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SPECTRAL PROPERTIES OF SELFADJOINT ORDINARY DIFFERENTIAL OPERATORS WITH AN INDEFINITE WEIGHT FUNCTION

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Abstract

Spectral properties of the equation $l(f)-\lambda rf=0$ with an indefinite weight function r are studied in $L^2_{|r|}$. The main tool is the theory of definitizable operators in Krein spaces. Under special assumptions on the weight function, for the operator associated with the problem, the regularity of the critical point infinity is proved. Some relations to full- and half-range expansions are indicated.

1. Basic Properties

1.1. We consider the formal differential expression $l(f)$ of order $2n$ on the interval (a,b), $-\infty \le a < b \le +\infty$:

$$l(f):=(-1)^n(p_0f^{(n)})^{(n)}+(-1)^{n-1}(p_1f^{(n-1)})^{(n-1)}+\ldots+p_nf,$$

where the functions $p_0,\ldots,p_n$ are real, $p_0>0$ a.e. on (a,b) and $1/p_0$, $p_1,\ldots,p_n\in L^1_{loc}(a,b)$. The exact meaning of $l(f)$ under this general assumption is that of the quasi-derivative of order $2n$ (see Krein [1947] and Naimark [1968]): $l(f):=f^{[2n]}$. We study the spectral properties of the equation

$$l(f)-\lambda rf=0, \tag{1.1}$$

where the real weight function $r\in L^1_{loc}(a,b)$ is *indefinite*, that is, the sets $\Delta_+:=\{x:r(x)>0\}$, $\Delta_-:=\{x:r(x)<0\}$ are both of positive Lebesgue measure. For the sake of simplicity we assume that $r\ne 0$ a.e. on (a,b). The problem (1.1) is called ***regular*** if $-\infty<a<b<\infty$ and $\frac{1}{p_0}$, $p_1,\ldots,p_n$, $r\in L^1(a,b)$; the boundary point a (b) is called ***singular*** if $a=-\infty$ $(b=\infty)$ or at least one of the functions $\frac{1}{p_0}$, $p_1,\ldots,p_n,r$ is not summable at a (b, respectively).

By L^2_r we denote the Krein space [Bognár, 1974; Langer, 1982] of all measurable functions f on (a,b) such that $\int_a^b|f|^2|r|\,dx<\infty$, equipped with the indefinite and definite inner products

$$[f,g]:=\int_a^b f\bar g r\,dx \quad\text{and}\quad (f,g):=\int_a^b f\bar g\,|r|\,dx,\ \text{resp.} \tag{1.2}$$

Evidently, the operator J

$$(Jf)(x):=(\operatorname{sgn} r(x))f(x)\qquad (x\in(a,b))$$

is the fundamental symmetry connecting the scalar products in (1.2).

By D^0 we denote the set of all $f\in L^2_r$ which vanish identically in neighborhoods of a and b and have absolutely continuous quasi-derivatives up to order

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$2n-1$ such that

$$f^{[2n]} = l(f) = |r|g$$

with some $g \in L_r^2$. On $\mathbf{D}^0$ we define the operators $B_{\min}^0$ and $A_{\min}^0$: $\mathbf{D}(A_{\min}^0) := \mathbf{D}(B_{\min}^0) := \mathbf{D}^0$

$$B_{\min}^0 f := g \text{ if } l(f) = |r|g, \ g \in L_r^2$$

and $A_{\min}^0 := JB_{\min}^0$. Evidently, $A_{\min}^0 f = g$ if and only if for $f \in \mathbf{D}^0$, $g \in L_r^2$ we have $l(f) = rg$. It is easy to see that the definition of these operators is correct. The closure of $A_{\min}^0$ in L_r^2 exists; it is denoted by $A_{\min}$ and called the *minimal operator* associated with the problem (1.1). It is easy to see that $A_{\min}^0$ and $A_{\min}$ are Hermitian with respect to the inner product $[\cdot,\cdot]$, that is, they are Hermitian in the Krein space L_r^2 (for the definition of Hermitian and selfadjoint operators in Krein spaces, we refer to Bognár [1974] and Langer [1982]).

1.2. Recall that an inner product on a linear space $\mathbf{L}$ is said to have a finite number κ of negative squares, if it is negative definite on a κ-dimensional subspace of $\mathbf{L}$ and there exists no $(\kappa+1)$-dimensional subspace with this property. In this paper we study the problem (1.1) under the following assumptions (A1) and (A2).

(A1). *The inner product* $\{\cdot,\cdot\}$, *defined on* $\mathbf{D}^0$ *by*

$$\{f,g\} := [A_{\min}^0 f, g] \ (= \sum_{j=0}^{n} \int_a^b p_{n-j} f^{(j)} \overline{g^{(j)}} dx)$$

has a finite number of negative squares.

PROPOSITION 1. *The condition (A1) is satisfied in each of the following cases:*

(a) *The problem (1.1) is regular.*

(b) *For each singular boundary point a or b of the problem (1.1), there exists $a' \in (a,b)$ or $b' \in (a,b)$ such that the inner product $\{\cdot,\cdot\}$ is nonnegative definite on the set of all functions $f \in \mathbf{D}^0$ which vanish outside of (a,a') or (b',b), respectively.*

To prove the first statement, we observe that $[A_{\min}^0 f, g] = (B_{\min}^0 f, g)$ $(f,g \in \mathbf{D}^0)$ and use M. G. Krein's results that $B_{\min}^0$ is bounded from below and that an arbitrary selfadjoint extension of $B_{\min}$ in $L_{|r|}^2$ has discrete spectrum (see Krein [1947] and Naimark [1968]). The second statement follows if we use the decomposition method of I. M. Glazman [1967], restricting $A_{\min}^0$ to all functions $f \in \mathbf{D}^0$ with the property $f(b') = f^{[1]}(b') = \cdots = f^{[2n-1]}(b') = 0$ (if, for example, b is singular), and use statement (a).

1.3. The Hermitian operator $A_{\min}$ in the Krein space L_r^2 has selfadjoint extensions in L_r^2. In fact, A is a selfadjoint extension of $A_{\min}$ if and only if the operator $B := JA$ is a selfadjoint extension of the Hermitian operator $B_{\min}$ in the Hilbert space $L_{|r|}^2$. Therefore the selfadjoint extensions A of $A_{\min}$ are completely described by boundary conditions at a and b which are the same for A and $B = JA$ and which can be found, for example, in Naimark [1968]. Now we formulate the second assumption.

(A2). *For some (and hence for all) selfadjoint extensions A of $A_{\min}^0$ in L_r^2, the resolvent set is nonempty.*

We mention (cf. Daho and Langer [1977]) that this condition is equivalent to each of the following:

(A2') *For some (and hence for all)* $\lambda \in \mathbf{C}$, *the range* $\mathbf{R}(A_{\min} - \lambda I)$ *is closed.*

(A2'') *For some selfadjoint extension* A *of* $A_{\min}$ *and for some* $\lambda \in \mathbf{C}$, *the range* $\mathbf{R}(A - \lambda I)$ *is closed.*

PROPOSITION 2. *The condition (A2) is satisfied in each of the following cases:*

(a) *The problem (1.1) is regular.*

(b) *For each singular boundary point* a *or* b *of the problem (1.1) there exists* $a' \in (a,b)$ *or* $b' \in (a,b)$ *such that the weight function* r *is of constant sign a.e. on* (a,a') *or* (b',b), *respectively.*

Here the statement (a) is a classical result, and (b) follows again from an application of Glazman's decomposition method [Glazman, 1967].

2. Definitizability of the Selfadjoint Extensions

2.1. Recall that a selfadjoint operator A in a Krein space $\mathbf{K}$ is said to be *definitizable* [Langer, 1982] if $\rho(A) \neq \phi$ and there exists a polynomial p such that $[p(A)f, f] \geq 0$ for all $f \in \mathbf{D}(A^k)$, where k is the degree of p.

THEOREM 1. *Suppose that the operator* $A^0_{\min}$ *in Section 1.1 satisfies the conditions (A1) and (A2). Then every selfadjoint extension* A *of* $A^0_{\min}$ *in* L^2_r *is definitizable.*

Indeed, it is easy to see that for such a selfadjoint extension A the inner product $[Af, g]$ $(f, g \in \mathbf{D}(A))$ has a finite number of negative squares. Hence we can apply Langer [1982: I.3(c)], and the statement follows.

Suppose, for example, that the problem (1.1) is regular. Then we have for $f, g \in \mathbf{D}(A)$

$$[Af, g] = \sum_{j=0}^{n} \int_a^b p_{n-j} f^{(j)} \overline{g^{(j)}} \mathrm{d}x + \mathbf{b}(f, g), \tag{2.1}$$

where $\mathbf{b}(f, g)$ ("the boundary form") is an inner product, depending for a regular (singular) boundary point only on the values of f, g and their first $2n-1$ quasi-derivatives at this point (in the neighborhood of this point, respectively). The number of negative squares of the inner product, given by the first term on the righthand side of (2.1), coincides with the number of negative squares of the inner product $[A^0_{\min} f, g]$ on $\mathbf{D}^0$. Thus, the number of negative squares of $[Af, g]$ $(f, g \in \mathbf{D}(A))$ is not greater than the sum of the negative squares of $[A^0_{\min} f, g]$ $(f, g \in \mathbf{D}^0)$ and of the boundary form $\mathbf{b}(f, g)$ $(f, g \in \mathbf{D}(A))$.

2.2. Here we suppose that the conditions of Theorem 1 are satisfied and A is an arbitrary selfadjoint extension of $A^0_{\min}$ in the Krein space L^2_r. By κ_A we denote the number of negative squares of the inner product $[Af, g]$ $(f, g \in \mathbf{D}(A))$. The following spectral properties of A are immediate consequences of the definitizability of A (see Langer [1982]).

(1) *The operator* A *has at least* κ_A *eigenvalues* λ *(counted according to their algebraic multiplicities) in the closed upper half-plane with the following property: If* $\lambda > 0$ $(\lambda < 0)$ *there exists an eigenelement* f *of* A *corresponding to* λ *such that* $[f, f] \leq 0$ $([f, f] \geq 0)$.

(2) *The nonreal spectrum of A consists of pairs of isolated eigenvalues $\lambda,\bar{\lambda}$; the linear span of the root spaces corresponding to these eigenvalues λ in the upper half-plane is neutral and hence of dimension $\le \kappa_A$.*

We mention that for any selfadjoint operator A in a Krein space the root spaces, corresponding to two eigenvalues λ,μ are orthogonal with respect to the indefinite inner product if $\lambda \ne \bar{\mu}$, and skewly linked if $\mu = \bar{\lambda}$ and λ,μ are isolated points of $\sigma(A)$.

(3) *The operator A has positive and negative spectrum, both of infinite multiplicity. If, in particular, $\sigma(A)$ is discrete, it contains infinitely many positive eigenvalues s_j^+ and infinitely many negative eigenvalues s_j^-, $j = 1,2,\dots$, and the root spaces, corresponding to the real eigenvalues of A are nondegenerated with respect to the indefinite inner product.*

We denote the signature of the root space corresponding to the real eigenvalue λ of A by $(\kappa_-(\lambda), \kappa_+(\lambda))$; for an arbitrary eigenvalue λ of A, its algebraic multiplicity is denoted by $\nu(\lambda)$.

(4) *If $\sigma(A)$ is discrete, we have*

$$\sum_j \kappa_+(s_j^-) + \sum_j \kappa_-(s_j^+) + \sum_{\substack{\lambda\in\sigma(A)\\ \mathrm{Im}(\lambda)>0}} \nu(\lambda) \le \kappa_A,$$

where the sign = holds if 0 is not an eigenvalue of A.

We mention that these statements imply some results of Mingarelli [1983a and 1983b].

The spectral theory of definitizable operators in Krein spaces yields the existence of a spectral function with critical points (see Langer [1982]) for A. It can also be shown that there exists a scalar or matrix spectral measure that has, possibly, certain singularities; in a special situation this spectral measure was considered by Daho and Langer [1977]. Moreover, expansions of arbitrary elements of L_r^2 with respect to eigenelements or generalized eigenelements of A hold. However, they become more complicated than in the case of a positive weight function as the integrals need a regularization at the singular critical points of A (see Daho and Langer [1977] for the case of second-order operators). In Section 3 we shall show that these expansions are "nice" if the spectrum of A is discrete, r has only finitely many turning points, and at these turning points some condition —going back to Beals [1984] —is satisfied. Recall that the points of $\bar{\Delta}_+\cap\bar{\Delta}_-$ are called the *turning points* of r.

2.3. The following result can also be proved by means of Glazman's decomposition method, using Theorem 1 of Jonas and Langer [1979].

PROPOSITION 3. *Suppose that the condition (A1) is satisfied and that r has only a finite number of turning points. If the set Δ_- has a positive distance from all the singular boundary points a or b, then $\sigma(A)\cap(-\infty,0)$ is discrete with the only accumulation point $-\infty$.*

For a special differential operator, this structure of $\sigma(A)$ was established in Mikulina [1971].

Finally, we mention that in the special case $\kappa_A = 0$ (that is, $[Af, f] \ge 0$ for $f \in \mathbf{D}(A)$) and $0 \notin \sigma(A)$ the eigenvalues $s_j^\pm$, $j = 1,2,\dots$, can be characterized by means of minimax principles (see Phillips [1970] and Textorius [1974]).

3. Regularity of the Critical Point Infinity

3.1. The turning point x_0 of the weight function r is said to be *n-simple* if there exists an interval I_0 around x_0 such that for $x \in I_0 \ \{x_0\}$ representation

$$r(x) = \operatorname{sgn}(x-x_0) \cdot |x-x_0|^\alpha \rho(x) \tag{3.1}$$

holds with some $\alpha > -\frac{1}{2}$ and a function ρ:

$$\rho(x) := \rho_+(x), \ x > x_0,$$

$$\rho(x) := \rho_-(x), \ x < x_0,$$

where $\rho_+(\rho_-)$ is defined and of class C^n on $I_0 \cap [x_0, \infty)$ $(I_0 \cap (-\infty, x_0]$, resp.), $0 \neq \operatorname{sgn}\rho_+(x_0) = \operatorname{sgn}\rho_-(x_0)$ and for the one-sided derivatives at x_0 we have

$$\rho_\pm'(x_0) = \rho_\pm''(x_0) = \ldots = \rho_\pm^{(n-1)}(x_0) = 0 \quad \text{if } n > 1.$$

THEOREM 2. *Suppose that the following conditions are satisfied:*

1. *The problem (1.1) is regular.*

2. *The weight function r has only a finite number of turning points that are all n-simple.*

3. *There exists a $\delta > 0$ such that for each turning point x_0 of r we have*

$$0 < \inf_{\substack{|x-x_0| \le \delta \\ x \in (a,b)}} p_0(x) \le \sup_{\substack{|x-x_0| \le \delta \\ x \in (a,b)}} p_0(x) < \infty.$$

Then infinity is not a singular critical point for every selfadjoint extension A of $A^0_{\min}$ in L^2_r.

We shall only sketch the proof. Propositions 1 and 2 and Theorem 1 imply that A is definitizable. We show that for A there exists an operator W with the properties given by Curgus [1984: Remark 3.6], and an application of a proposition given in that paper [Curgus, 1984: Proposition 3.5] yields the desired result.

To simplify the construction of W, we suppose $a < 0 < b$ and that $x_0 = 0$ is the only turning point of r. Let $\delta > 0$ be such that $(-\delta, \delta) \subseteq (a, b)$ and I_0 in (3.1) can be chosen to be $(-\delta, \delta)$.

We choose $2n$ mutually distinct points $t_1, \ldots, t_{2n} \in (1,2)$ and define the functions

$$h_j(x) := \frac{1}{t_j^\alpha} \frac{\rho(x)}{\rho(-t_j x)} \quad (x \in [-\frac{\delta}{2}, \frac{\delta}{2}], \ x \neq 0).$$

By **D** we denote the set of all functions $f \in L_r^2$ which have an absolutely continuous $(n-1)$-st derivative and for which

$$\int_a^b p_0 |f^{(n)}|^2 |r| \, dx < \infty.$$

Further, we choose $\varphi \in C^n(a,b)$, which is constant in a neighborhood of zero, $\varphi(0) = 1$, and $\operatorname{supp}\varphi \subseteq [-\frac{\delta}{2}, \frac{\delta}{2}]$.

Now we define linear operators $X_\pm$, $Y_\pm$ in L_r^2 as follows:

$$(X_+u)(x):=u(x), \qquad x\in(0,b]$$

$$(X_+u)(x):=\varphi(x)\sum_{j=1}^{2n}\alpha_j t_j u(-t_j x), \quad x\in[a,0),$$

$$(Y_+u)(x):=0, \qquad x\in[a,0),$$

$$u(x)+\sum_{j=1}^{2n}\alpha_j(\varphi h_j u)(-\frac{x}{t_j}), \quad x\in(0,b],$$

where $\alpha_1, \ldots, \alpha_{2n}$ are reals to be chosen below. It is not hard to see that X_+, Y_+ are bounded in L_r^2. Moreover, the numbers $\alpha_1, \ldots, \alpha_{2n}$ can be chosen such that X_+, Y_+ map $\mathbf{D}$ into itself. In order to see this with $u\in\mathbf{D}$, we form the first n derivatives of X_+u on $[a,0)$ and on $(0,b]$. Then X_+u will have $n-1$ absolutely continuous derivatives on $[a,b]$ if and only if for the first $n-1$ derivatives the limits from the left and from the right at zero coincide, which is equivalent to the equations

$$\sum_{j=1}^{2n}\alpha_j t_j^{k+1}=(-1)^k \qquad (k=0,1,\ldots,n-1). \tag{3.2}$$

A similar reasoning for Y_+ yields the equations

$$\sum_{j=1}^{2n}\alpha_j t_j^{-k-a}=(-1)^{k+1}\frac{\rho_+(0)}{\rho_-(0)} \quad (k=0,1,\ldots,n-1). \tag{3.3}$$

The system (3.2), (3.3) determines the numbers α_j, $j=1,2,\ldots,2n$ uniquely. It is easy to check that the operators X_+, Y_+ satisfy the relation $X_+=Y_+^*J$ where * denotes the adjoint in $L^2_{|r|}$.

In the same way, exchanging the roles of $[a,0)$ and $(0,b]$, operators X_-, Y_- with similar properties are defined. Finally, put

$$W:=Y_+X_++Y_-X_-.$$

As in Curgus [1984: Remark 3.6], it follows that W is positive, bounded, and boundedly invertible in the Krein space L_r^2. Moreover,

$$(Wu)(x)=u(x) \quad \text{if } x\in[a,b]\ (-\frac{\delta}{2},\frac{\delta}{2}).$$

We mention that $X_\pm$, $Y_\pm$ here do not necessarily have the property (a) given in Curgus [1984: Remark 3.6].

The set $\mathbf{D}[JA]$ (see Krein [1947] and Curgus [1984]) consists of those functions of $\mathbf{D}$ that satisfy the essential boundary conditions. As the function Wu coincides near a and b with u, it satisfies the same boundary conditions as u; hence $W\mathbf{D}[JA]\subset\mathbf{D}[JA]$. Thus W has all the desired properties.

The construction of the operators $X_\pm$, $Y_\pm$ follows [Beals, 1984: Lemma 1].

3.2. Under the conditions of Theorem 2 we denote by $P_{j,\pm}$ the orthogonal projection in the Krein space L_r^2 onto the root space of A corresponding to $s_j^\pm$, $j=1,2,\ldots$, and by P_0 the orthogonal projection onto the (finite dimensional) span of the root spaces corresponding to the (possible) eigenvalue zero and to the nonreal eigenvalues of A.

COROLLARY. *Under the conditions of Theorem 2 we have for arbitrary* $f\in L_r^2$

$$f = P_0 f + \sum_{j=1}^{\infty} P_{j,+} f + \sum_{j=1}^{\infty} P_{j,-} f,$$

where both sums converge in the norm of $L^2_{|r|}$.

We mention that for all the points $s_j^{\pm}$ with the property $\kappa_+(s_j^+) = \kappa_+(s_j^-) = 0$ there can be chosen an orthogonal basis of eigenvectors $e_{j,k}^{\pm}$ in $P_{j,\pm} L_r^2$, $k = 1,\ldots,\nu_j^{\pm}$, $j = 1,2,\ldots$, such that

$$P_{j,\pm} = \sum_{k=1}^{\nu_j^{\pm}} \frac{[\,\cdot\,, e_{j,k}^{\pm}]}{[e_{j,k}^{\pm}, e_{j,k}^{\pm}]} .$$

The corollary contains, for example, the full-range expansions of the "regular" examples in Kaper, Kwong, Lekkerkerker, and Zettl [1984]. We mention that the above construction of W and hence the statement of Theorem 2 can also be extended to some singular operators. This extension will be considered elsewhere.

3.3. Suppose now for a moment that (under the conditions of Theorem 2) we have $\kappa_A = 0$ and $0 \notin \sigma_p(A)$. Then $\sigma(A)$ consists of the two sequences (s_j^+), (s_j^-), and we have $\kappa_-(s_j^+) = \kappa_+(s_j^-) = 0$, $j = 1,2,\ldots$. Moreover, the subspace

$$c.l.s.\,\{P_{j,+} L_r^2 : j = 1,2,\ldots\}$$

is a maximal nonnegative subspace of the Krein space L_r^2 (see Bognár [1974] and Langer [1982]). If we denote by $\mathbf{K}_+$ the subspace

$$\mathbf{K}_+ := \{f \in L_r^2 : f(x) = 0 \quad \text{if } x \in \Delta_-\}$$

and by P_+ the orthogonal projection onto $\mathbf{K}_+$ in L_r^2, it follows that for arbitrary $f_+ \in \mathbf{K}_+$ we have

$$f_+ = \sum_{j=1}^{\infty} P_+ P_{j,+} f_+, \tag{3.4}$$

where the series converges again in the norm of $L^2_{|r|}$. This is an abstract form of the half-range expansion considered, for example, in Beals [1984] and Kaper, Kwong, Lekkerkerker, and Zettl [1984].

If $\kappa_A \geq 0$ we consider for arbitrary $f_+ \in \mathbf{K}_+$ the sum

$$\sum_{j:\kappa_-(s_j^+)=0} P_+ P_{j,+} f_+ .$$

It converges in the norm of $L^2_{|r|}$; however, it equals f_+ only for $f_+ \in \mathbf{K}_+'$, where $\mathbf{K}_+'$ is a subspace of K_+ such that $\dim \mathbf{K}_+ / \mathbf{K}_+' < \infty$. To expand arbitrary elements of $\mathbf{K}_+$, we have to add to the elements of $P_+ P_{j,+} L_r^2$, $\kappa_-(s_j^+) = 0$, finitely many elements h_k^+ that are the projections onto K_+ of root vectors h_k of A corresponding to the possible eigenvalue zero, the eigenvalues $s_j^{\pm}$ with $\kappa_-(s_j^+) > 0$, $\kappa_+(s_j^-) > 0$ and to nonreal eigenvalues. A minimal set of such elements h_k^+ which have to be added can easily be found from the condition that the linear span of these root vectors and of c.l.s. $\{P_{j,+} L_r^2 : \kappa_-(s_j^+) = 0,\ j = 1,2,\ldots\}$ is a maximal nonnegative subspace of the Krein space L_r^2.

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