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## Optical Tomography for Media with Variable Index of Refraction

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### ABSTRACT

Optical tomography is the use of near-infrared light to determine the optical absorption and scattering properties of a medium  $M \subset \mathbb{R}^n$ . If the refractive index is constant throughout the medium, the steady-state case is modeled by the stationary linear transport equation in terms of the Euclidean metric and photons which do not get absorbed or scatter travel along straight lines. In this expository article we consider the case of variable refractive index where the dynamics are modeled by writing the transport equation in terms of a Riemannian metric; in the absence of interaction, photons follow the geodesics of this metric. The data one has is the measurement of the out-going flux of photons leaving the body at the boundary. This may be knowledge of both the locations and directions of the exiting photons (fully angularly resolved measurements) or some kind of average over direction (angularly averaged measurements). We discuss the results known for both types of measurements in all spatial dimensions.

### RESUMEN

Tomografía óptica es el uso de luz infrarroja próxima para determinar la absorción óptica y las propiedades de dispersión de un medio  $M \subset \mathbb{R}^n$ . Si el índice de refracción es constante a través del medio, el caso estado “steady” es modelado por la ecuación de transporte lineal

estacionaria en terminos de la métrica Euclideana y factores los cuales não são absorvidos o dispersados viajando a lo largo de lineas retas. En este artículo expositivo consideramos el caso de indice de refracción variable donde las dinámicas son modeladas escribiendo la ecuación de transporte e terminos de la metrica Riemanniana; en la ausencia de interacción, fotones siguen las geodesicas de esta métrica. Los datos que uno tiene es la medida del flujo out-going de fotones dejando el cuerpo en la frontera. Esto puede ser conocido de la localización y las direcciones de los fotones existentes (mediciones completamente angulares) o alguna clase de promedio sobre dirección ( mediciones angularmente promediadas). Nosotros discutimos los resultados conocidos para ambos tipos de mediciones en todas las dimensiones espaciales.

**Key words and phrases:** *Boltzmann equation, integral geometry, optical tomography, Riemannian metric, transport equation.*

**Math. Subj. Class.:** *35R35, 35Q99.*

## 1 Introduction

The stationary linear transport equation models, among other things, the propagation of the energy density of waves in heterogeneous media [B3, Cha, RPK, vanR-Nh], neutrons in nuclear reactors [Mo], and near-infrared photons in tissue. The term “linear” refers to the fact that the equation models only scattering of particles from the material and assumes that the density of particles is low enough that particle-to-particle interaction may be neglected. Recently, photon propagation has been applied in optical tomography for use in medical imaging [A, Ch-Al, RBH].

In the absence of scattering, propagation is usually assumed to be along straight lines, as would be the case for light propagation in a medium with constant index of refraction. This amounts to writing the transport equation in terms of the Euclidean metric. Here we consider the situation where, in the absence of scattering, propagation is along geodesics of a Riemannian metric. We consider this a model for optical tomography in a medium with continuously varying refractive index.

When prescribed in-going, and measured out-going, fluxes are allowed to depend on both position and direction (in other words, measurements are fully “angularly resolved”), the problem of optical tomography in the Euclidean setting has been well studied. In many instances restrictions are placed on the scattering kernel, for example that it be small, or that it be independent of direction. For recent results see [B1, SU, T1, T2]. For dimensions three and greater, these restrictions are relaxed in [CS2] in the stationary case, and [CS1] in the time-dependent case. Analogous results are proven in the Riemannian setting in [M1, M2]; see also [KLU]. Stability of the reconstruction of material parameters has been studied in [BJ, SU, W] for the Euclidean case.

Such measurements are, however, unrealistic since angular resolution of out-going flux is difficult to achieve. In [L], Langmore introduces an angularly averaged measurement operator and proves that angularly averaged measurements taken at every point on the boundary of the medium enables reconstruction of the extinction coefficient  $\sigma$ ; once  $\sigma$  is known, measurements made at a

single boundary point allow unique determination of the scattering kernel under some additional assumptions. In particular, it is assumed that scattering is supported away from the boundary of the body, the scattering phase function is known, and the scattering kernel is small. It is also assumed that  $\sigma$  and the phase function are “close to” real-analytic. These results are extended to the Riemannian case in [LM].

In this article we shall present results for the inverse problem of optical tomography in the presence of a Riemannian metric. These results have appeared in [LM, M1, M2].

Let  $M \subset \mathbb{R}^n$  be an open bounded set with smooth boundary  $\partial M$  and let  $g$  be a smooth Riemannian metric on  $\bar{M}$ . We shall make geometric assumptions on the Riemannian manifold  $(M, g)$  in due course. If  $u(x, v)$  represents the density of particles at position  $x$  with velocity  $v$  in the unit tangent sphere at  $x$ ,  $\Omega_x M$ , then the stationary linear transport equation is

$$Tu(x, v) = -\mathcal{D}u(x, v) - \sigma(x, v)u(x, v) + \int_{\Omega_x M} k(x, v', v)u(x, v') dv' = 0. \quad (1.1)$$

The functions  $\sigma$  and  $k$  are the material specific parameters we seek to determine from boundary measurements. The first of these,  $\sigma$  is the extinction coefficient and  $\sigma(x, v)u(x, v)$  accounts for loss of energy in direction  $v$  due to both absorption at  $x$  as well as scattering into other directions. The second,  $k(x, v', v)$  is the scattering kernel and is proportional to the probability of a particle with position  $x$  and velocity  $v' \in \Omega_x M$  being scattered to velocity  $v \in \Omega_x M$ . The operator  $\mathcal{D}$  is the derivative along the geodesic flow (see (2.2) below) which in the case of  $g$  being Euclidean is simply  $\mathcal{D}u(x, v) = v \cdot \nabla_x u(x, v)$ . The measure  $dv'$  is the volume form on  $\Omega_x M$  induced from the Euclidean volume form on  $T_x M$  determined by  $g$  at  $x$ ; here  $T_x M$  is the full tangent space to  $M$  at  $x$ . While  $u$  is strictly speaking a density, for large numbers of particles it is reasonable to represent  $u$  as, say, an  $L^1$  function.

In [B2] a transport equation is derived as a limiting case of Maxwell’s equations with non-constant, but isotropic, permeability resulting in a varying refractive index  $n(x)$ . If the metric of (1.1) is conformal to the Euclidean metric,  $g_{ij}(x) = c_0^{-2}n(x)^2\delta_{ij}$  with  $c_0$  the speed of light in a vacuum, then (1.1) can be shown to correctly describe energy density propagation in isotropic material. While it has not been shown that a general anisotropic index of refraction leads to a limiting transport equation of the form (1.1), we consider it as a model for this case. We point out that the same model appears in [Sh2].

Define the “incoming” and “outgoing” bundles on  $\partial M$ , the boundary of  $M$ ,

$$\Gamma_{\pm} = \{(x, v) : x \in \partial M, \text{ and } \pm \langle v, \nu_x \rangle > 0\}$$

where  $\nu_x$  is the unit outer normal vector to the boundary  $\partial M$  at  $x$  and  $\langle \cdot, \cdot \rangle$  is the inner product, each with respect to  $g$  at  $x$ .

If  $u_-$  is an incoming flux of particles on  $\Gamma_-$  let  $u$  be the solution, should it exist, to  $Tu = 0$  with the boundary condition  $u|_{\Gamma_-} = u_-$ . Denote this solution operator by  $T^{-1}$ , i.e.  $u = T^{-1}u_-$ . The measurement operator corresponding to fully angularly dependent measurements is the albedo

operator

$$\mathcal{A} : u_- \mapsto u|_{\Gamma_+}.$$

It is from knowledge of  $\mathcal{A}$  that we seek to determine uniquely the material parameters  $\sigma$  and  $k$ . In section 5 we will present the corresponding operator for angularly averaged measurements.

Determination of  $\sigma$  and  $k$  relies upon invertibility of certain geodesic ray transforms (integrals along geodesics). To ensure injectivity of these transforms we assume that the metric is “simple”:

**Assumption 1.1.**  $(M, g)$  is simple:  $M$  is strictly convex, and for any  $x \in \bar{M}$  the exponential map  $\exp_x : \exp_x^{-1}(\bar{M}) \rightarrow \bar{M}$  is a diffeomorphism (and consequently  $M$  is diffeomorphic to a ball).

We must assume that  $\sigma$  depends only on position. If  $k \equiv 0$  and  $g$  is Euclidean then determination of  $\sigma$  from  $\mathcal{A}$  is simply the usual X-ray transform. If  $\sigma$  depends on direction  $v$ , then it is not uniquely determined from its X-ray transform (see the introduction of [CS2]). Characterization of non-uniqueness when  $\sigma$  depends on both  $x$  and  $v$  has recently been investigated in [ST]. We will also assume that  $0 \leq \sigma \in L^\infty(M)$  and  $0 \leq k \in L^\infty(\Omega^2 M)$  are bounded functions. Here we introduce the somewhat unconventional notation

$$\Omega^2 M := \{(x, v, w) : x \in M, v, w \in \Omega_x M\}.$$

The forward problem is not necessarily well-posed without some subcriticality assumptions (see [RS]). Such conditions will be satisfied by smallness assumptions on  $k$  to be stated precisely in the following section.

The content of this article has appeared in [LM, M1, M2]. The purpose here is to collect together these results into a single summarizing exposition. In section 2 we describe the solution to the forward problem, thus enabling the definition of the measurement operator  $\mathcal{A}$ , and expressing it as a series. This series can be understood intuitively as contributions from ballistic particles which do not scatter, singularly scattered particles, and multiply scattered particles. In section 3 we prove unique identifiability of  $\sigma$  for all dimensions and of quite general  $k$  for dimensions three and higher. In section 4 a finer analysis in two dimensions provides determination of scattering kernels  $k$  which are *a-priori* known to be small relative to  $\sigma$ . The results of sections 3 and 4 assume knowledge of angularly resolved measurements. In section 5 we restrict to angularly averaged measurements and prove identifiability of  $\sigma$  and a restricted class of  $k$  in arbitrary dimensions.

## 2 The Forward Problem and the Albedo Operator

The approach utilized in the results of this article is to obtain a series expansion for the distribution kernel of the boundary measurement operator. In this section we prove solveability of the forward problem, and in the process obtain such a series expansion for  $\mathcal{A}$ .

If  $(x, v) \in \Omega M$  we shall denote by  $\gamma_{(x,v)}(t)$  the geodesic satisfying  $\gamma_{(x,v)}(0) = x$  and  $\dot{\gamma}_{(x,v)}(0) = v$ ; we shall also use the compressed notation

$$\vec{\gamma}_{(x,v)}(t) = (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)).$$

Define the “distance-to-boundary” functions  $\tau_{\pm} : \Omega M \rightarrow \mathbb{R}^+$  by

$$\tau_{\pm}(x, v) = \min\{t > 0 : \gamma_{(x,v)}(\pm t) \in \partial M\}$$

and set  $\tau = \tau_- + \tau_+$ .

The volume forms on  $\Omega_x M$  and on  $\Gamma_{\pm}$  are the following: on  $M$  we have the naturally defined volume form of the metric. At any  $x \in M$ , the volume form  $dv$  on  $\Omega_x M$  is the form induced from the Euclidean volume form on  $T_x M$  defined by the metric  $g$  at  $x$ . The resulting form on  $\Omega M$  is the Liouville form and is preserved under the geodesic flow of  $g$ . We denote by  $d\mu$  the induced volume form on  $\Gamma_{\pm}$  which has the property that  $dt d\mu(x, v)$  is the pull-back of the Liouville form by the geodesic flow. Equivalently, we have the induced volume form of  $\partial M$  included in  $\bar{M}$ ; if  $(x, v)$  are local coordinates for  $\partial M$  and  $dx$  is this volume form on  $\partial M$ , then it holds that

$$d\mu(x, v) = |\langle v, \nu_x \rangle| dv dx.$$

The operator  $\mathcal{D}$  in (1.1) is the derivative along the geodesic flow and is defined by

$$\mathcal{D}u(x, v) = \left. \frac{\partial}{\partial t} \right|_{t=0} u(\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)). \tag{2.2}$$

If  $(x^i, y^i)_{i=1}^n$  are local coordinates for  $\Omega M$  with the  $(y^i)$  with respect to the natural basis  $(\frac{\partial}{\partial x^i})$  then in these coordinates

$$\mathcal{D}f = \frac{\partial f}{\partial x^i} y^i + \frac{\partial f}{\partial y^i} (-y^j y^k \Gamma_{jk}^i)$$

where  $\Gamma_{jk}^i$  are the Christoffel symbols of the Levi-Civita connection of  $g$ . If  $x, y \in M$  then simplicity of  $(M, g)$  ensures there exists a unique geodesic from  $x$  to  $y$ ; let  $d(x, y)$  be the geodesic distance between  $x$  and  $y$ , and let  $v(x, y)$  be the tangent vector at  $x$  of this geodesic. Define

$$E(x, y) := \exp\left\{-\int_0^{d(x,y)} \sigma(\gamma_{(x,v(x,y))}(t)) dt\right\}.$$

Note, using the fact  $\gamma_{(y,v(y,x))}(d(y, x) - s) = \gamma_{(x,v(x,y))}(s)$  we have  $E(x, y) = E(y, x)$ .

Define

$$T_0 u = -\mathcal{D}u - \sigma u, \quad T_1 u(x, v) = \int_{\Omega_x M} k(x, v', v) u(x, v) dv',$$

so that  $T = T_0 + T_1$ . We begin by re-writing this as an integral equation. In the absence of scattering, the homogeneous boundary value problem  $T_0 u = 0$ ,  $u|_{\Gamma_-} = u_-$  has solution  $Ju_-$  where

$$Ju_-(x, v) = E(x, \gamma_{(x,v)}(-\tau_-(x, v))) u_-(\gamma_{(x,v)}(-\tau_-(x, v))).$$

The inhomogeneous Dirichlet problem  $T_0 u = f$ ,  $u|_{\Gamma_-} = 0$  has solution

$$u(x, v) = T_0^{-1} f(x, v) = \int_0^{\tau_-(x, v)} E(x, \gamma_{(x, v)}(t - \tau_-(x, v))) u(\vec{\gamma}_{(x, v)}(t - \tau_-(x, v))) dt.$$

Defining

$$Ku(x, v) = \int_0^{\tau_-(x, v)} E(x, \gamma_{(x, v)}(t - \tau_-(x, v))) T_1 u(\vec{\gamma}_{(x, v)}(t - \tau_-(x, v))) dt,$$

we obtain the solution to  $Tu = 0$ ,  $u|_{\Gamma_-} = u_-$  satisfies

$$(I - K)u = Ju_-. \quad (2.3)$$

We shall make repeated use of the following immediate lemma.

**Lemma 2.1** (Change of Variables). *If  $u \in L^1(\Omega M)$  then*

$$\int_M \int_{\Omega_x M} u(x, v) dv_x dx = \int_{\Gamma_{\pm}} \int_0^{\tau_{\mp}(x', v')} u(\gamma_{(x', v')}(t), \dot{\gamma}_{(x', v')}(t)) dt d\mu(x', v').$$

Assume the following smallness condition on  $k$ :

$$\text{with } \sigma_p(x, v') := \int_{\Omega_x M} k(x, v', v) dv, \quad \|\tau\sigma_p\|_{L^\infty(\Omega M, dv dx)} < 1. \quad (2.4)$$

This subcriticality condition will ensure well-posedness of the boundary value problem (see [RS]).

**Proposition 2.1.** *The operator  $K$  is bounded on  $L^1(\Omega M, \tau^{-1} dv dx)$  with operator norm bounded by  $\|\tau\sigma_p\| < 1$  and so  $(I - K)$  is invertible on this space. Equation (2.3) and hence (1.1) is uniquely solvable for  $u_- \in L^1(\Gamma_-, d\mu)$  and the solution  $u$  has a well-defined trace  $u|_{\Gamma_-}$ . The albedo operator  $\mathcal{A} : L^1(\Gamma_-, d\mu) \rightarrow L^1(\Gamma_+, d\mu)$  is a bounded map.*

Since we have  $\|\tau\sigma_p\| < 1$  we may express the solution to (1.1) as a Neumann series,

$$u = \sum_{j=0}^{\infty} K^j Ju_- = Ju_- + KJu_- + (I - K)^{-1} K^2 Ju_-. \quad (2.5)$$

We seek the solution, in the sense of distributions, to (1.1) with singular boundary condition  $u|_{\Gamma_-} = \delta_{\{x_0, v_0\}}(x', v')$ ; the right hand side is the distribution on  $\Gamma_-$  defined by

$$(\delta_{\{x_0, v_0\}}, \varphi) = \int_{\Gamma_-} \delta_{\{x_0, v_0\}}(x', v') \varphi(x', v') d\mu(x', v') = \varphi(x_0, v_0)$$

for  $\varphi \in C_0^\infty(\Gamma_-)$ . It is convenient to use parallel translation in what follows: for  $x, y \in M$  define  $\mathcal{P} : \Omega_x M \rightarrow \Omega_y M$ ,  $\mathcal{P}(v; x, y)$  to be the parallel translation of  $v$  from  $x$  to  $y$  (along the unique geodesic joining  $x$  to  $y$ ).

**Proposition 2.2.** *The three terms in (2.5) can be written*

$$K^j J u_-(x, v) = \int_{\Gamma_-} u_j(x, v, x', v') u_-(x', v') d\mu(x', v')$$

with

$$\begin{aligned} u_0(x, v, x', v') &= \int_0^{\tau_+(x', v')} E(x, \gamma_{(x, v)}(-\tau_-(x, v))) \delta_{(x, v)}(\vec{\gamma}_{(x', v')}(t)) dt \\ u_1(x, v, x', v') &= \int_0^{\tau_+(x', v')} \int_0^{\tau_-(x, v)} E(x, y(s)) E(x', z(t)) k(z(t), \dot{z}(t), \mathcal{P}(\dot{y}(s); y(s), z(t))) \delta_{\{y(s)\}}(z(t)) ds dt \\ &\quad y(s) = \gamma_{(x, v)}(s - \tau_-(x, v)), \quad z(t) = \gamma_{(x', v')}(t) \\ u_2 &\in L^\infty(\Gamma_-, \mathcal{W}), \quad \mathcal{W} = \{f : \mathcal{D}f \in L^1(\Omega M), \tau^{-1}f \in L^1(\Omega M)\}. \end{aligned}$$

*Proof.* We present the straight-forward proof for  $u_0$  and refer the reader to [M1] for the more involved proofs for  $u_1$  and  $u_2$ . If  $\varphi \in C_0^\infty(\Omega M)$  then

$$\begin{aligned} (J u_-, \varphi) &= \int_{\Gamma_-} u_-(x', v') \int_0^{\tau_+(x', v')} E(x', \gamma_{(x', v')}(t)) \varphi(\vec{\gamma}_{(x', v')}(t)) dt d\mu(x', v') \\ &= \int_M \int_{\Omega_x M} \int_{\Gamma_-} u_-(x', v') \int_0^{\tau_+(x', v')} \delta_{(x, v)}(\vec{\gamma}_{(x', v')}(t)) dt d\mu(x', v') \varphi(x, v) dv dx. \end{aligned}$$

□

From Proposition 2.2 we obtain the distribution kernel  $\alpha(x, v, x', v')$  of the albedo operator  $\mathcal{A}$ .

**Theorem 2.1.** [M1] *The distribution kernel  $\alpha(x, v, x', v')$  of  $\mathcal{A}$  is  $\alpha = \alpha_0 + \alpha_1 + \alpha_2$  with*

$$\begin{aligned} \alpha_0 &= E(x, \gamma_{(x, v)}(-\tau_-(x, v))) \text{bigr} \delta_{\{\vec{\gamma}_{(x, v)}(-\tau_-(x, v))\}}(x', v'), \\ \alpha_1 &= u_1|_{(x, v) \in \Gamma_+}, \\ \alpha_2 &\in L^\infty(\Gamma_-; L^1(\Gamma_+, d\mu)). \end{aligned}$$

*Proof.* If  $\varphi_- \in C_0^\infty(\Gamma_-)$  then changing variables to  $(y, w) = \vec{\gamma}_{(x', v')}(t)$ ,

$$\begin{aligned} \int_{\Gamma_-} u_0(x, v, x', v') \varphi_-(x', v') d\mu(x', v') &= \int_M \int_{\Omega_y M} E(x, \gamma_{(x, v)}(-\tau_-(x, v))) \delta_{(x, v)}(y, w) \varphi_-(\vec{\gamma}_{(y, w)}(-\tau_-(y, w))) dw dy \\ &= E(x, \gamma_{(x, v)}(-\tau_-(x, v))) \varphi_-(\vec{\gamma}_{(x, v)}(-\tau_-(x, v))) \\ &= \int_{\Gamma_-} \alpha_0(x, v, x', v') \varphi_-(x', v') d\mu(x', v'). \end{aligned}$$

The claim for  $\alpha_2$  follows from [M1] Theorem 2.3 which shows that the trace on  $\Gamma_\pm$  is continuous from  $\mathcal{W}$  into  $L^1(\Gamma_\pm, d\mu)$ . □



### 3 Full Measurements in Arbitrary Dimensions

We show here that both the extinction coefficient and the scattering kernel are uniquely determined in dimensions three and greater. This is achieved by isolating terms in the kernel of the albedo operator  $\mathcal{A}$  that differ in strength of singularity. Briefly,  $\alpha_0$  determines  $\sigma$  and  $\alpha_1$  determines  $k$ . In dimension two, however,  $\alpha_2$  is in fact a locally  $L^1$  function and is not immediately distinguishable from  $\alpha_2$ . Thus, the method demonstrated in this section fails to determine  $k$  in dimension two. We address the solution to this problem in the next section.

#### 3.1 Determination of $\sigma$

We construct an appropriate approximate identity. Let  $\psi \in C_0^\infty([0, \infty))$  be such that  $\psi(0) = 1$  and  $\int_0^\infty \psi(t) dt = 1$ ; define  $\psi_\varepsilon(x) = \psi(x/\varepsilon)$ .

**Proposition 3.1.** *The following limit holds in  $L^1(\Gamma_+, d\mu(x, v))$ :*

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_-} \alpha(x, v, x', v') \psi_\varepsilon(d(x', \gamma_{(x,v)}(-\tau_-(x, v)))) d\mu(x', v') = E(x, \gamma_{(x,v)}(-\tau_-(x, v))).$$

*Proof.* When  $\alpha$  is replaced by  $\alpha_0$  the result is immediate. When  $\alpha$  is replaced by  $\alpha_1$ ,

$$\begin{aligned} 0 &\leq \int_{\Gamma_+} \int_{\Gamma_-} \int_0^{\tau_+(x', v')} \int_0^{\tau_-(x, v)} k(z(t), \dot{z}(t), \mathcal{P}(\dot{y}(s); y(s), z(t))) \delta_{\{y(s)\}}(z(t)) \psi_\varepsilon(d(x', \gamma_{(x,v)}(-\tau_-(x, v)))) \\ &\quad \times ds dt d\mu(x', v') d\mu(x, v) \\ &= \int_M \int_{\Omega_M^2} k(y, w', w) \psi_\varepsilon(d(\gamma_{(y,w')}(-\tau_-(y, w')), \gamma_{(y,w)}(\tau_+(y, w)))) dw' dw dy \end{aligned}$$

where  $(y, w') = \vec{\gamma}_{(x', v')}(t)$ ,  $(y, w) = \vec{\gamma}_{(x, v)}(s - \tau_-(x, v))$ . Now there exists constant  $C$  such that

$$\text{supp } \psi_\varepsilon(d(\gamma_{(y,w')}(-\tau_-(y, w')), \gamma_{(y,w)}(\tau_+(y, w)))) \subset W_\varepsilon = \{(y, w', w) \in \Omega^2 M : \|w' - w\|_g < C\varepsilon\}$$

and so

$$0 \leq \int_{\Gamma_+} \int_{\Gamma_-} \alpha_1(x, v, x', v') \psi_\varepsilon(d(x', \gamma_{(x,v)}(-\tau_-(x, v)))) d\mu(x', v') d\mu(x, v) = \int_{W_\varepsilon} k(y, w', w) dw' dw dy \rightarrow 0$$

as  $\varepsilon \rightarrow 0$  since  $k \in L^1(\Omega^2 M)$  and the measure of  $W_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Finally, for  $\alpha_2$

$$\begin{aligned} 0 &\leq \int_{\Gamma_+} \left| \int_{\Gamma_-} \alpha_1(x, v, x', v') \psi_\varepsilon(d(x', \gamma_{(x,v)}(-\tau_-(x, v)))) d\mu(x', v') \right| d\mu(x, v) \\ &\leq \int_{V_\varepsilon} |\alpha_2(x, v, x', v')| d\mu(x', v') d\mu(x, v) \rightarrow 0 \end{aligned}$$

as  $\varepsilon \rightarrow 0$ . Here

$$\text{supp } \psi_\varepsilon(x, v, x', v') \subset V_\varepsilon = \{(x, v, x', v') \in \Gamma_+ \times \Gamma_- : \|w' - w\|_g < C\varepsilon\}$$

and the limit above holds since by Theorem 2.1  $\alpha_2 \in L^1(\Gamma_+ \times \Gamma_-)$  and the measure of  $V_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .  $\square$

Proposition 3.1 enables us to obtain from  $\mathcal{A}$  the integral of  $\sigma$  along the geodesic between any two points in the boundary of  $M$ . That is, we may determine the geodesic X-ray transform of  $\sigma$ . For simple Riemannian manifolds this transform is known to be invertible (see [Sh1]). We have thus proven the following theorem.

**Theorem 3.1.** [M1] *Let  $M \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a bounded domain with smooth boundary and let  $g$  be a known simple Riemannian metric on  $M$ . Let  $0 \leq \sigma(x) \in L^\infty(M)$  depend only on  $x$ , and  $0 \leq k \in L^\infty(\Omega^2 M)$  satisfy (2.4). Then we may determine  $\sigma$  from  $\mathcal{A}$ .*

### 3.2 Determination of $k$

Toward determining the scattering kernel  $k$ , fix  $(y, w, w') \in \Omega^2 M$ ,  $w \neq w'$ . Let  $\exp_{w'} : T_{w'}\Omega_y M \rightarrow \Omega_y M$  be the exponential map of the unit tangent sphere based at  $w' \in \Omega_y M$ . Denote by  $\hat{v}(v) = \exp_{w'}(v)$ , and let  $\mathcal{J}_{(y,w')}(v)$  be the determinant of the Jacobian of this change of variables. Let  $\varphi_1 \in C_0^\infty(\mathbb{R}^n)$  be such that  $0 \leq \varphi_1 \leq 1$ ,  $\varphi_1(0) = 0$ ,  $\varphi_1(\hat{v}) = 0$  for  $|\hat{v}| > \varepsilon_0$  for sufficiently small  $\varepsilon_0$ , and  $\int_{\mathbb{R}^{n-1}} \varphi_1(\hat{v}) d\hat{v} = 1$ . Now define  $\psi_\varepsilon : \Omega_y M \rightarrow \mathbb{R}$  by

$$\psi_\varepsilon(v) = \frac{1}{\varepsilon^{n-1}} \varphi_1\left(\frac{\exp_{w'}^{-1}(v)}{\varepsilon}\right).$$

Note that if  $f : \Omega_y M \rightarrow \mathbb{R}$  is continuous at  $w'$  then

$$\int_{\Omega_y M} f(v) \psi_\varepsilon(v) dv = \int_{\mathbb{R}^{n-1}} f(\exp_{w'}^{-1}(\hat{v})) \frac{1}{\varepsilon^{n-1}} \varphi_1\left(\frac{\hat{v}}{\varepsilon}\right) \mathcal{J}_{(y,w')}(v) d\hat{v} \rightarrow f(\exp_{w'}^{-1}(0)) \mathcal{J}_{(y,w')}(0) = f(w')$$

as  $\varepsilon \rightarrow 0$ .

Define  $y(s) = \gamma_{(y,w)}(s - \tau_-(y, w))$ ,  $-\tau_+(y, w) \leq s \leq \tau_-(y, w)$ . Define  $\beta(s) \in \partial M$  to be the unique point in the boundary for which it holds that  $\gamma_{(\beta(s), \mathcal{P}(w'; y, \beta(s)))}(\cdot)$  contains the point  $y(s)$ , and define  $v(s) \in \Omega_{y(s)} M$  to be its tangent vector there. Since  $w$  and  $w'$  are independent,  $\beta'(0) \neq 0$ . Now let  $\varphi \in C_0^\infty(\mathbb{R})$  be such that  $0 \leq \varphi \leq 1$ ,  $\varphi(0) = 1$ ,  $\int_{\mathbb{R}} \varphi(x) dx = \|\beta'(0)\|_g^{-2}$ ; let

$$\varphi_\eta(x) = \frac{1}{\eta} \varphi\left(\frac{x}{\eta}\right).$$

Define  $h_1 : \partial M \rightarrow \mathbb{R}$  by  $h_1(x') = \langle \exp_{\beta(0)}^{-1}(x'), \beta'(0) \rangle$ . Note that, restricted to the curve  $\beta(s)$ , for sufficiently small  $s$ ,  $h_1(\beta(s)) = 0$  if and only if  $s = 0$ . Furthermore,

$$\left. \frac{d}{ds} \right|_{s=0} h_1(\beta(s)) = \|\beta'(0)\|^2.$$

The construction of  $\beta(s)$  for fixed  $(y, w, w')$  is now repeated for arbitrary  $(z, \xi, \xi') \in \Omega^2 M$  and we denote this by  $\beta_{(z, \xi, \xi')}(s)$ . Define  $h_2 : \partial M \times \Omega^2 M$  by  $h_2(x', z, \xi, \xi') = \text{dist}(x', \beta_{(z, \xi, \xi')}(\cdot))$ . Let  $\mu \in C_0^\infty(\mathbb{R})$  with  $\mu(0) = 1$  and define  $\mu_\delta(t) = \mu(t/\delta)$ . Let

$$W = \{(z, \xi, \xi') \in \Omega^2 M : \xi \neq \xi'\}.$$

**Proposition 3.2.** *If  $n \geq 3$  and  $(y, w, w') \in \Omega^2 M$  with  $w \neq w'$  then*

$$\begin{aligned} \lim_{\eta \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_-} \psi_\varepsilon(\mathcal{P}(v'; x', y)) \varphi_\eta(h_1(x')) \mu_\delta(h_2(x', y, w, w')) \alpha(\vec{\gamma}_{(y,w)}(\tau_+(y, w)), x', v') d\mu(x', v') \\ = E(y, \gamma_{(y,w)}(\tau_+(y, w))) E(y, \gamma_{(y,w')}(-\tau_-(y, w'))) k(y, w', w) \end{aligned}$$

the limit holding in  $L^1(W)$ .

*Proof.* Replacing  $\alpha$  by  $\alpha_0$  and integrating with respect to  $d\mu(x', v')$ , the integrand gets evaluated at  $(x', v') = \vec{\gamma}_{(y,w)}(-\tau_-(y, w))$  and so we obtain a multiple of

$$\psi_\varepsilon(\mathcal{P}(\vec{\gamma}_{(y,w)}(-\tau_-(y, w))); \gamma_{(y,w)}(-\tau_-(y, w)), y) = \psi_\varepsilon(w) = 0$$

for all sufficiently small  $\varepsilon$  since  $w \neq w'$ . Replacing  $\alpha$  by  $\alpha_1$ ,

$$\begin{aligned} I_1 &:= \int_{\Gamma_-} \psi_\varepsilon(\mathcal{P}(v'; x', y)) \varphi_\eta(h_1(x')) \mu_\delta(h_2(x', y, w, w')) \alpha_1(\vec{\gamma}_{(y,w)}(\tau_+(y, w)), x', v') d\mu(x', v') \\ &= \int_0^{\tau(y,w)} \int_{\Omega_{y(s)M}} \psi_\varepsilon(\mathcal{P}(\dot{\gamma}_{(y(s),\hat{v})}(-\tau_-(y(s), \hat{v}))); \gamma_{(y(s),\hat{v})}(-\tau_-(y(s), \hat{v})), y) \\ &\quad \times \varphi_\eta(h_1(\gamma_{(y(s),\hat{v})}(-\tau_-(y(s), \hat{v})))) \mu_\delta(h_2(\gamma_{(y(s),\hat{v})}(-\tau_-(y(s), \hat{v})), y, w, w')) \\ &\quad \times E(y(s), \gamma_{(y,w)}(\tau_+(y, w))) E(y(s), \gamma_{(y(s),\hat{v})}(-\tau_-(y(s), \hat{v}))) k(y(s), \hat{v}, \dot{y}(s)) d\hat{v} ds \end{aligned}$$

Now define  $\tilde{v}(\hat{v}, s) = \mathcal{P}(\dot{\gamma}_{(y(s),\hat{v})}(-\tau_-(y(s), \hat{v}))); \gamma_{(y(s),\hat{v})}(-\tau_-(y(s), \hat{v})), y) \in \Omega_y M$  and denote by  $\frac{d\hat{v}}{d\tilde{v}}$  the change of volume element of this change of variables. We obtain

$$\begin{aligned} I_1 &= \int_0^{\tau(y,w)} \int_{\Omega_y M} \psi_\varepsilon(\tilde{v}) \varphi_\eta(h_1(\gamma_{(y(s),\tilde{v}(\tilde{v},s))}(-\tau_-(y(s), \tilde{v}(\tilde{v}, s)))) \\ &\quad \times \mu_\delta(h_2(\gamma_{(y(s),\tilde{v}(\tilde{v},s))}(-\tau_-(y(s), \tilde{v}(\tilde{v}, s))), y, w, w')) E(\cdot) E(\cdot) k(y(s), \tilde{v}(\tilde{v}, s), \dot{y}(s)) \frac{d\hat{v}}{d\tilde{v}} d\tilde{v} ds \\ &\rightarrow I_2 := \int_0^{\tau(y,w)} \varphi_\eta(h_1(\beta_{(y,w,w')}(s))) \mu_\delta(h_2(\beta_{(y,w,w')}(s), y, w, w')) E(\cdot) E(\cdot) \\ &\quad \times k(y(s), v(s), \dot{y}(s)) \frac{d\hat{v}}{d\tilde{v}} \Big|_{\hat{v}=v(s)} ds \end{aligned}$$

as  $\varepsilon \rightarrow 0$ . Note that  $\mu_\delta(h_2(\beta_{(y,w,w')}(s), y, w, w')) = 1$ . Now define  $\tilde{s}(s) = h_1(\beta_{(y,w,w')}(s))$ . Then  $\tilde{s} = 0$  if and only if  $s = 0$  (for sufficiently small  $s$ ), and  $\frac{d\tilde{s}}{ds} \Big|_{s=0} = \|\beta'_{(y,w,w')}(0)\|^2$ . So, for sufficiently small  $\tilde{s}_0$ ,

$$\begin{aligned} I_2 &= \int_{-\tilde{s}_0}^{\tilde{s}_0} \varphi_\eta(\tilde{s}) E(\cdot) E(\cdot) k(y(s(\tilde{s})), v(s(\tilde{s})), \dot{y}(s(\tilde{s}))) \frac{d\hat{v}}{d\tilde{v}} \Big|_{\hat{v}=v(s)} \frac{ds}{d\tilde{s}} d\tilde{s} \\ &\rightarrow E(y, \gamma_{(y,w)}(\tau_+(y, w))) E(y, \gamma_{(y,w')}(-\tau_-(y, w'))) k(y, w', w) \end{aligned}$$

as  $\eta \rightarrow 0$  since  $\int_{\mathbb{R}} \varphi^2(x) dx = \|\beta'_{(y,w,w')}(0)\|_g^{-2}$ .

Finally, we must show that the integral vanishes when we replace  $\alpha$  by  $\alpha_2$ . Let  $\chi(z, \xi, \xi') \in C_0^\infty(W)$ . Then

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_M \int_{\Omega_2^2 M} \left| \int_{\Gamma_-} \psi_\varepsilon(\mathcal{P}(v'; x', z)) \varphi_\eta(h_1(x')) \mu_\delta(h_2(x', z, \xi, \xi')) \alpha_2(\vec{\gamma}_{(z, \xi)}(\tau_+(z, \xi)), x', v') \chi(z, \xi, \xi') \right. \\ & \quad \times d\mu(x', v') \Big| d\xi' d\xi dz \\ & \leq \frac{1}{\eta} \int_{\Gamma_+} \int_0^{\tau_-(x, v)} \int_{\Omega_{\vec{\gamma}_{(x, v)}(s - \tau_-(x, v))} M} \int_{\partial M} \mu_\delta(h_2(x', \vec{\gamma}_{(x, v)}(s - \tau_-(x, v)), \xi')) \\ & \quad \times |\chi(\vec{\gamma}_{(x, v)}(s - \tau_-(x, v)), \xi')| |\langle \mathcal{P}(\xi'; \vec{\gamma}_{(x, v)}(s - \tau_-(x, v)), x'), \nu_{x'} \rangle| dx' d\xi' ds d\mu(x, v) \\ & \rightarrow 0 \end{aligned}$$

as  $\delta \rightarrow 0$  since the support of  $\mu_\delta$  is a  $(3n - 1)$ -dimensional variety in the  $(4n - 3)$ -dimensional domain of integration, and since  $\alpha_2 \in L^\infty(\Gamma_-; L^1(\Gamma_+, d\mu))$ .  $\square$

Combining Proposition 3.2 with Theorem 3.1 we first determine the function  $E$ , and then obtain  $k$ . We thus have the following.

**Theorem 3.2.** [M1] *Let  $M \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded domain with smooth boundary and let  $g$  be a known simple Riemannian metric on  $M$ . Let  $0 \leq \sigma(x) \in L^\infty(M)$  depend only on  $x$ , and  $0 \leq k \in L^\infty(\Omega^2 M)$  satisfy (2.4). Then we may determine  $k$  from  $\mathcal{A}$ .*

## 4 Full Measurements in Dimension Two

In this section we present the results of [M2] where it is proven that knowledge of the albedo operator on a simple Riemannian surface uniquely determines the unknown metric  $g$ , and the coefficients  $\sigma$  and  $k$ . We shall refer the reader to [M2] for most of the proofs. We impose two additional conditions on the manifold  $(M, g)$ .

**Assumption 4.1.** If the maximal sectional curvature  $\kappa_0$  of  $(M, g)$  is positive then we assume that there are no *focal points*. That is, for every geodesic  $\gamma : [a, b] \rightarrow M$  and every non-zero Jacobi field  $J(t)$  along  $\gamma$  with  $J(a) = 0$  it holds that  $\|J(t)\|$  is strictly increasing on  $[a, b]$ .

**Assumption 4.2.** In the case that  $\kappa_0 > 0$  we assume that the diameter of  $(M, g)$  satisfies  $\text{diam}(M, g) < \pi/(2\sqrt{\kappa_0})$ .

**Theorem 4.1.** [M2] *Let  $(M, g)$  be a two-dimensional Riemannian manifold satisfying assumptions 4.1 and 4.2. Then the albedo operator  $\mathcal{A}$  determines the metric  $g$  up to a diffeomorphic change of coordinates which is the identity on  $\partial M$ .*

*Proof.* From Theorem 3.1 we see that  $\mathcal{A}$  determines the so-called *scattering relation* of  $(M, g)$ , that is, the set  $\{(x, v, \gamma_{(x, v)}(-\tau_-(x, v)), \dot{\gamma}_{(x, v)}(-\tau_-(x, v))\}$ . It is a result of [PU1] (see also [PU2]) that for simple manifolds with no focal points,  $g$  is uniquely determined by this scattering relation.  $\square$

We now present the precise statement for the determination of  $\sigma$  and  $k$ . Let

$$\mathcal{U}_{\Sigma, \varepsilon} = \{(\sigma(x), k(x, w', w)) : \|\sigma\|_{L^\infty} \leq \Sigma, \|k\|_{L^\infty} \leq \varepsilon\}.$$

**Theorem 4.2.** [M2] *Let  $(M, g)$  be a two-dimensional Riemannian manifold satisfying assumptions 1.1 and 4.2. Given  $\Sigma > 0$  there exists  $\varepsilon > 0$  such that any pair  $(\sigma, k) \in \mathcal{U}_{\Sigma, \varepsilon}$  is uniquely determined, within  $(\sigma, k) \in \mathcal{U}_{\Sigma, \varepsilon}$ , by the associated albedo operator  $\mathcal{A}$ . One may take  $\varepsilon = Ce^{-2 \text{diam}(M, g)\Sigma}$  where  $C$  depends only on  $(M, g)$ .*

From Theorem 3.1 we may assume that  $\sigma$  (and hence  $E$ ) is known. The main starting point for determination of  $k$  is a more precise expression for  $\alpha_1$ . Throughout we are restricting to the case  $n = 2$ . We will frequently omit the exact points of evaluation of  $E$  writing instead “ $E(\cdot)$ .” We will not need the omitted information.

**Proposition 4.1.** *The second term  $\alpha_1$  in the series expansion for the kernel of  $\mathcal{A}$  has the expression*

$$\begin{aligned} \alpha_1(x, v, x', v') &= \chi(x, v, x', v') E(\cdot) E(\cdot) \mathcal{J}(x, v, x', v') \\ &\quad \times \frac{k(\vec{\gamma}_{(x', v')}(s(x', v')), \dot{\gamma}_{(x, v)}(t(x', v') - \tau_-(x, v)))}{|\sin \psi|} \end{aligned}$$

where  $\chi = 1$  if the geodesics  $\gamma_{(x', v')}(s(x', v'))$  and  $\gamma_{(x, v)}(t(x', v') - \tau_-(x, v))$  intersect for  $s = s(x', v') > 0$  and  $t = t(x', v') > 0$ , and  $\chi = 0$  otherwise, and where  $\psi$  is the angle between the tangent vectors at this point of intersection. The function  $\mathcal{J}$  is uniformly bounded  $0 < m_1 \leq \mathcal{J} \leq m_2$ .

*Proof.* (Sketch) By definition, if  $\varphi_- \in C_0^\infty(\Gamma_-)$  then

$$KJ\varphi_-(x, v) = \int_0^{\tau_-(x, v)} E(\cdot) \int_{\Omega_{y(t)} M} k(y(t), w, \dot{y}(t)) E(\cdot) \varphi_-(\vec{\gamma}_{(y(t), w)}(-\tau_-(y(t), w))) dw dt$$

where  $y(t) = \gamma_{(x, v)}(t - \tau_-(x, v))$ . We re-write this integral in terms of an integral over  $\Gamma_-$ . Define the family of indicator functions  $\chi : \Omega M \rightarrow \{0, 1\}$ , parameterized by  $(x, v)$ , by

$$\chi(x, v, x', v') = \begin{cases} 1 & \text{if } \gamma_{(x', v')}(s) = \gamma_{(x, v)}(t - \tau_-(x, v)) = y(t) \text{ for some } s > 0, t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

On the support of  $\chi$  we have well-defined functions  $s(x', v')$ ,  $t(x', v')$ ; on this support the change of variables

$$(t, w) = \Phi(x', v') = (t(x', v'), \dot{\gamma}_{(x', v')}(s(x', v')))$$

is well-defined and smooth. In [M2] it is shown that if  $J_\Phi$  is the Jacobian of this change of variables then

$$|\det J_\Phi| = \frac{|(v', \nu_{x'})|}{|\sin \psi|} \mathcal{J}(x, v, x', v')$$

where  $\mathcal{J} \neq 0$  and  $\mathcal{J}$  is bounded as in the statement of the proposition. The expression for  $\alpha_1$  follows immediately.  $\square$

Suppose now that we have two identical manifolds  $(M, g)$  with material parameters  $(\sigma, k)$  and  $(\sigma, \tilde{k})$ . Let  $\mathcal{A}$  and  $\tilde{\mathcal{A}}$  be their respective albedo operators and let  $\alpha_j, \tilde{\alpha}_j$  be the terms in their series expansions. Then it follows that

$$(\alpha_2 - \alpha_1)(x, v, x', v') = \chi(x, v, x', v') E(\cdot) E(\cdot) \mathcal{J}(\cdot) \frac{(\tilde{k} - k)(\tilde{\gamma}_{(x', v')}(s(x', v')), \dot{\gamma}_{(x, v)}(t(x', v') - \tau_-(x, v)))}{|\sin \psi|},$$

and this in turn implies

$$\chi|(\tilde{k} - k)(y, w, \hat{w})| \leq C e^{2 \text{diam}(M, g) \Sigma} \chi |\sin \psi| |(\alpha_2 - \tilde{\alpha}_2)(x, v, x', v')| \quad \text{a.e.}, \quad (4.6)$$

where  $y$  is the point of intersection of the geodesics  $\gamma_{(x', v')}$  and  $\gamma_{(x, v)}$  and  $w, w'$  are their tangent vectors there. In what follows we outline the proof of another estimate of  $\alpha_2 - \tilde{\alpha}_2$  of the form

$$\|\sin \psi (\alpha_2 - \tilde{\alpha}_2)\|_{L^\infty} \leq C \varepsilon \|k - \tilde{k}\|_{L^\infty}. \quad (4.7)$$

Once this is established, we combine it with (4.6) to obtain

$$\|k - \tilde{k}\|_{L^\infty} \leq C \varepsilon \|k - \tilde{k}\|_{L^\infty}$$

so that for sufficiently small  $\varepsilon$  we have  $k = \tilde{k}$ . This completes the proof of Theorem 4.2.

Toward proving (4.7) let  $\varphi_-$  be the Dirac delta distribution on  $\Gamma_-$  with respect to the measure  $d\mu(x', v')$ :

$$\varphi_-(x', v') = \frac{1}{|\langle v', \nu_{x'} \rangle|} \delta_{(x_0, v_0)}(x', v').$$

Then

$$\begin{aligned} \alpha_2 - \tilde{\alpha}_2 &= (I - K)^{-1} K^2 J \varphi_- - (I - \tilde{K})^{-1} \tilde{K}^2 J \varphi_- \\ &= (I - K)^{-1} [K(K - \tilde{K}) + (K - \tilde{K})\tilde{K}] J \varphi_- + (I - \tilde{K})^{-1} (K - \tilde{K}) (I - K)^{-1} \tilde{K}^2 J \varphi_-. \end{aligned} \quad (4.8)$$

From this we see that we must obtain estimates for  $K^2$  and  $K^3$ . These are obtained in [M2] and we present them here without proof.

**Proposition 4.2.** *For almost every  $(x, v, x'_0, v'_0) \in \Gamma_+ \times \Gamma_-$  such that there exist  $s > 0, t > 0$  with  $\gamma_{(x, v)}(s - \tau_-(x, v)) = \gamma_{(x'_0, v'_0)}(t)$*

$$|\tilde{K} K J \varphi_-(x, v, x'_0, v'_0)| \leq \begin{cases} \frac{C \|\tilde{k}\|_{L^\infty} \|k\|_{L^\infty}}{\cos(\sqrt{\kappa_0} \text{diam}(M, g))} \left(1 + \log \frac{1}{|\sin \psi|}\right) & \kappa_0 > 0, \\ C \|\tilde{k}\|_{L^\infty} \|k\|_{L^\infty} \left(1 + \log \frac{1}{|\sin \psi|}\right) & \kappa_0 \leq 0, \end{cases}$$

where  $\kappa_0$  is the maximal sectional curvature of  $(M, g)$  and where  $\psi$  is the angle between the tangent vectors of the intersecting geodesics at the point of intersection.

Although the proof of this proposition is not presented here, we point out that a recurring theme in the proof is comparison of geodesic triangles in  $(M, g)$  to triangles with (for example) the same side-angle-side property on either the sphere or the hyperbolic plane of constant curvature  $\kappa_0$ . Such constant curvature manifolds represent the “worst case” scenario with regard to how intersecting geodesics might get close to intersecting a second time. The final estimate needed is the following.

**Proposition 4.3.** *It holds that  $K^3 J\varphi_- \in L^\infty(\Gamma_+ \times \Gamma_-)$  with norm bounded by  $C\|k\|_{L^\infty}^3$ .*

We now complete the proof of (4.7) via (4.8).

**Lemma 4.1.** *For almost every  $(x, v, x'_0, v'_0) \in \Gamma_+ \times \Gamma_-$  such that  $\gamma_{(x,v)}$  and  $\gamma_{(x'_0,v'_0)}$  we have*

$$|(I - K)^{-1}(K^2 - \tilde{K}^2)J\varphi_-(x, v, x'_0, v'_0)| \leq C\|k - \tilde{k}\|_{L^\infty}(\|k\|_{L^\infty} + \|\tilde{k}\|_{L^\infty})\left(1 + \log \frac{1}{|\sin \psi|}\right)$$

and

$$|(I - \tilde{K})^{-1}(K - \tilde{K})(I - K)^{-1}\tilde{K}^2 J\varphi_-(x, v, x'_0, v'_0)| \leq C\|k - \tilde{k}\|_{L^\infty}\|\tilde{k}\|_{L^\infty}^2(1 + \|k\|_{L^\infty}).$$

*Proof.* For the first estimate, we re-write

$$\begin{aligned} (I - K)^{-1}[K(K - \tilde{K}) + (K - \tilde{K})\tilde{K}]J\varphi_- &= (I + (I - K)^{-1}K)[K(K - \tilde{K}) + (K - \tilde{K})\tilde{K}]J\varphi_- \\ &= [K(K - \tilde{K}) + (K - \tilde{K})\tilde{K}]J\varphi_- + (I - K)^{-1}[K^2(K - \tilde{K}) + K(K - \tilde{K})\tilde{K}]J\varphi_-. \end{aligned}$$

The contribution from the first term is estimated by Proposition 4.2 (with  $K$  or  $\tilde{K}$  replaced by  $K - \tilde{K}$ ). For the final term,  $K^2(K - \tilde{K}) + K(K - \tilde{K})\tilde{K}J\varphi_-$  is an  $L^\infty$  function by Proposition 4.3, and  $(I - K)^{-1}$  preserves this space.

For the second estimate, express  $(I - K)^{-1}\tilde{K}^2 = \tilde{K}^2 + (I - K)^{-1}K\tilde{K}^2$  and apply Proposition 4.3.  $\square$

Combining the estimates of Lemma 4.1 with (4.8) we obtain

$$C\chi|\sin \psi| |(\alpha_2 - \tilde{\alpha}_2)(x, v, x', v')| \leq C'\varepsilon\|k - \tilde{k}\|_{L^\infty},$$

and this together with (4.6) yields

$$\|k - \tilde{k}\|_{L^\infty(\Omega^2 M)} \leq C\varepsilon\|k - \tilde{k}\|_{L^\infty(\Omega^2 M)}. \quad (4.9)$$

Thus, for sufficiently small  $\varepsilon$ , it must hold that  $k = \tilde{k}$ . Although it has not been carefully tracked in this article, in [M2] it is shown that  $\varepsilon$  can be taken to be  $\varepsilon = Ce^{-2\text{diam}(M,g)\Sigma}$  with  $C$  depending only on  $(M, g)$ .

## 5 Angularly Averaged Measurements

We now consider the problem of parameter determination with less information. The results presented here appear in [LM]. Instead of knowing the angularly resolved measurement  $u(x, v)$  on  $\Gamma_+$  we will assume only knowledge of an *average* over outgoing directions of  $u$  at  $x \in \partial M$ . This is motivated in part by the fact that in practice it is very difficult and perhaps impossible to measure the angular resolution of exiting photons. To be more precise, for  $x \in \partial M$ , define

$$\Omega_x^\pm M = \{v \in \Omega_x M : \pm \langle v, \nu_x \rangle > 0\}$$

(and so  $\Gamma_\pm$  are the disjoint unions of the  $\Omega_x^\pm M$  over  $x \in \partial M$ ).

**Definition 5.1.** The data of which we will be assuming knowledge consists of *angularly averaged* measurements on  $\partial M$ , weighted with respect to a prescribed function  $m(x, v)$ . Specifically, given  $u_-$  on  $\Gamma_-$  and  $u = T^{-1}u_-$  we define  $\mathcal{M} : L^1(\Gamma_-, d\mu) \rightarrow L^1(\partial M)$  by

$$\mathcal{M}u_-(x) := \int_{\Omega_x^+ M} u(x, v)m(x, v) dv.$$

We require that  $m \in L^\infty(\Gamma_+)$ , that  $m$  does not vanish on  $\Gamma_+$ , and that  $|m(x, v)/\langle v, \nu_x \rangle|$  be bounded on  $\Gamma_+$ .

The function  $m$  corresponds to the limitations of the measurement apparatus. It may represent a limited aperture or, for example, when  $m(x, v) = \langle v, \nu_x \rangle$ , the measurement is power flux exiting the boundary.

We assume here that the metric  $g$  is known. With this limited data, we are still able to determine the extinction coefficient  $\sigma$ , but are only able to determine scattering kernels of a more restricted class. We assume that  $k$  is of the form  $k(x)\Theta(x, w, w')$  where  $\Theta$  is assumed to be *known*. This corresponds to knowing the angular behavior of the scattering events, but not the density of scatterers (which is quantified by  $k(x)$ ). There are also assumptions of analyticity of various coefficients. To be precise we present the main theorems.

If  $\mathcal{H} \subset \Gamma_-$ , then by  $\Gamma(\mathcal{H}) = \{\gamma_{(x', v')}(t) : (x', v') \in \mathcal{H}, 0 \leq t \leq \tau_+(x', v')\}$  we mean the set of geodesics with initial data in  $\mathcal{H}$ . With constants  $C_{\kappa_m}$  and  $C_{\kappa_M}$  to be defined later, we have the following.

**Theorem 5.1.** [LM] Suppose that

$$\|k\|_{L^\infty} < \min \{[(C_{\kappa_m} C_{\kappa_M})^{n-1} \text{diam } M |\mathbb{S}^{n-1}|]^{-1}, [\text{diam } M |\mathbb{S}^{n-1}|]^{-1}\},$$

and that  $\mathcal{H}_\sigma \subset \Gamma_-$  is open and such that the geodesic X-ray transform restricted to  $\Gamma(\mathcal{H}_\sigma)$  is injective. Suppose that  $\sigma = \sigma(x)$  depends on position only. Then  $\sigma$  is uniquely determined by  $\{(u_-, \mathcal{M}u_-) : u_- \in L^1(\mathcal{H}_\sigma)\}$ .

Fixing  $D > 0$  and making the definition  $\mathcal{K}_\varepsilon^D := \{k \in L^\infty(M) : \text{dist}(\text{supp}(k), \partial M) > D, \|k\|_{L^\infty} \leq \varepsilon\}$  we also have the following uniqueness result for  $k$ .



**Theorem 5.2.** *Suppose the scattering kernel has the form  $k(x)\Theta(x, v', v)$ , with  $(g, m, \sigma, \Theta)$  known and real analytic, and with both  $m$  and  $\Theta$  non-vanishing. Let  $\mathcal{H}_k \subset \Gamma_-$  be open, and assume that for every  $(x, v) \in TM \setminus \{0\}$ , there exists  $\gamma \in \Gamma(\mathcal{H}_k)$  through  $x$  and normal to  $v$  at  $x$ . Then there exists  $\varepsilon$  sufficiently small such that for a.e.  $x \in \partial M$ , knowledge of  $\{(u_-, \mathcal{M}u_-(x)) : u_- \in L^1(\mathcal{H}_k)\}$  uniquely determines  $k$  within the class  $\mathcal{K}_\varepsilon^D$ .*

*Furthermore,  $\varepsilon$  may be chosen such that this result holds in some  $C^2$  neighborhood of  $(m, \sigma)$ , and some  $C^{3n}$  neighborhood of  $(\Theta, g)$ .*

We begin by deriving an expression for the averaged albedo operator  $\mathcal{M}$  as an infinite sum of the operators  $\mathcal{M}_i$  (see Definition 5.2), and where we give expressions for the Schwartz kernel of  $\mathcal{M}_i$ ,  $i \geq 1$ . In order to simplify the presentation of what follows, we introduce some notation. If  $x_1, \dots, x_j$  are points in  $M$ , then

$$E(x_1, x_2, \dots, x_j) := E(x_1, x_2)E(x_2, x_3) \cdots E(x_{j-1}, x_j) = \prod_{i=1}^{j-1} E(x_i, x_{i+1}). \quad (5.10)$$

If  $y \in M$ , consider  $z = z_y(t, v) = \gamma_{(y, v)}(t)$  defined from the ‘‘polar’’ coordinates  $(t, v) \in \mathbb{R} \times \Omega_y M$ . Let  $J_y(z)$  denote the Jacobian determinant  $|\det \partial z / \partial(t, v)|^{-1}$  of this change of variables. For a given  $z$ , let  $(t, v)$  be its polar coordinate expression, and let  $\{v^1, \dots, v^n\}$  be an orthonormal basis for  $T_y M$  with  $v^1 = v$ . Let  $Y_{y, z, i}$  be the Jacobi field along  $\gamma_{(y, v)}(\cdot)$  with  $Y_{y, z, i}(0) = 0$ ,  $\dot{Y}_{y, z, i} = v^i$ ,  $2 \leq i \leq n$ . Then  $J_y(z)$  is given by the expression

$$J_y(z) = \prod_{i=2}^n |Y_{y, z, i}(d(y, z))|^{-1}. \quad (5.11)$$

Analogous to (5.10) we introduce the notation

$$\mathcal{J}(y_1, \dots, y_j) := \prod_{i=2}^j J_{y_i}(y_{i-1}). \quad (5.12)$$

By considering comparison theorems for Jacobi fields, one easily obtains the following.

**Lemma 5.1.** *With  $J_y(z)$  defined as above, there exists a constant  $C_{\kappa_M}$ , such that for all  $y, z \in M$*

$$J_y(z) \leq \frac{C_{\kappa_M}}{d(y, z)^{n-1}}. \quad (5.13)$$

*Further, there exists a constant  $C_{\kappa_m}$  such that if  $Y$  is any Jacobi field along a geodesic  $\gamma(t)$  with  $Y(0) = 0$  and  $\dot{Y}(0) \in \dot{\gamma}^\perp(0) \subset \Omega_{\gamma(0)} M$ , then*

$$\frac{|Y(t)|}{t} \leq C_{\kappa_m} \quad (5.14)$$

*for all  $0 \leq t \leq \tau_+(\gamma(0), \dot{\gamma}(0))$ .*

The notation  $C_{\kappa_M}$  and  $C_{\kappa_m}$  is chosen because these constants are determined in terms of minimal and maximal sectional curvatures of  $(M, g)$ .

**Definition 5.2.** For each  $i \geq 0$  we define the operator  $\mathcal{M}_i$  by

$$\mathcal{M}_i u_-(x) := \int_{\Omega_x^+ M} K^i J u_-(x, v) m(x, v) dv.$$

**Theorem 5.3.** [LM] Let  $\|k\|_{L^\infty(\Omega^2 M)} < (|\mathbb{S}^{n-1}| \text{diam } M)^{-1}$ . Then  $\mathcal{M}_i, \mathcal{M} : L^1(\Gamma_-, d\mu) \rightarrow L^1(\partial M)$  continuously, and for almost every  $x \in \partial M$ ,  $\mathcal{M}$  has the expansion

$$\mathcal{M} u_-(x) = \sum_{j=0}^{\infty} \mathcal{M}_j u_-(x) = \sum_{j=0}^{\infty} \int_{\Gamma_-} \alpha_j(x, x', v') u_-(x', v') d\mu(x', v')$$

where  $\alpha_0(x, \cdot, \cdot)$  is a distribution supported on a manifold of dimension  $n - 1$ , and for  $j \geq 1$ ,  $\alpha_j(x, x', v')$  are the following Schwartz kernels: when  $j = 1$ ,

$$\alpha_1(x, x', v') = \int_0^{\tau_+(x', v')} k(\vec{\gamma}_{(x', v')}(t), \bar{w}_1) E(x', \gamma_{(x', v')}(t), x) m(x, \hat{w}_1) J_x(\gamma_{(x', v')}(t)) dt \quad (5.15)$$

where  $\bar{w}_1, \hat{w}_1$  are the initial and final tangent vectors of the geodesic from  $\gamma_{(x', v')}(t)$  to  $x$ , and  $E$  and  $J_x$  are defined in (5.10), and (5.11); for  $j \geq 2$ ,

$$\alpha_j(x, x', v') = \int_0^{\tau_+(x', v')} \int_M \cdots \int_M k(\vec{\gamma}_{(x', v')}(t), \bar{w}_1) \prod_{i=2}^j k(y_i, \hat{w}_{i-1}, \bar{w}_i) E(x', \gamma_{(x', v')}(t), y_2, \dots, y_j, x) \mathcal{J} m(x, \hat{w}_j) dy_j \cdots dy_2 dt \quad (5.16)$$

where:

- $\bar{w}_1, \hat{w}_1$  are the initial and final tangent vectors of the geodesic joining  $\gamma_{(x', v')}(t)$  to  $y_2$ ,
- $\bar{w}_i, \hat{w}_i$  are the initial and final tangent vectors of the geodesic joining  $y_i$  to  $y_{i+1}$  for  $i = 2, \dots, j - 1$ ,
- $\bar{w}_j, \hat{w}_j$  are the initial and final tangent vectors of the geodesic joining  $y_j$  to  $x$ , and
- $\mathcal{J} = \mathcal{J}(\gamma_{(x', v')}(t), y_2, \dots, y_j, x)$ .

*Proof.* We refer the reader to [LM] for the full proof of this theorem. To give a flavor of the proof,

we present the derivation of  $\mathcal{M}_1$  here. Let  $\phi_-$  be a function on  $\Gamma_-$ . We have

$$\begin{aligned}
 \mathcal{M}_1\phi_-(x) &= \int_{\Omega_x^+ M} KJ\phi_-(x, v) dv \\
 &= \int_{\Omega_x^+ M} \int_0^{\tau_-(x, v)} E(x, \gamma_{(x, v)}(t - \tau_-(x, v))) T_1 J\phi_-(\vec{\gamma}_{(x, v)}(t - \tau_-(x, v))) dt m(x, v) dv \\
 &= \int_{\Omega_x^+ M} \int_0^{\tau_-(x, v)} E(x, \gamma_{(x, v)}(t - \tau_-(x, v))) \\
 &\quad \times \int_{\Omega_y M} k(y, \hat{w}, \bar{w}_1) E(\gamma_{(x, v)}(t - \tau_-(x, v)), \gamma_{(y, \hat{w})}(-\tau_-(y, \hat{w}))) \\
 &\quad \times \phi_-(\vec{\gamma}_{(y, \hat{w})}(-\tau_-(y, \hat{w}))) d\hat{w} dt m(x, v) dv
 \end{aligned}$$

where  $(y, \bar{w}_1) = \vec{\gamma}_{(x, v)}(t - \tau_-(x, v))$

$$\begin{aligned}
 &= \int_M \int_{\Omega_y M} E(x, y, \gamma_{(y, \hat{w})}(-\tau_-(y, \hat{w}))) k(y, \hat{w}, \bar{w}_1) \phi_-(\vec{\gamma}_{(y, \hat{w})}(-\tau_-(y, \hat{w}))) d\hat{w} \\
 &\quad \times m(x, \hat{w}_1) J_x(y) dy
 \end{aligned}$$

where  $\bar{w}_1, \hat{w}_1$  are the initial and final tangent vectors (respectively) of the geodesic from  $y$  to  $x$ ,

$$\begin{aligned}
 &= \int_{\Gamma_-} \int_0^{\tau_+(x', v')} k(\vec{\gamma}_{(x', v')}(t), \bar{w}_1) E(x', \gamma_{(x', v')}(t), x) m(x, \hat{w}_1) J_x(\gamma_{(x', v')}(t)) dt \\
 &\quad \times \phi_-(x', v') d\mu(x', v').
 \end{aligned}$$

This proves (5.15).  $\square$

As seen in the (partial) proof of Theorem 5.3, there appear weakly singular integrals due to the presence of the functions  $J_x$  (see also (5.13)).

**Definition 5.3.** Let  $\mathcal{T}$  be the operator with kernel  $d(x, \cdot)^{n-1}$ , that is

$$\mathcal{T}f(x) := \int_M \frac{f(y)}{d(x, y)^{n-1}} dy. \tag{5.17}$$

From (5.13) one has  $\mathcal{T} : L^p(M) \rightarrow L^p(M)$  continuously,  $1 \leq p \leq \infty$  with  $\|\mathcal{T}\| \leq (C_{\kappa_m})^{n-1} |\mathbb{S}^{n-1}| \text{diam } M$  (see [T], Prop. 5.1, Appendix A).

We also define the analogous operator  $\tilde{\mathcal{T}}$ ,

$$\tilde{\mathcal{T}}f(x) := \int_M f(y) J_y(x) dy \tag{5.18}$$

for which it holds  $\tilde{\mathcal{T}} : L^p(M) \rightarrow L^p(M)$ ,  $1 \leq p \leq \infty$ , with  $\|\tilde{\mathcal{T}}\| \leq |\mathbb{S}^{n-1}| \text{diam } M$ .

We have seen that  $\alpha_0$  is more singular than the remaining  $\alpha_j$ . In the following proposition we estimate the contribution from the terms for  $j \geq 1$ .

**Proposition 5.1.** *Let  $p = (n - \mu)/(n - 1)$ ,  $0 < \mu < 1$ ,  $q = (1 - 1/p)^{-1}$ . With  $C_{\kappa_m}, C_{\kappa_M}$  defined as in Lemma 5.1, let  $\|k\|_{L^\infty(\Omega^2 M)} < [(C_{\kappa_m} C_{\kappa_M})^{n-1} \text{diam } M |\mathbb{S}^{n-1}|]^{-1}$ . Then for almost every  $x \in \partial M$ ,*

$$\left| \sum_{j=1}^{\infty} \int_{\Gamma_-} \alpha_j(x, x', v') f(x', v') d\mu(x', v') \right| \leq C_0 \|f\|_{L^q(\Gamma_-, d\mu)} \quad (5.19)$$

where  $C_0 > 0$  depends on  $\kappa_m, \kappa_M, \|k\|_{L^\infty(\Omega^2 M)}, \|m\|_{L^\infty(\Omega M)}$  and  $\text{diam } M$ .

*Proof.* When  $j = 1$ ,

$$\begin{aligned} & \left| \int_{\Gamma_-} \alpha_1(x, x', v') f(x', v') d\mu(x', v') \right| \\ &= \left| \int_{\Gamma_-} \int_0^{\tau_+(x', v')} k(\tilde{\gamma}_{(x', v')}(t), \bar{w}) E(\cdot) m(x, \hat{w}_1) J_x(\gamma_{(x', v')}(t)) dt f(x', v') d\mu(x', v') \right| \\ &\leq \|k\|_\infty \|m\|_\infty (C_{\kappa_M})^{n-1} \int_M \int_{\Omega_M} \frac{|f(\tilde{\gamma}_{(y, v)}(-\tau_-(y, v)))|}{d(x, y)^{n-1}} dv dy \quad \text{by (5.13)} \\ &\leq (C_{\kappa_M})^{n-1} \|k\|_\infty \|m\|_\infty \left( \int_{\Omega_M} \frac{1}{d(x, y)^{(n-1)p}} dv dy \right)^{\frac{1}{p}} \\ &\quad \times \left( \int_{\Omega_M} |f(\tilde{\gamma}_{(y, v)}(-\tau_-(y, v)))|^q dv dy \right)^{\frac{1}{q}} \quad \text{by Hölder's inequality} \\ &\leq (C_{\kappa_M})^{n-1} \|k\|_\infty \|m\|_\infty |\mathbb{S}^{n-1}|^{1/p} C_d^{1/p} (\text{diam } M)^{1/q} \|f\|_{L^q(\Gamma_-, d\mu)} \quad \text{see (5.17)} \\ &= C \|k\|_\infty \|m\|_\infty (\text{diam } M)^{1/q} \|f\|_{L^q(\Gamma_-, d\mu)}, \text{ say.} \end{aligned}$$

The computation for the terms  $j \geq 2$  are slightly different from the above. We refer the reader to [LM] for the details, where it is also proven that the infinite series converges in  $L^1$ .  $\square$

## 5.1 Determination of $\sigma$

The determination of  $\sigma$  is attained via a limiting argument exactly as in Proposition 3.1. The construction of the approximate identity is slightly different but the spirit is the same. The details are in [LM]. Let  $\mathcal{H}_\sigma \subset \Gamma_-$  be such that the geodesic X-ray transform restricted to geodesics in  $\Gamma(\mathcal{H}_\sigma)$  is invertible.

**Theorem 5.4.** [LM] *Let  $(\sigma, k, m)$  satisfy the hypothesis of Theorem 5.1. For almost every  $(x^*, v^*) \in \mathcal{H}_\sigma$*

$$\lim_{\eta \rightarrow 0} \mathcal{M}f_\eta(x^*) = E(\gamma_{(x^*, v^*)}(-\tau_-(x^*, v^*)), x^*) m(x^*, v^*)$$

where  $f_\eta$  concentrates to  $(x', v') = \tilde{\gamma}_{(x^*, v^*)}(-\tau_-(x^*, v^*))$  as  $\eta \rightarrow 0$ .

Since  $m$  is known we thus know the integrals of  $\sigma$  along every geodesic in  $\Gamma(\mathcal{H}_\sigma)$  and hence  $\sigma$  is uniquely determined.

## 5.2 Determination of $k$

Throughout this section, the measurement point  $x$  will be fixed. As mentioned earlier we assume that the scattering kernel is of the form  $k(x)\Theta(x, v', v)$ , where  $\Theta(x, v', v)$  is *a-priori* known. We prove in this setting that the spatial distribution  $k(x)$  is uniquely determined by the averaged albedo operator  $\mathcal{M}$  from measurements at the single point  $x$ .

**Definition 5.4.** Given a complete Riemannian manifold  $(M, g)$  with geodesics  $\gamma_{(x,v)}(t)$ , and functions  $\eta : \Omega M \rightarrow \mathbb{R}$ , and  $\beta \in C^\infty(\Gamma_-)$  we may define the *weighted geodesic transform* by, for  $f : M \rightarrow \mathbb{R}$ ,

$$I_{\eta,\beta}f(x', v') := \beta(x', v') \int_0^{\tau_+(x', v')} f(\gamma_{(x', v')}(t)) \eta(\vec{\gamma}_{(x', v')}(t)) dt. \quad (5.20)$$

We also have the  $L^2$  adjoint, for  $f : \Gamma_- \rightarrow \mathbb{R}$ ,

$$I_{\eta,\beta}^*f(x) = \int_{\Omega_x M} f(\vec{\gamma}_{(x,v)}(-\tau_-(x, v))) \beta(\vec{\gamma}_{(x,v)}(-\tau_-(x, v))) \eta(x, v) dv. \quad (5.21)$$

Let  $\chi \in C_0^\infty(M)$  with  $\chi \equiv 1$  on a neighborhood of  $\{y \in M : \text{dist}(y, \partial M) \geq D\}$ . If  $x_1 \in M$ , let  $\bar{v} = \bar{v}(x_1) \in \Omega_{x_1} M$  and  $v = v(x_1) \in \Omega_x^+ M$  be the initial and final tangent vectors, respectively, of the geodesic joining  $x_1$  to  $x$ . Then for  $v_1 \in \Omega_{x_1} M$  define the weight function

$$w(x_1, v_1) := \Theta(x_1, v_1, \bar{v}) E(\gamma_{(x_1, v_1)}(-\tau_-(x_1, v_1)), x_1, x) m(x, v) J_x(x_1) \chi(x_1). \quad (5.22)$$

From (5.15) and (5.20) we see that  $\alpha_1(x, x', v') = I_{w,1}k(x', v')$ . If  $g, \Theta, \sigma$  and  $m$  are real-analytic, then  $w$  is a real-analytic, non-vanishing weight function in  $\{y \in M : \text{dist}(y, \partial M) \geq D\}$ . It is injectivity of this weighted ray transform which will allow determination of  $k(x)$ . We make use of injectivity results of [FSU]. These results allow a possibly proper subsets of geodesics on which the transform is known. It is the inclusion of the factor  $\beta$  in definition (5.20) that restricts the transform to a such a subset geodesics.

**Definition 5.5.** We say that  $\Gamma$  is a *regular* family of curves (for the metric  $g$ ) if for any  $(x, v) \in T^*M \setminus \{0\}$  there exists  $\gamma \in \Gamma$  through  $x$ , normal to  $v$ , and such that  $\gamma$  has no conjugate points (automatically satisfied for simple metrics).

We say that  $\beta \in C^\infty(\Gamma_-)$  is *regular* if there exists a set  $\mathcal{H} \subset \{(x', v') : \beta(x', v') \neq 0\}$  such that  $\Gamma(\mathcal{H})$  is a regular family.

Suppose that  $k(x)\Theta(x, v, v')$  and  $\tilde{k}(x)\Theta(x, v, v')$  are two scattering kernels with  $k, \tilde{k} \in \mathcal{K}_\varepsilon^D$ ; let  $\mathcal{M}, \tilde{\mathcal{M}}$ , and  $\alpha, \tilde{\alpha}$  be the averaged albedo operators and Schwarz kernels associated to  $k$  and  $\tilde{k}$  respectively. We set  $\Delta k = k - \tilde{k}$  and  $\Delta \alpha_j = \alpha_j - \tilde{\alpha}_j$ ,  $j = 1, 2, \dots$

**Lemma 5.2.** [FSU] *Let  $(m, \Theta, \sigma, g)$  be fixed and real analytic, and suppose  $\beta \in C^\infty(\Gamma_-)$  is a regular function for  $g$ . Then there exists  $C > 0$ , independent of  $k, \tilde{k} \in \mathcal{K}_\varepsilon^D$  such that*

$$\|\Delta k\|_{L^2(M)} \leq C \|I_{w,\beta}^* I_{w,\beta} \Delta k\|_{H^1(M)}, \quad (5.23)$$

*with the above estimate holding in a  $C^2$  neighborhood of  $(m, \Theta, \sigma)$ , and a  $C^3$  neighborhood of  $g$ .*

**Proposition 5.2.** *Let  $(m, \Theta, \sigma, g)$  be fixed and real analytic, and suppose  $\beta \in C^\infty(\Gamma_-)$  is a regular function for  $g$ . Furthermore, suppose  $\mathcal{H}_k \subset \{(x', v') : \beta(x', v') \neq 0\}$  is such that  $\Gamma(\mathcal{H}_k)$  is a regular set of geodesics. Suppose that  $\mathcal{M} = \widetilde{\mathcal{M}}$  on  $L^1(\mathcal{H}_k, d\mu)$ . Then there exists  $C > 0$  such that for all  $k, \tilde{k} \in \mathcal{K}_\varepsilon^D$*

$$\|\Delta k\|_{L^2(M)} \leq C \|I_{w,\beta}^* I_{w,\beta} \Delta k\|_{H^1(M)} = C \|I_{w,\beta}^* \beta \Delta \alpha_1\|_{H^1(M)} = C \left\| I_{w,\beta}^* \beta \sum_{j=2}^{\infty} \Delta \alpha_j \right\|_{H^1(M)},$$

with the above estimate holding in a  $C^2$  neighborhood of  $(m, \sigma, \Theta)$  and a  $C^3$  neighborhood of  $g$ .

*Proof.* The first inequality is (5.23); next,  $\beta I_{w,1} = I_{w,\beta}$  and  $w$  has been defined so that  $I_{w,1} \Delta k = \Delta \alpha_1$ ; finally, since  $\sigma = \tilde{\sigma}$ ,  $\alpha_0 = \tilde{\alpha}_0$  and so  $\mathcal{M} = \widetilde{\mathcal{M}}$  implies that  $\sum_{j=1}^{\infty} \Delta \alpha_j = 0$  which proves the final equality.  $\square$

Analogous to (4.9) we proceed to prove that  $\|\Delta k\|_{L^2(M)} \leq \varepsilon C \|\Delta k\|_{L^2(M)}$ , and so for sufficiently small  $\varepsilon > 0$ , we will have  $\Delta k = 0$ . The following proposition is an extension of Proposition 4 of [FSU] to more general weight functions  $\eta_j$  and restrictions  $\beta_j$ . The proof is essentially the same and further details are given in [LM].

**Proposition 5.3.** *There is  $C > 0$  such that for all  $f \in L^2(M)$  with  $\text{supp } f \subset \{y \in M : d(x, \partial M) > D\}$ ,*

$$\|I_{\eta_1, \beta_1}^* I_{\eta_2, \beta_2} f\|_{H^1(M)} \leq C \|\eta_1\|_{C^2(\Omega M)} \|\eta_2\|_{C^2(\Omega M)} \|\beta_1 \beta_2\|_{C^2(\Gamma_-)} \|f\|_{L^2(M)}, \quad (5.24)$$

with  $C$  depending continuously on the  $C^4$  norm of  $g$ .

We wish to apply (5.24) to each  $\Delta \alpha_i$ ,  $i > 1$ , and to do so we express each as a sum of weighted X-ray transforms. This is achieved by expanding one instance of the kernel  $\Theta$  occurring in  $\Delta \alpha_i$  in a manner based on spherical harmonic expansions of functions on  $\mathbb{S}^{n-1}$ . This expansion is the content of Lemma 5.3. The proof is postponed to the end of this section.

**Lemma 5.3.** *Let  $\Theta(x, v', v) \in C^{3n}(\Omega^2 M)$  and  $g \in C^{3n}(M)$ . There exist  $\Theta_j(x, v')$ ,  $\varphi_j(x, v) \in C^{3n}(\Omega M)$  such that*

$$\Theta(x, v', v) = \sum_{j=1}^{\infty} \Theta_j(x, v') \varphi_j(x, v)$$

with

$$\|\Theta_j\|_{C^2(\Omega M)} \leq \frac{C}{1+j^2}, \quad \text{and} \quad \|\varphi_j(x, v)\|_{L^\infty(\Omega M)} \leq 1,$$

the above estimate holding in a  $C^{3n}$  neighborhood of  $(\Theta, g)$ .

**Proposition 5.4.** Fix  $(m, g, \Theta, \sigma)$  with  $m, \sigma \in C^2$ , and  $g, \Theta \in C^{3n}$ . Suppose that  $\|k\|_\infty, \|\tilde{k}\|_\infty < [ \|\Theta\|_\infty \text{diam } M |\mathbb{S}^{n-1}| ]^{-1}$ ,  $k, \tilde{k} \in \mathcal{K}_\varepsilon^D$ , and  $\beta \in C^\infty(\Gamma_-)$ . Then there is  $C > 0$  such that

$$\left\| I_{w, \beta}^* \sum_{i=2}^{\infty} \Delta \alpha_i \right\|_{H^1(M)} \leq C \varepsilon \|\Delta k\|_{L^2(M)},$$

with the above estimate holding in a  $C^2$  neighborhood of  $(m, \sigma)$ , and a  $C^{3n}$  neighborhood of  $(\Theta, g)$ .

*Proof.* For  $(x, v) \in \Omega M$  we define  $\mathcal{E}(x, v) = E(\gamma_{(x, v)}(-\tau_-(x, v)), x)$ . From Theorem 5.3,

$$\begin{aligned} \Delta \alpha_2(x, x', v') &= \int_0^{\tau_+(x', v')} E(x', \gamma_{(x', v')}(t)) \int_M [\Delta k(\gamma_{(x', v')}(t))k(y_2) - \tilde{k}(\gamma_{(x', v')}(t))\Delta k(y_2)] \\ &\quad \times \Theta(\tilde{\gamma}_{(x', v')}(t), \bar{w}_1) \Theta(y_2, \hat{w}_1, \bar{w}_2) E(\gamma_{(x', v')}(t), y_2) E(y_2, x) \mathcal{J}(\gamma_{(x', v')}(t), y_2, x) dy_2 dt \end{aligned}$$

and from Lemma 5.3 we have (formally at least),

$$\begin{aligned} \Delta \alpha_2(x, x', v') &= \sum_{l=1}^{\infty} \int_0^{\tau_+(x', v')} \mathcal{E}(\tilde{\gamma}_{(x', v')}(t)) \Theta_l(\tilde{\gamma}_{(x', v')}(t)) \\ &\quad \times \int_M [\Delta k(\gamma_{(x', v')}(t))k(y_2) + \tilde{k}(\gamma_{(x', v')}(t))\Delta k(y_2)] \phi_l(\gamma_{(x', v')}(t), \bar{w}_1) \Theta(y_2, \hat{w}_1, \bar{w}_2) \\ &\quad \times E(\gamma_{(x', v')}(t), y_2) E(y_2, x) \mathcal{J}(\gamma_{(x', v')}(t), y_2, x) m(x, \hat{w}_2) dy_2 dt \\ &= \sum_{l=1}^{\infty} I_{\Theta_l \mathcal{E}, 1} [\Psi_{2,1,l} + \Psi_{2,2,l}](x', v') \end{aligned}$$

where

$$\begin{aligned} \Psi_{2,1,l}(z) &= \int_M \Delta k(z) k(y_2) \varphi_l(z, \bar{w}_1(z, y_2)) \Theta(y_2, \hat{w}_1(z, y_2), \bar{w}_2(y_2, x)) E(z, y_2) E(y_2, x) \\ &\quad \times \mathcal{J}(z, y_2, x) m(x, \hat{w}_2(y_2, x)) dy_2, \\ \Psi_{2,2,l}(z) &= \int_M \tilde{k}(z) \Delta k(y_2) \varphi_l(z, \bar{w}_1(z, y_2)) \Theta(y_2, \hat{w}_1(z, y_2), \bar{w}_2(y_2, x)) E(z, y_2) E(y_2, x) \\ &\quad \times \mathcal{J}(z, y_2, x) m(x, \hat{w}_2(y_2, x)) dy_2. \end{aligned}$$

In a similar manner, with  $y_1 = \gamma_{(x',v')}(t)$ ,

$$\begin{aligned} \Delta\alpha_j(x, x', v') &= \sum_{l=1}^{\infty} \int_0^{\tau_+(x',v')} \mathcal{E}(\tilde{\gamma}_{(x',v')}(t)) \Theta_l(\tilde{\gamma}_{(x',v')}(t)) \\ &\quad \times \int_M \cdots \int_M \left( \sum_{i=1}^j \tilde{k}(y_1) \cdots \tilde{k}(y_{i-1}) \Delta k(y_i) k(y_{i+1}) \cdots k(y_j) \right) \varphi_l(\gamma_{(x',v')}(t), \bar{w}_1) \\ &\quad \times \left( \prod_{i=2}^j \Theta(y_i, \hat{w}_{i-1}, \bar{w}_i) E(y_{i-1}, y_i) \right) E(y_j, x) \mathcal{J}(\gamma_{(x',v')}(t), y_2, \dots, y_j, x) \\ &\quad \times m(x, \hat{w}_j) dy_j \cdots dy_2 dt \\ &= \sum_{l=1}^{\infty} \sum_{i=1}^j I_{\Theta_l \mathcal{E}, 1} \Psi_{j,i,l}(x', v'). \end{aligned}$$

Explicitly,

$$\begin{aligned} \Psi_{j,i,l}(y_1) &= \int_M \cdots \int_M \tilde{k}(y_1) \cdots \tilde{k}(y_{i-1}) \Delta k(y_i) k(y_{i+1}) \cdots k(y_j) \varphi_l(y_1, \bar{w}_1) \\ &\quad \times \left( \prod_{i=2}^j \Theta(y_i, \hat{w}_{i-1}, \bar{w}_i) E(y_{i-1}, y_i) \right) E(y_j, x) \mathcal{J}(\cdot) m(x, \hat{w}_j) dy_j \cdots dy_2. \end{aligned}$$

One may estimate  $\|\Psi_{j,i,l}\|_{L^2(M)}$  (see [LM]):

$$\|\Psi_{j,i,l}\|_{L^2(M)} \leq C\varepsilon(\varepsilon\|\Theta\|_{\infty}(\text{diam } M)|\mathbb{S}^{n-1}|)^{j-2} \|\Delta k\|_{L^2(M)}. \quad (5.25)$$

Now, using the relation  $\beta I_{\Theta_j \mathcal{E}, 1} = I_{\Theta_j \mathcal{E}, \beta}$ , we have

$$\begin{aligned} \left\| I_{w,\beta}^* \sum_{j=2}^{\infty} \Delta\alpha_j \right\|_{H^1(M)} &\leq \sum_{j=2}^{\infty} \sum_{i=1}^j \sum_{l=1}^{\infty} \|I_{w,\beta}^* I_{\Theta_j \mathcal{E}, \beta} \Psi_{j,i,l}\|_{H^1(M)} \\ &\leq \sum_{j=2}^{\infty} \sum_{i=1}^j \sum_{l=1}^{\infty} \frac{C}{1+l^2} \|\Psi_{j,i,l}\|_{L^2(M)} \\ &\leq \sum_{j=2}^{\infty} \sum_{i=1}^j \sum_{l=1}^{\infty} \frac{C'}{1+l^2} \varepsilon(\varepsilon\|\Theta\|_{\infty}(\text{diam } M)|\mathbb{S}^{n-1}|)^{j-2} \|\Delta k\|_{L^2(M)} \\ &\leq \varepsilon C'' \|\Delta k\|_{L^2(M)} \sum_{j=2}^{\infty} j(\varepsilon\|\Theta\|_{\infty}(\text{diam } M)|\mathbb{S}^{n-1}|)^{j-2}. \end{aligned}$$

This follows from Proposition 5.3 and Lemma 5.3, with  $C$  depending continuously on the  $C^{3n}$  norm of  $g$  and  $\Theta$ , and the  $C^2$  norms of  $\sigma$  and  $m$  as well. We see that for sufficiently small  $\varepsilon$  this series converges (thus justifying the formal computations performed above).  $\square$

*Proof of theorem 5.2.* Fix real analytic  $(m, g, \sigma, \Theta)$ . Given the hypothesis of Theorem 5.2 we are ensured that both Proposition 5.2 and Proposition 5.4 hold. Combining them, we have the existence



of a constant  $C$ , depending continuously on the  $C^2$  norms of  $(m, \sigma)$ , and the  $C^{3n}$  norms of  $(\Theta, g)$  such that

$$\|\Delta k\|_{L^2(M)} \leq C\varepsilon \|\Delta k\|_{L^2(M)}$$

and so for  $0 \leq \varepsilon < C^{-1}$ , we must have  $k = \tilde{k}$ .  $\square$

*Proof of Lemma 5.3.* In a fixed coordinate system for  $M$  we define a smooth bijection  $\Omega_x M \rightarrow \mathbb{S}^{n-1}$  and define  $\tilde{\Theta} \in C^\infty(\Omega M \times \mathbb{S}^{n-1})$

$$\tilde{\Theta}(x, v', \theta) = \Theta(x, v', v(\theta)).$$

For  $f \in H_k$ , the space of spherical harmonics of order  $k$ , the Laplacian  $\Delta_S$  on  $\mathbb{S}^{n-1}$  is  $\Delta_S f = -k(k+n-2)f$ . Denote by  $Z_k^x(\theta)$  the so-called zonal harmonics for which  $f(x) = \langle f, Z_k^x(\theta) \rangle_{L^2(\mathbb{S}^{n-1})}$  for all  $f \in H_k$ . Then one has (see, for example, [F]),

$$\begin{aligned} \dim(H_k) &= d_k \leq c_n(k^{n-2} + 1) \\ \|Z_k^x\|_{L^2(\mathbb{S}^{n-1})} &= c'_n \sqrt{d_k} \quad \text{for all } x \in \mathbb{S}^{n-1}. \end{aligned}$$

Let  $\{\tilde{\psi}_{kl}\}_{l=1}^{d_k} \in L^\infty(\mathbb{S}^{n-1})$  be an  $L^2(\mathbb{S}^{n-1})$  orthonormal basis for  $H_k$  (so  $Z_k^x = \sum_{l=1}^{d_k} \tilde{\psi}_{kl}(\theta) \tilde{\psi}_{kl}(x)$ ). Define  $\psi_{kl} := \|\tilde{\psi}_{kl}\|_{L^\infty(\mathbb{S}^{n-1})}^{-1} \tilde{\psi}_{kl}$ . Then for each  $(x, v') \in \Omega M$ ,

$$\tilde{\Theta}(x, v', \theta) = \sum_{k=0}^{\infty} \sum_l^{d_k} \Theta_{kl}(x, v') \psi_{kl}(\theta)$$

with

$$\Theta_{kl}(x, v') = \|\tilde{\psi}_{kl}\|_{L^\infty(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \tilde{\Theta}(x, v', \theta) \tilde{\psi}_{kl}(\theta) d\theta.$$

For each  $\theta$ ,

$$|\tilde{\psi}_{kl}(\theta)| = |\langle \tilde{\psi}_{kl}, Z_k^\theta \rangle| \leq \|\tilde{\psi}_{kl}\|_{L^2(\mathbb{S}^{n-1})} \|Z_k^\theta\|_{L^2(\mathbb{S}^{n-1})} = c'_n \sqrt{d_k}.$$

Next, since  $\tilde{\psi}_{kl} \in H_k$ , for any  $N \in \mathbb{N}$  and  $k \geq 1$ ,

$$\begin{aligned} |\Theta_{kl}(x, v')| &= \|\tilde{\psi}_{kl}\|_{L^\infty(\mathbb{S}^{n-1})} \left| \int_{\mathbb{S}^{n-1}} \tilde{\Theta}(x, v', \theta) \tilde{\psi}_{kl}(\theta) d\theta \right| \\ &= \|\tilde{\psi}_{kl}\|_{L^\infty(\mathbb{S}^{n-1})} \left| \int_{\mathbb{S}^{n-1}} \tilde{\Theta}(x, v', \theta) \frac{(\Delta_S)^N \tilde{\psi}_{kl}(\theta)}{(-k(k+n-2))^N} d\theta \right| \\ &\leq \frac{c'_n \sqrt{d_k}}{(k(k+n-2))^N} \left| \int_{\mathbb{S}^{n-1}} \tilde{\psi}_{kl} (\Delta_S)^N \tilde{\Theta}(x, v', \theta) d\theta \right| \\ &\leq \frac{c'_n k^{(n-2)/2}}{(k(k+n-2))^N} \|(\Delta_S)^N \tilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})}, \end{aligned}$$

where  $c''_n$  depends only on the dimension  $n$ . Thus for sufficiently large  $N$  (in fact  $N \geq 5n/2$ ), there is  $c_{n,N}$  such that

$$|\Theta_{kl}(x, v')| \leq \frac{c_{n,N}}{1+k^{2n}} \|(\Delta_S)^N \tilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})}.$$

Now renumber the collection of coefficient functions and define new basis functions as follows: with  $j = d_0 + d_1 + \dots + d_{k-1} + l$ , set

$$\Theta_j(x, v') = \Theta_{kl}(x, v'), \text{ and } \varphi_j(x, v) := \psi_{kl}(\theta_x(v)).$$

Then

$$\Theta(x, v', v) = \sum_{j=1}^{\infty} \Theta_j(x, v') \varphi_j(x, v).$$

Now

$$j \leq \sum_{m=0}^k d_m \leq c_n(k+1)(k^{n-2} + 1) \leq \hat{c}_n k^n$$

so

$$\begin{aligned} |\Theta_j(x, v')| = |\Theta_{kl}(x, v')| &\leq \frac{c_{n,N}}{1+k^{2n}} \|(\Delta_S)^N \tilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})} \\ &\leq \frac{\tilde{c}_{n,N}}{1+j^2} \|(\Delta_S)^N \tilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})}. \end{aligned}$$

If one applies the same decomposition to  $\partial_x^\alpha \Theta(x, v', v)$ ,  $\alpha \in \{0, 1, 2\}^n$ , one finds that the coefficients are nothing more than  $\partial_x^\alpha \Theta_j(x, v')$  and these satisfy

$$|\partial_x^\alpha \Theta_j(x, v')| \leq \frac{\tilde{c}_{n,N}}{1+j^2} \|(\Delta_S)^N \partial_x^\alpha \tilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})} \leq \frac{C}{1+j^2}.$$

Note that  $C$  depends on  $2N$  derivatives of  $\tilde{\Theta}$  in the last variable, and hence  $2N$  derivatives of  $g$  due to the change of variables introduced earlier. This proves the claim of the lemma.  $\square$

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