1982

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Branko Ćurgus
Western Washington University, branko.curgus@wwu.edu

Harry I. Miller

BRANKO ĆURGUS and HARRY I. MILLER

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(Accepted by the Department of Natural and Mathematical Sciences at the Session on 21. V 1979. according to the reviews of Prof. M. Bajraktarević and Prof. V. Perić)

INTRODUCTION

Many proofs of the fact that there exist *Lebesgue* nonmeasurable subsets of the real line are known. The oldest proof of this result is due to Vitali [4]. The cosets (under addition) of Q, the set of rational numbers, constitute a partition of the line into an uncountable family of disjoint sets, each congruent to Q under translation, Vitali's proof shows that V is nonmeasurable, if V is a set having one and only one element in common with each of these cosets.

Another proof of the existence of a nonmeasurable set that is widely quoted in text books, for example in *Real Functions*, by Goffman [2], makes use of the fact that if S is a measurable subset of the real line, then either S or its complement has a nonempty perfect subset. The proof makes use of the well-ordering theorem and the continuum hypothesis to construct two transfinite sequences.

The proof mentioned in the first paragraph depended on group-theoretic properties of Lebesgue measure and the proof in the second paragraph used topological properties. Ulam [6], using purely set-theoretic arguments showed that a finite measure μ defined for all subsets of a set X of power $\aleph_1$ vanishes identically if it is equal to zero for every one element subset.

Pairs of transfinite sequences have been useful in many proofs in real analysis, for example see C. A. Rodgers [5, pg. 74] or H. I. Miller [3]. In this paper we use pairs of transfinite sequences to prove two theorems. We

* Work supported by the Republican Council for Scientific Work of Bosnia and Herzegovina.

will give a new proof of the existence of a nonmeasurable subset of the real line and will show that if U is any uncountable subset of a separable metric space, then U contains an uncountable subset V which has no uncountable closed subsets.

RESULTS

$S \subset (0, 1)$ is Lebesgue measurable (see [2]) if and only if, for every $\varepsilon < 0$, there are open sets G and H, subsets of $(0, 1)$, such that $G \supset S$, $H \supset C(S)$, and $m(G) + m(H) < 1 + \varepsilon$. Here $C(S)$ denotes the complement of S relative to $(0, 1)$, i. e. $C(S) = (0, 1)\setminus S$ and m denotes Lebesgue measure.

Theorem 1: *There exists a set S, $S \subset (0, 1)$, which is not Lebesgue measurable.*

Proof: Let τ denote the collection of pairs of open sets given by the following formula:

$$\tau = \left\{(G, H);\ G \text{ and } H \text{ open subsets of } (0, 1),\ m(G) + m(H) < \frac{3}{2}\right\}.$$

The set τ has c, the cardinal number of the continuum, elements. Using the well-ordering theorem (every set can be well ordered) and the continuum hypothesis ($c = \aleph_1$), τ can be written in the form

$$\tau = \{(G_\alpha, H_\alpha);\ \alpha < \Omega\},$$

where Ω denotes the first uncountable ordinal.

We will use transfinite induction to construct two transfinite sequences of real numbers, $\{x_\alpha\}_{\alpha<\Omega}$ and $\{y_\alpha\}_{\alpha<\Omega}$. Pick

$$x_1 \in (0, 1), \quad y_1 \in (0, 1), \quad x_1 \notin G_1, \quad y_1 \notin H_1, \quad x_1 \neq y_1.$$

This will not be possible in case $G_1 = (0, 1)$ or $H_1 = (0, 1)$ ($G_1 = H_1 = (0, 1)$ is impossible as $m(G_1) + m(H_1) < \frac{3}{2}$). In this case pick

$$x_1 = 2 \quad \text{and} \quad y_1 \in (0, 1), \quad y_1 \notin H_1 \quad \text{if} \quad G_1 = (0, 1)$$

or pick

$$y_1 = 2 \quad \text{and} \quad x_1 \in (0, 1), \quad x_1 \notin G_1 \quad \text{if} \quad H_1 = (0, 1)$$

Suppose for every $\beta < \alpha$, where $\alpha < \Omega$, we have chosen x_β and y_β, in such a way that:

1) $(\{x_\beta;\ \beta < \alpha\} \setminus \{2\}) \cap (\{y_\beta;\ \beta < \alpha\} \setminus \{2\}) = \varnothing$

2) $(\{x_\beta;\ \beta < \alpha\} \setminus \{2\}) \cup (\{y_\beta;\ \beta < \alpha\} \setminus \{2\}) \subset (0, 1)$

3) for every $\beta < \alpha$, either $x_\beta \in C(G_\beta)$ or $y_\beta \in C(H_\beta)$.

Here, as before, $C(A) = (0, 1) \setminus A$. Since either $m(C(G_\alpha)) > 0$ or $m(C(H_\alpha)) > 0$, it follows that either $C(G_\alpha)$ or $C(H_\alpha)$ is an uncountable set. Moreover, the set

$$(\{x_\beta;\ \beta < \alpha\} \setminus \{2\}) \cup (\{y_\beta;\ \beta < \alpha\} \setminus \{2\})$$

is a countable set, as $\alpha < \Omega$. This insures that we can pick x_α, y_β, in such a way that 1), 2) and 3) hold with the symbol $<$, replaced everywhere by the symbol $\leqslant$. It follows, by transfinite induction, that we obtain two transfinite sequences $\{x_\alpha\}_{\alpha<\Omega}$ and $\{y_\alpha\}_{\alpha<\Omega}$, such that 1), 2) and 3) hold for every $\alpha < \Omega$.

Let

$$S = \{x_\alpha;\ \alpha < \Omega\} \setminus \{2\}.$$

The fact that

$$\{y_\alpha;\ \alpha < \Omega\} \setminus \{2\} \subset C(S),$$

follows from 1) and 2); for if $y_{\alpha_1} = x_{\alpha_2} \in (0, 1)$, for some $\alpha_1, \alpha_2 < \Omega$, then

$$(\{x_\beta;\ \beta < \alpha\} \setminus \{2\}) \cap (\{y_\beta;\ \beta < \alpha\} \setminus \{2\}) \neq \varnothing, \quad \forall_{\alpha > max(\alpha_1, \alpha_2)}$$

Furthermore, 3) guarantees that for every $\alpha < \Omega$, either

$$G_\alpha \not\supset S \quad \text{or} \quad H_\alpha \not\supset C(S).$$

By the remarks proceeding the statement of Theorem 1 it follows that S is nonmeasurable.

Theorem 2: *If (X, d) is a separable metric space and $U \subset X$ is an uncountable set, then there exists an uncountable set V, $V \subset U$, which does not contain any uncountable closed set.*

Proof: Let $\mathcal{U}$ denote the collection of uncountable closed subsets of U, i. e.

$$\mathcal{U} = \{F \subset U;\ F \text{ closed and uncountable}\}.$$

If $\mathcal{U} = \varnothing$, then set $V = U$ and the proof is complete. If $\mathcal{U} \neq \varnothing$, then, by the separabilty of (X, d) and the *Cantor-Bendixon* theorem [5], it follows that $k(\mathcal{U})$, the cardinal number of $\mathcal{U}$, satisfies the equality $k(\mathcal{U}) = c$, where c denotes the cardinal number of the continuum. If we assume the continuum hypothesis ($c = \aleph_1$), we get $k(\mathcal{U}) = \aleph_1$. Therefore, by the well-ordering theorem, $\mathcal{U}$ can be written in the form

$$\mathcal{U} = \{F_\alpha;\ \alpha < \Omega\},$$

where Ω denotes the first uncountable ordinal number. We now proceed by transfinite induction to construct two transfinite sequences in X, $\{x_\alpha\}_{\alpha<\Omega}$ and $\{y_\alpha\}_{\alpha<\Omega}$. Pick $x_1, y_1 \in F_1$, such that $x_1 \neq y_1$. Suppose that for every $\beta < \alpha$, where $\alpha < \Omega$, we have chosen x_β and y_β in such a way that:

1) $x_{\beta_1} \neq x_{\beta_2}$, $y_{\beta_1} \neq y_{\beta_2}$ $(\forall \beta_1 < \beta_2 < \alpha)$

2) $\{x_\beta;\ \beta < \alpha\} \cap \{y_\beta;\ \beta < \alpha\} = \emptyset$.

The set

$$\{x_\beta;\ \beta < \alpha\} \cup \{y_\beta;\ \beta < \alpha\}$$

is countable, as $\alpha < \Omega$, so that there exists

$$x_\alpha,\ y_\alpha \in F_\alpha \setminus (\{x_\beta;\ \beta < \alpha\} \cup \{y_\beta;\ \beta < \alpha\}),\quad x_\alpha \neq y_\alpha\,,$$

as F_α is an uncountable set. This insures that 1) and 2) hold with the symbol $<$, replaced everywhere by the symbol $\leqslant$. It follows, by transfinite induction, that we obtain two transfinite sequences $\{x_\alpha\}_{\alpha<\Omega}$ and $\{x_\alpha\}_{\alpha<\Omega}$, such that 1) and 2) hold for every $\alpha < \Omega$. Let

$$V = \{x_\alpha;\ \alpha < \Omega\}.$$

The fact that V is uncountable follows from 1) and 2) implies that

$$V \cap \{y_\alpha;\ \alpha < \Omega\} = \emptyset,$$

which in turn implies that V contains no uncountable closed set, as $F_\alpha \not\subset V (\forall \alpha < \Omega)$. V is clearly a subset of U. This completes the proof.

Remark 1: The proof of the Theorem 2 is essentially the same as the proof of a theorem *F. Bernstein*. Bernstein's theorem states that there exists a set B of real numbers, such that both B and B', the complement of B, meet every uncountable closed subset of the real line (see [4], pg 23.)

Remark 2: Theorem 2 is true for any topological space having a countable base.

Remark 3: Suppose that (X, d) is a metric space and that the set of uncountable closed subsets of X has cardinality c, the cardinal number of the continuum. If (X, M, m) is a probability space with the properties:

a) the σ-field M contains each open subset of X,

b) m is a regular measure (see [1]) with respect to the topology induced by d,

c) $m(\{x\}) = 0$ for every $x \in X$,

then by the argument in the proof of the Theorem 2 and the definition of a regular measure, X (playing the role of U is uncountable by c)) contains a set V, such that $V \notin M$, i. e. V is nonmeasurable.

Remark 4: If A is *Lebesgue* measurable subset of the real line, then by the regularity of the Lebesgue measure m, A contains an uncountable closed set F if $m(A) > 0$. However Theorem 2 shows that if $m(A) = 0$, then A need not contain an uncountable closed set F, even if A is uncountable.

ĆURGUS, B. i MILLER, H. I.

NEIZMJERIVI SKUPOVI I PAROVI TRANSFINITNIH NIZOVA

KRATAK SADRŽAJ

Parovi transfinitnih nizova korišteni su u mnogim dokazima u realnoj analizi. U ovom radu upotrijebljeni su parovi transfinitnih nizova za dokaz dva teorema. Dat je novi dokaz egzistencije *Lebesguevog* neizmjerivog podskupa realne prave i pokazano je da ako je U neprebrojiv podskup separabilnog metričkog prostora tada U sadrži neprebrojiv podskup V koji ne sadrži neprebrojiv zatvoren skup.

REFERENCES

[1] Billingsly, P., *Convergence of Probability Measures*, John Wiley and Sons, Inc., New York, 1968.

[2] Goffman, C., *Real Functions;* Holt, Rinehart and Winston, New York, 1961.

[3] Miller, H. I., *A universall null set whose Steinnaus distance set is* [0,1], Radovi Akademije Nauka B i H. LXI, 1978, pg 37—40.

[4] Oxtoby, J. C., *Measure and Category*, Springer-Verlag, New York 1971.

[5] Rodgers, C. A., *Hausdorff Measure*. Cambridge University Press, Cambridge, 1970.

[6] Ulam, S. M., *Zur masstheorie in der allgemeinen mengenlehre*, Fund. Math. 16,141-150 (1930).

Department of Mathematics University of Sarajevo
71000 Sarajevo
Yugoslavia