, and laid the groundwork for numerous related arguments in the following centuries.[7] Two notable contemporaries of Thomas Aquinas who took up this argument were Alexander Nequam[36] and Richard Fishacre.[37] The proofs that these men provided for God's infinite nature were notable for their explicit invocation of mathematics.[38]

Furthermore, Aquinas claimed that "Things other than God can be relatively infinite, but not absolutely infinite."[15] For example, a line may be considered relatively infinite, as a Geometer may extend the line to whatever length he requires. However, according to Aquinas, no line in existance is truly infinite. While these beliefs were developed in the Middle Ages, they remained prevalent up to the seventeenth century. Related to these claims, Aquinas discussed the infinite extension of man's intellect.

> "The fact that the power of the intellect extends itself in a way to infinite things, is because the intellect is a form not in matter, but either wholly separated from matter, as is the angelic substance, or at least an intellectual power, which is not the act of any organ, in the intellectual soul joined to a body."[15]

This distinction between objects of the intellect and the material world is extremely significant to later mathematicians, as it provided justification for an investigation of the infinite, despite its material limitations.

It should also be noted that Aquinas did not push this idea as far as his successors. Rather, he maintained an Aristotelian perspective and argued that man's understanding of both physical and mathematical bodies (i.e. bodies that can be conceived by the intellect) is restricted to the finite.[39]

> "We must therefore observe that a body, which is a complete magnitude, can be considered in two ways; mathematically, in respect to its quantity only; and naturally, as regards its matter and form. Now it is manifest that a natural body cannot be actually infinite. For every natural body has some determined substantial form ... The same applies to a mathematical body. For if we imagine a mathematical body actually existing, we must imagine it under some form, because nothing is actual except by its form ..." [15]

[36]Nequam was a twelfth century English theologian who, despite being extremely interested in Aristotle's philosophy, employed mathematics in order to support his arguments for the infinitude of God. He lived from 1157-1217.

[37]Fishacre was a thirteenth century Dominican who lived from 1200 to 1248. He was the Dominican chair at Oxford and is well known for his extensive investigation of and commentaries regarding Peter Lombard's Sentences.

[38]While not central to this paper, Anne Davenport provides an extensive study dealing with these two theologians and their use of mathematics to explain the infinitude of God in her study *The Catholics, the Cathars, and the Concept of Infinity in the Thirteenth Century.*[2]

[39]Aristotle also distinguished between physical and mathematical bodies. However, he argues that neither of these may be infinite. After considering "mathematical", "sensible", and "intelligible" objects (which are presented as exhaustive cases that may qualify all potential "objects") Aristotle concludes that "It is evident from what has been said, then, that no infinite body exists in actuality."[1]

However, these notions about infinity began to change shortly after Aquinas' *Summa Theologica* was published, as seen in William of Ockham's *Exposito* on Aristotle's *Physics*.[40]

While Ockham shared Aquinas's views that God was infinite by nature, he did not restrict this actual infinitude to God. He argued that "Every continuum is actually existent. Therefore any of its parts is really existent in nature. But the parts of the continuum are infinite because there are not so many that there are not more, and therefore the infinite parts are actually existent."[12] Ockham therefore drops Aquinas's qualification that a continuum is only potentially indivisible an infinite number of times, it is now actually so. This belief necessitates the existence of arbitrarily small positive distances. However one must keep in mind that Ockham did not explicitly state the existence of these distances nor relate the results of his claims to limits or continuity beyond the above statement. His works reveal a very early step in the direction of limits, however these ideas were not fully developed for hundreds of years.

Ockham also discussed man's ability and limitations when attempting to comprehend the infinite. Ultimately he concluded that human intellect is limited in its capacity to grasp the infinite, but not in its ability to create justified proofs of God's infinitude.

> "It is true that Ockham argues we cannot demonstrate the proposition, "God exists" via causality. Similarly, he does not think that we can demonstrate the propositions, "there is only one God," "God is intensively infinite;" he does diminish the reach of human reason in this respect. But, he thinks that it is probable or plausible that God exists and that we can construct proofs for this conclusion that are rationally persuasive."[12]

The idea that man's intellect is limited in grasping the infinite was maintained by many, including a large group of seventeenth century mathematicians. Ockham's discussion of these limitations is particularly noteworthy due to the fact that he explains that man is only able to understand infinity through motion or causality.[12] This notion is striking when one considers the fact that the infinite occurs in Descates' and Fermat's proofs in the context of a dynamic system, and is subsequently shut out of the proof when the mathematicians choose to limit their thinking to a stagnant geometric system.

**The Seventeenth Century.** The seventeenth century saw the rise of two major distinct schools of thought when dealing with the concept of infinity.

The negative perspective, typically held by empiricists, argued against the existence of an infinite. Instead, they claimed that when one spoke of the "infinite" they were really discussing nothing more than that which reached beyond their intellectual capacity, something for which we can conceive no limits. For example, Gassendi[41] argued that "We call infinite that thing whose limits we have not perceived, and so by that word we do not signify what we understand about a thing, but rather what we do not understand."[11] He even argued with Descartes on this matter, and claimed that "Infinity either of place or of perfection cannot be understood."[11]

In contrast, the positivists argued that the infinite does exist, even if mankind's understanding of such is limited. Some of the most notable mathematicians who shared these

[40] William of Ockham was a Franciscan friar from England who lived from 1287 to 1347. He was extremely interested in philosophy, theology and logic, and contributed written works in each field.

[41] Pierre Gassendi lived from 1592 to 1655. He was a moderate skeptic and empiricist. He also spoke in defense of atomism, and was known for his conflicts with Descartes, who believed in the infinite.

views include Descartes, Spinoza, Leibniz, and Malebranche. While these men all argued in favor of infinity's existence, they did not conceptualize the infinite the way mathematicians do today. On the contrary, even the positivists upheld that the infinite was incomprehensible.

Descartes was one of these thinkers who, while firmly believing in the existence of the infinite, was extremely critical of man's ability to comprehend such. He argued that "Since we are finite, it would be absurd for us to determine anything concerning the infinite; for this would be to attempt to limit it and grasp it ..."[11] This idea was central to Descartes' proof of God's existence. This proof is predicated on the notion that all else besides God is finite and thus incapable of understanding, much less generating, the idea of infinity. Thus there must exist an infinite generator, i.e. God.[42]

At the time that Descartes and Fermat were developing their mathematics these were the general arguments surrounding the notions of infinity and, subsequently, limits. The plausibility of infinity was still under debate and, even among those who did believe in the existence of this concept, it was widely believed that man's intellect was incapable of grasping this concept in full. Consequently subjects predicated on the notion of infinity, like limits, were highly debated and infrequently utilized by seventeenth century mathematicians.

## Conclusion

While the notions of infinity, continuity and limits are widely accepted today, they were the cause of much turmoil for thousands of years. Philosophers, theologians, and mathematicians largely viewed the infinite as a vast unknown or impossibility. The repercussions of this turmoil and the stagnation that mathematics was faced with are apparent in the works of Fermat and Descartes, whose methods came extremely close to considering limits but fell short. These methods were created just prior to the adaptation of a more definite understanding of the infinite which can be seen in the later development of the calculus. [43]

Discussions of the infinite existed among the Greeks, though they were extremely limited after Aristotle's distinction between the potential and actual infinite. Interest in the subject was revived in the Medieval period with the widespread belief of an infinite God. While these theological investigations opened the door for consideration of the infinite, they also appear to have had a negative, "shutting down effect" in later years, as they perpetuated the belief that God is the only thing to have an actually infinite nature. The result of these claims was that great thinkers were hesitant to publish their work for fear of the religious backlash they might face or, as was the case for Descartes, some of them even feared that their proofs could shed doubt on their religious beliefs.[44]

In some senses mathematics stagnated as a result of these concerns, and was not able to overcome this stagnation until the development of calculus some thirty years later. These

[42] "By the name God I understand a substance that is infinite, independent, all-knowing, all-powerful, and by which I myself and everything else, if anything else does exist, has been created."[5]

[43] While this confusion largely dealt with infinity on a large scale in terms of multitude or extension, these concepts are inherently tied to notions of the infinitely small. When contemplating the possibility of dividing a distance into an infinite multitude of parts one must necessarily consider the existence of an infinitely small distance, as apparent in the aforementioned discussions of continuity.

[44] "Descartes began writing his *Treatise on the Universe* in 1629. The work was ready for publication in 1633. In June of that year, Galileo was condemned by the Inquisition for his doctrine that the earth moves. This doctrine was also central to Descartes' cosmology. When Descartes heard of the condemnation, he decided not to publish his treatise. Descartes feared censure by the Church. His fear was not ill-founded, since his philosophy was condemned by Rome in 1663, sixteen years after his death."[14]

limitations can be seen in the works of Fermat and Descartes, both great thinkers in the seventeenth century whose solutions to the tangent line problem proved far less useful than calculus. The history behind this problem and the avoidance of arbitrarily small positive numbers and limits in their tangent line methods should demonstrate just how recent and significant the development of infinity, limits and, subsequently, the calculus has been.

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