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M-addition

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Abstract

This study builds upon the work of Gardner, Hug and Weil [1, Section 6] by further exploring the properties of M-addition. It is shown that several well-known theorems on Minkowski addition have M-addition parallels, including results involving intersections, the valuation property and the convex hull. The last of these enables us to detail sufficient conditions for when the M-sum of convex polytopes is a convex polytope. Nested operations of M-addition are also examined and an M-addition generalization of the Shapley-Folkman Lemma and a related bound are offered.

Acknowledgments

I am greatly indebted in the genesis, direction and outcome of this thesis to my adviser Richard Gardner. The genesis, in that he proposed that I study his recent research paper [1] and try to explore several of the questions it raised. The direction, because he continually suggested avenues for investigation and target results to strive for. And the outcome, because of his encouragement, support and indefatigable editorial oversight. I am deeply appreciative for the great amount of time and energy he invested in helping me see this work to completion. I would also like to express gratitude for David Hartenstine's and Branko Ćurgus' comments on an earlier draft of the manuscript. The thesis is better for having first passed beneath their eyes.

Contents

1 Introduction

Many are the ways to combine sets, and this study examines a generalization of one of the most significant, Minkowski addition. The Minkowski sum of subsets $A_1, \dots, A_m$ of a vector space V over a field $\mathbb{F}$ has the familiar definition

$$\sum_{j=1}^{m} A_j \;=\; \left\{ \sum_{j=1}^{m} v_j \;:\; v_j \in A_j \right\}.$$

The powerful generalization we study in these pages is *M-addition*. This operation expands the idea of Minkowski summation by weighting the sets with m-tuples of scalars $(\alpha_1, \dots, \alpha_m) \in M \subset \mathbb{F}^m$. Thus for arbitrary nonempty $M \subset \mathbb{F}^m$ and arbitrary nonempty $A_1, \dots, A_m \subset V$,

$$\oplus_M(A_1, \dots, A_m) \;=\; \bigcup_{(\alpha_1,\dots,\alpha_m)\in M} \alpha_1 A_1 + \cdots + \alpha_m A_m \;=\; \bigcup_{(\alpha_1,\dots,\alpha_m)\in M} \sum_{j=1}^{m} \alpha_j A_j \tag{1}$$

$$= \left\{ \sum_{j=1}^{m} \alpha_j v_j \;:\; v_j \in A_j,\, (\alpha_1, \dots, \alpha_m) \in M \right\}. \tag{2}$$

The A_j's are the *summands* while the *M-set* is the collection of scaling m-tuples, in this case M. If for $M \subset \mathbb{F}^m$ there exists $k \in \{1, \dots, m\}$ such that $\alpha_k = 0$ for every $(\alpha_1, \dots, \alpha_m) \in M$, then the corresponding A_k is a *dummy summand* since it makes no contribution to the sum. When dealing with just two summands A_1 and A_2, we will interchangeably write $\oplus_M(A_1, A_2)$ and $A_1 \oplus_M A_2$, the latter imitating the Minkowski sum notation $A_1 + A_2$.

1.1 Examples

The operation of M-addition is powerful and general because it forms a collection of linear combinations where we have control over where each vector comes from *and* over what condition each ordered set of weights satisfies (i.e. what set M it belongs to). As the following examples illustrate, this idea of what we might call "controlled linear combinations" is significant, if not ubiquitous, in linear algebra and convex geometry. Using the above definition we can rephrase familiar instances of it in terms of M-addition.

In Minkowski summation $\sum_{j=1}^{m} A_j$, we take all sums $\sum_{j=1}^{m} v_j$ where $v_j \in A_j$. Thus from (1) this is simply M-addition with $M = \{(1, \dots, 1)\}$.

For $v_1, \dots, v_m \in V$, $\operatorname{span}\{v_1, \dots, v_m\}$ is the set of all linear combinations where the v_j are fixed but the weights are arbitrary elements of $\mathbb{F}$. From (2) we see that this is the same as $\oplus_{\mathbb{F}^m}\big(\{v_1\}, \dots, \{v_m\}\big)$. In the more general setting of a ring R, the analogous object to the spanning set is the ideal $\langle v_1, \dots, v_m \rangle$ generated by $v_1, \dots, v_m \in$

R. Definition (2) also makes sense in this context when each $A_j \subset R$ and $M \subset R^m$, and so we similarly have

$$\langle v_1, \ldots, v_m \rangle \;=\; \oplus_{R^m}\big(\{v_1\}, \ldots, \{v_m\}\big).$$

If $M = \{e_1, \ldots, e_m\}$ is the set of the standard basis vectors of $\mathbb{F}^m$, then for any $A_1, \ldots, A_m \subset V$ we see from (1) that

$$\oplus_M(A_1, \ldots, A_m) \;=\; \bigcup_{j=1}^{m} A_j.$$

Recall $C \subset \mathbb{R}^n$ is *convex* if $(1-\lambda)v + \lambda w \in C$ for all $\lambda \in [0,1]$ and $v, w \in C$, which is equivalent to saying

$$\{\,(1-\lambda)v + \lambda w \;:\; \lambda \in [0,1],\; v, w \in C\,\} \;=\; \bigcup_{\lambda\in[0,1]} (1-\lambda)C + \lambda C \;\subset\; C.$$

Comparing this to (1), we see that if $M \subset \mathbb{R}^2$ is the line segment with endpoints $e_1 = (1,0)$ and $e_2 = (0,1)$, then C is convex if and only if

$$\oplus_M(C, C) \;\subset\; C.$$

For any $A \subset \mathbb{R}^n$, the intersection

$$\bigcap_{\substack{K \text{ convex} \\ A\subset K}} K \;=\; \operatorname{conv} A$$

is known as the *convex hull of* A and is the smallest convex set containing A. One can show that

$$\operatorname{conv} A \;=\; \left\{ \sum_{j=1}^{n} \lambda_j v_j \;:\; n \in \mathbb{N},\, v_j \in A,\, \lambda_j \geq 0,\, \sum_{j=1}^{n} \lambda_j = 1 \right\} \tag{3}$$

(see [4, Theorem 1.1.2, p. 27]). This can also be formulated in terms of M-addition by means of Carathéodory's theorem. This classic result in convexity theory states that it suffices to take linear combinations of $n+1$ or fewer elements of $A \subset \mathbb{R}^n$ in (3) to generate $\operatorname{conv} A$ [4, Theorem 1.1.4]. Thus if we set

$$M \;=\; \left\{ (\lambda_1, \ldots, \lambda_{n+1}) \in \mathbb{R}^{n+1} \;:\; \lambda_j \geq 0, \quad \sum_{j=1}^{n+1} \lambda_j = 1 \right\},$$

then by comparing (3) with (2) we see that

$$\operatorname{conv} A \;=\; \oplus_M(\underbrace{A, A, \ldots, A}_{n+1 \text{ times}}).$$

Thus operations as diverse as the Minkowski sum of sets, the span of a set of vectors, the union of sets and the convex hull of a set are all merely specific instances of M-addition.

1.2 Historical background

M-addition first appeared in a 1997 paper on normed algebras by Protasov [3], who combined two origin-symmetric compact convex sets in $\mathbb{R}^n$ using a convex $M \subset \mathbb{R}^2$ symmetric with respect to both axes.

Gardner, Hug and Weil later independently discovered M-addition in the course of their investigations into operations between compact convex sets [1]. To explain how they arrived at the operation, we recall that for a set K belonging to the collection $\mathcal{K}^n$ of all compact convex sets of $\mathbb{R}^n$, the *support function* is the map $h_K : \mathbb{R}^n \to \mathbb{R}$ defined by

$$h_K(x) \ = \ \max\{x \cdot v \ : \ v \in K\}. \tag{4}$$

Figure 1 illustrates this in the case of a unit vector input. Since support functions uniquely determine compact convex sets (as explained in Section 2), one can attempt to define an operation $K * L$ implicitly by means of the output of its support function $h_{K*L}(x)$. Gardner, Hug and Weil considered the formula

$$h_{K*L}(x) \ = \ h_M\big(h_K(x), h_L(x)\big) \tag{5}$$

for all $x \in \mathbb{R}^n$ and for some fixed $M \subset \mathbb{R}^2$. This created an "M-addition," or a way of combining elements of $\mathcal{K}^n$ which would potentially depend upon each specific M. They discovered that the right-hand side of (5) is a support function for all $K, L \in \mathcal{K}^n$ if and only if M is an element of $\mathcal{K}^2$ contained in the positive quadrant [1, Corollary 6.6], and in this case $*$ is $\oplus_M$ as defined in (1) or (2) with $m = 2$. They also proved that if both $K, L \in \mathcal{K}^n_s$, the class of origin-symmetric compact convex sets in $\mathbb{R}^n$, and $M \in \mathcal{K}^2$ is symmetric with respect to each axis, then (5) gives the support function of Protasov's $K \oplus_M L$ [1, Theorem 6.5(ii)].

Having made the connection between the implicit definition (5) and the explicit operation $K \oplus_M L$, Gardner, Hug and Weil went on to define $\oplus_M(A_1, \ldots, A_m)$ for arbitrary $M \subset \mathbb{R}^m$ and arbitrary $A_1, \ldots, A_m \subset \mathbb{R}^n$ as in (1) and (2). They also initiated a formal study of the properties of this operation, the fruits of which became Section 6 of [1]. We will restate several of these important results in Section 3.1 and liberally use them throughout the study.

Beyond inaugurating systematic research into M-addition, the authors of [1] also discovered a special link between projection covariant operations and M-addition. By *projection covariant*, we mean an operation $*$ satisfying

$$(K|S) * (L|S) \ = \ (K * L)|S$$

for all appropriate K and L and any subspace S of $\mathbb{R}^n$, where $A|S$ denotes the orthogonal projection of $A \subset \mathbb{R}^n$ onto S. They proved that a binary operation $* : (\mathcal{K}^n_s)^2 \to \mathcal{K}^n$ is projection covariant if and only if it is M-addition for some $M \in \mathcal{K}^2$ which is symmetric with respect to both axes [1, Theorem 7.6].

Two slightly more distant links to M-addition also emerged, both in the context of projection covariance. (We recall that a map $K \mapsto \Diamond K$ which acts on individual sets is projection covariant if

$$(\Diamond K)|S \;=\; \Diamond(K|S)$$

for all appropriate $K \subset \mathbb{R}^n$ and any subspace S of $\mathbb{R}^n$.) The first was that when $n \geq 2$, a map $\Diamond : \mathcal{K}^n \to \mathcal{K}^n_s$ (known as an *origin symmetrization*) is projection covariant if and only if there exists $M \in \mathcal{K}^2$ that is symmetric in the line $x_1 = x_2$ such that

$$h_{\Diamond K}(x) \;=\; h_M\big(h_K(x), h_{-K}(x)\big) \tag{6}$$

for all $K \in \mathcal{K}^n$ and all $x \in \mathbb{R}^n$ [1, Theorem 8.2]. The second result stated that a binary map $* : (\mathcal{K}^n)^2 \to \mathcal{K}^n$, $n \geq 2$, is projection covariant if and only if there is a closed convex $M \subset \mathbb{R}^4$ such that

$$h_{K*L}(x) \;=\; h_M\big(h_{-K}(x), h_K(x), h_{-L}(x), h_L(x)\big) \tag{7}$$

for all $K, L \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$ [1, Theorem 9.7]. The connection to M-addition here lies in the similarity between these support functions and (5), which we saw was the support function of $K \oplus_M L$ under certain conditions on M. Indeed, we might be tempted to conclude from (5) that the explicit descriptions of the operations in (6) and (7) are $K \oplus_M (-K)$ and $\oplus_M(-K, K, -L, L)$, respectively. But the precise link to M-addition here is not this simple because $h_{\oplus_M(K_1,\ldots,K_m)}(x)$ does not always equal $h_M(h_{K_1}(x), \ldots, h_{K_m}(x))$, such as when M is not contained in the positive orthant (see Section 3.3 for more details on the support function of an M-sum). In fact, it is currently unknown for what $M \subset \mathbb{R}^2$

$$h_{K \oplus_M (-K)}(x) \;=\; h_M\big(h_K(x), h_{-K}(x)\big)$$

and for what $M \subset \mathbb{R}^4$

$$h_{\oplus_M(-K,K,-L,L)}(x) \;=\; h_M\big(h_{-K}(x), h_K(x), h_{-L}(x), h_L(x)\big).$$

Thus these two results suggest a connection to M-addition, but the precise details remain unclear.

2 Definitions and preliminaries

Prior to embarking upon our study, we pause for a moment to ensure that the author and reader are speaking the same mathematical dialect. For the sake of having all notation defined in one section, we take the liberty to repeat several definitions made in the introduction.

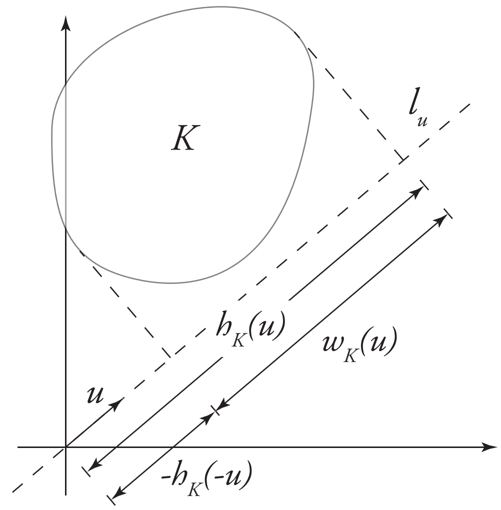


Figure 1: The support function.

All our sets are nonempty, and we draw the indices m, n and p from the natural numbers. The unit sphere in $\mathbb{R}^n$ is S^{n-1}. A *closed orthant* of $\mathbb{R}^n$ is one of the 2^n regions where for any $j \in \{1, \ldots, n\}$, the sign of α_j is the same or $\alpha_j = 0$ for all $(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_m)$ in the set. We write $\{e_1, \ldots, e_n\}$ for the standard basis of $\mathbb{R}^n$, o for the origin, and use $[v, w]$ for the line segment with endpoints $v, w \in \mathbb{R}^n$. If $v \in \mathbb{R}^n \backslash \{o\}$, $v^\perp = \{w \in \mathbb{R}^n : w \cdot v = 0\}$ and $\ell_v = \operatorname{span}\{v\}$. For $A \subset \mathbb{R}^n$, ∂A is the *boundary* of A, and if S is a subspace of $\mathbb{R}^n$ and $v \in \mathbb{R}^n$, then $A|S$ and $v|S$ are, respectively, the orthogonal projection of A onto S and the orthogonal projection of v onto S. The number of elements in a finite set I is $|I|$.

The collection of all nonempty compact convex subsets of $\mathbb{R}^n$ is $\mathcal{K}^n$, while $\mathcal{K}^n_o \subset \mathcal{K}^n$ and $\mathcal{K}^n_s \subset \mathcal{K}^n$ are the subclasses of elements that contain the origin or are origin symmetric, respectively. Recall that A is *origin symmetric* or *o-symmetric* when $v \in A$ implies $-v \in A$ for all $v \in A$. The class of nonempty compact subsets of $\mathbb{R}^n$ is $\mathcal{C}^n$. The *convex hull* $\operatorname{conv} A$ of a set A is the intersection of all convex sets containing A. We call an element of $\mathcal{K}^n$ whose interior is nonempty a *convex body*, while a *convex polytope* is the convex hull of a finite collection of points.

Convex combination scalars are finite collections of nonnegative constants $\lambda_1, \ldots, \lambda_n$ that sum to one. A linear combination $\sum_{j=1}^n \lambda_j v_j$ using such weights is called a *convex combination* of $v_1, \ldots, v_n$. Hence (3) says that $\operatorname{conv} A$ is the set of all convex combinations of elements of A.

One of the principal analytic tools at the disposal of the geometer to study a compact convex set K is the *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ defined by (4). In a direction $u \in S^{n-1}$, we informally think of $h_K(u)$ as providing the signed distance along ℓ_u from o to the nearest or furthest edge of $K|\ell_u$, as illustrated in Figure 1 (this explanation can be made precise through using the notion of supporting hyperplanes, but our current study does not demand these technicalities).

The support function enjoys several useful properties. First, for any $\alpha_1, \ldots, \alpha_m \geq 0$, $K_1, \ldots, K_m \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$,

$$h_{\sum_{j=1}^m \alpha_j K_j}(x) \;=\; \sum_{j=1}^{m} \alpha_j h_{K_j}(x). \tag{8}$$

This follows from the definition and the fact that $\max(A+B) = \max A + \max B$ for any $A, B \subset \mathbb{R}$ whose maxima exist. Other immediate consequences of definition (4) include that $h_{\beta K}(x) = h_K(\beta x)$ for any $\beta \in \mathbb{R}$, and that $h_K(\beta x) = \beta h_K(x)$ if $\beta \geq 0$. The latter implies that h_K is entirely determined by its values on S^{n-1}. Also note

$$K \;\subset\; L \quad \text{ if and only if } \quad h_K(x) \;\leq\; h_L(x) \tag{9}$$

for all $x \in \mathbb{R}^n$, as we may infer from considering the definition from a geometric point of view. Thus the support function uniquely determines a compact convex set.

The *width function* w_K of $K \in \mathcal{K}^n$ is built from the support function via the formula $w_K(u) = h_K(u) + h_K(-u)$. This measures the length of $K|\ell_u$ in a direction $u \in S^{n-1}$, as also illustrated in Figure 1. Notice that

$$\begin{aligned} h_K(u) + h_K(-u) \;&=\; \max\{u \cdot v \;:\; v \in K\} + \max\{-u \cdot v \;:\; v \in K\} \\ &=\; \max\{u \cdot v \;:\; v \in K\} - \min\{u \cdot v \;:\; v \in K\}, \end{aligned}$$

and so $w_K \geq 0$.

As the reader undoubtedly surmised from definition (1) of M-addition, Minkowski addition and the *dilatation of a set by* $\alpha \in \mathbb{R}$, defined as $\alpha A = \{\alpha v \;:\; v \in A\}$, will be central to our investigations. We therefore do well at the outset to clarify the somewhat counterintuitive relationship between them. One always has $\alpha(A+B) = \alpha A + \alpha B$, but we can only say

$$(\alpha+\beta)A \subset \alpha A + \beta A \quad \text{ or, more generally, } \quad \left(\sum_{j=1}^{m} \alpha_j\right) A \;\subset\; \sum_{j=1}^{m} \alpha_j A, \tag{10}$$

where $\alpha, \beta, \alpha_1, \ldots, \alpha_m \in \mathbb{R}$. To see an example where containment is proper, consider $\alpha = 1, \beta = -1$ and $K = [0,1]^2$. Then $(1-1)K = 0K = \{o\}$, while

$$1K - 1K \;=\; K - K \;=\; \{\, (\alpha_1, \alpha_2) - (\beta_1, \beta_2) \;:\; 0 \leq \alpha_1, \alpha_2, \beta_1, \beta_2 \leq 1 \,\} \;=\; [-1,1]^2.$$

Not only can we only generally ensure containment instead of equality in (10), under certain circumstances this containment is always proper.

Lemma 1. *Let $K \in \mathcal{K}^n$ be a convex body. If $\alpha, \beta > 0$, then*

$$(\beta - \alpha)K \subsetneq \beta K + (-\alpha)K. \tag{11}$$

Proof. If $\alpha = \beta$, then $(\beta - \alpha)K = \{o\}$. However, since there exists $v, w \in K$ such that $v \neq w$, $o \neq \beta v - \alpha w \in \beta K + (-\alpha)K$, showing (11) holds. Suppose $\alpha < \beta$. Then for $u \in S^{n-1}$,

$$\begin{aligned} h_{\beta K+(-\alpha K)}(u) &= \beta h_K(u) + \alpha h_K(-u) \\ &= (\beta - \alpha)h_K(u) + \alpha h_K(u) + \alpha h_K(-u) \;=\; h_{(\beta-\alpha)K}(u) + \alpha w_K(u). \end{aligned}$$

Since $0 \leq w_K(u)$, we see that for all $u \in S^{n-1}$,

$$h_{(\beta-\alpha)K}(u) \;\leq\; h_{(\beta-\alpha)K}(u) + \alpha w_K(u) \;=\; h_{\beta K+(-\alpha K)}(u), \tag{12}$$

and so $(\beta - \alpha)K \subset \beta K + (-\alpha)K$ by (9). Since K is not a singleton set, $0 < w_K(u_0)$ for some $u_0 \in S^{n-1}$, and hence strict inequality holds in (12) with u replaced by u_0, implying $(\beta - \alpha)K \subsetneq \beta K + (-\alpha)K$.

If $\beta < \alpha$, then since $-K \in \mathcal{K}^n$ and for any $\gamma, \delta \in \mathbb{R}$ one has $\gamma(\delta K) = (\gamma\delta)K$, by the above

$$(\beta - \alpha)K \;=\; (\alpha - \beta)(-K) \;\subsetneq\; \alpha(-K) + (-\beta)(-K) \;=\; \beta K + (-\alpha)K.$$

□

This addresses the case when the scalars in (10) have opposite signs. When α and β are both nonnegative or both nonpositive,

$$(\alpha + \beta)K \;=\; \alpha K + \beta K \tag{13}$$

if and only if K is convex. Schneider [4, Remark 1.1.1] proves this assertion in the case of nonnegative scalars, and for nonpositive ones, start with $\alpha, \beta \geq 0$ and consider the equation $(-\alpha - \beta)K = -\alpha K + (-\beta)K$. Multiplying through by -1 yields

$$-(-\alpha - \beta)K \;=\; (\alpha + \beta)K \;=\; -(-\alpha K + (-\beta)K) \;=\; \alpha K + \beta K,$$

and so this is equivalent to the nonnegative case. Thus (13) holds for nonpositive α and β precisely when K is convex.

Minkowski addition is a special case of L_p *addition*. For $1 \leq p < \infty$, the L_p sum of $K, L \in \mathcal{K}^n_o$ is usually defined implicitly via support functions to be the set $K +_p L$ satisfying

$$h_{K+_pL}(u)^p \;=\; h_K(u)^p + h_L(u)^p \tag{14}$$

for all $u \in S^{n-1}$. The case $p = 1$ corresponds to Minkowski addition and here K and L need not contain the origin. When $p = \infty$, we define

$$h_{K+_\infty L}(u) \;=\; \max\{h_K(u), h_L(u)\}, \tag{15}$$

and here again $K, L \in \mathcal{K}^n$ do not have to contain the origin. For $1 \leq p < \infty$ and $K, L \in \mathcal{K}^n_o$, Lutwak, Yang and Zhang [2] recently clarified the nature of this operation by providing the explicit pointwise formula

$$K +_p L \; = \; \{ (1-\alpha)^{1/q} v + \alpha^{1/q} w \; : \; 0 \leq \alpha \leq 1, \, v \in K, \, w \in L \}, \tag{16}$$

where q is the Hölder conjugate of p, i.e.

$$\frac{1}{p} + \frac{1}{q} \; = \; 1.$$

(When $p = 1$ and $q = \infty$, we say $1/q = 0$.) Comparing (16) with (2) shows the similarity of this new formulation to M-addition. Indeed, (16) is $K \oplus_M L$ for

$$M \; = \; \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 \; : \; \alpha_1, \alpha_2 \geq 0 \quad \text{and} \quad \alpha_1^q + \alpha_2^q = 1\}.$$

Generalizing this to the m-ary case, we see that for $K_1, \ldots, K_m \in \mathcal{K}^n_o$ and $1 \leq p < \infty$,

$$K_1 +_p \cdots +_p K_m \; = \; \oplus_M (K_1, \ldots, K_m)$$

for

$$M \; = \; \left\{ (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m \; : \; \alpha_1, \ldots, \alpha_m \geq 0 \quad \text{and} \quad \sum_{j=1}^{m} \alpha_j^q = 1 \right\}. \tag{17}$$

One consequence of the current study is that this pointwise definition is also valid when $p = \infty$, $q = 1$ and $K_1, \ldots, K_m \in \mathcal{K}^n$, as discussed in Section 4.3.

Four flavors of symmetry will repeatedly surface in our investigations. We have already encountered the first, namely origin symmetry or o-symmetry. A close relative is *central symmetry*, which belongs to a set when a translate of it is o-symmetric. Secondly, $A \subset \mathbb{R}^n$ is *1-unconditional* if

$$\{(\varepsilon_1 v_1, \ldots, \varepsilon_n v_n) \; : \; \varepsilon_j \in \{-1, 1\}, \, (v_1, \ldots, v_n) \in A\} \; \subset \; A.$$

Such sets are thus symmetric with respect to each coordinate hyperplane and are always o-symmetric. The related *1-unconditional hull* of A is

$$\widehat{A} \; = \; \{(\alpha_1 v_1, \ldots, \alpha_n v_n) \; : \; -1 \leq \alpha_j \leq 1, \, (v_1, \ldots, v_m) \in A\}.$$

Note that if C is 1-unconditional and convex, then $\widehat{C} = C$ since convex sets contain the line segments between their points. The notation for the 1-unconditional hull of $\oplus_M(A_1, \ldots, A_m)$ is $\widehat{\oplus}_M(A_1, \ldots, A_m)$.

We coin the third and fourth varieties of symmetry to describe circumstances significant in M-addition. *Positive symmetry* belongs to $A \subset \mathbb{R}^n$ when

$$\{(|v_1|, \ldots, |v_m|) \; : \; (v_1, \ldots, v_m) \in A\} \; \subset \; A,$$

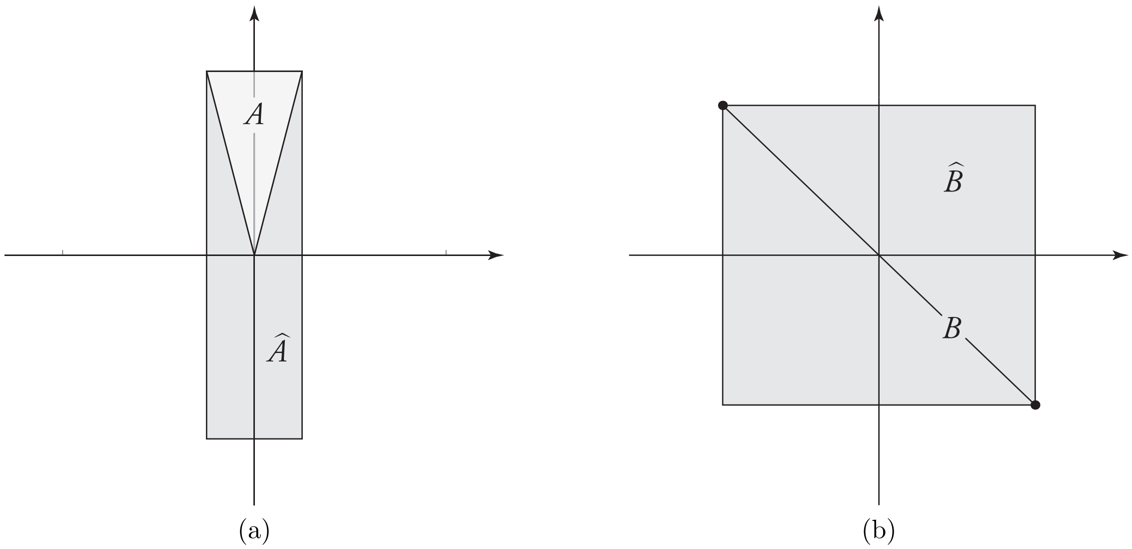


Figure 2: Positive (and negative) symmetry is distinct from o-symmetry.

and A has *negative symmetry* when

$$\{(-|v_1|, \ldots, -|v_m|) \;:\; (v_1, \ldots, v_m) \in A\} \;\subset\; A.$$

In words, A has positive or negative symmetry when the reflection of any $v \in A$ into the positive or negative orthant, respectively, still belongs to A. One-unconditionality is again stronger than both positive and negative symmetry, whereas o-symmetry is distinct. The latter is illustrated in Figure 2, where set A in Figure 2a has positive symmetry but is not o-symmetric, while the line segment B in Figure 2b is o-symmetric without possessing either positive or negative symmetry. Figures 2a and 2b also display the 1-unconditional hulls $\widehat{A}$ and $\widehat{B}$ of these sets.

We will occasionally use the *Hausdorff metric* to measure the distance from $C \in \mathcal{C}^n$ to its convex hull, and thus obtain a sense of "how far" C is from being convex. We can formulate the definition of this metric in terms of dilatations of the *closed unit ball* $B^n = \{x \in \mathbb{R}^n \;:\; |x| \leq 1\}$, where $|x|$ is the *Euclidean norm* of x. The Hausdorff distance between $C, D \in \mathcal{C}^n$ is then

$$\delta(C, D) \;=\; \min\{\, \lambda \geq 0 \;:\; C \subset D + \lambda B^n \;\text{ and }\; D \subset C + \lambda B^n \,\}. \tag{18}$$

Note that $C + \lambda B^n$ is simply the union of all closed λ-neighborhoods around elements of C,

$$C + \lambda B^n \;=\; \bigcup_{x \in C} B(x, \lambda) \;=\; \bigcup_{x \in C} \{\, y \in \mathbb{R}^n \;:\; |x - y| \leq \lambda \,\}.$$

We will distinguish results from other papers from those original to this thesis by calling the former "Propositions" and the latter "Theorems." While clarifying our contributions, this has the acknowledged downside of ascribing the more distinguished name of "Theorem" to some results which scarcely seem to merit it from the standpoint of difficulty.

3 Foundational properties

In this section we detail some of the fundamental properties of M-addition. While several of these characteristics are basic and have simple proofs, they warrant statement because of their utility to our future investigations. Some more complex properties will not be used again but rather suggest avenues for further research in M-addition theory.

3.1 Results from *Operations between sets in geometry*

Before stating any new results, however, we restate some of the conclusions from [1, Section 6] that we will repeatedly draw upon.

Proposition 2. *[1, Theorem 6.1 (i)] Let $m \geq 2$ and let M be a subset of $\mathbb{R}^m$. If $m \leq n$, the operation $\oplus_M$ maps $(\mathcal{K}^n)^m$ to $\mathcal{K}^n$ if and only if $M \in \mathcal{K}^m$ and M is contained in one of the 2^m closed orthants of $\mathbb{R}^m$. (The assumption $m \leq n$ is needed only to conclude that $M \in \mathcal{K}^m$)*

Proposition 3. *[1, Lemma 6.2] If $M \subset \mathbb{R}^m$, then for any o-symmetric convex $C_1, \ldots, C_m \subset \mathbb{R}^n$,*

$$\oplus_M(C_1, \ldots, C_m) \;=\; \oplus_{\widehat{M}}(C_1, \ldots, C_m).$$

The actual statement of the above [1, Lemma 6.2] takes the C_j's to also be compact and hence elements of $\mathcal{K}^n_s$. However, the proof only uses o-symmetry and convexity.

Proposition 4. *[1, Theorem 6.3] Let $2 \leq m \leq n$ and let M be a compact subset of $\mathbb{R}^m$. Then $\oplus_M : (\mathcal{K}^n_s)^m \to \mathcal{K}^n_s$ if and only if the 1-unconditional hull $\widehat{M}$ of M is convex. (The assumption $m \leq n$ is not needed in the "if" direction.)*

3.2 Linear transformations

We commence by considering the interplay between M-addition and linear transformations.

Theorem 5. *If $M \subset \mathbb{R}^m$, $A_1, \ldots, A_m \subset \mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}^p$ is linear, then*

$$\oplus_M(\phi A_1, \ldots, \phi A_m) \;=\; \phi\big(\oplus_M\,(A_1, \ldots, A_m)\big).$$

Proof. The union definition (1) of M-addition and the linearity of ϕ yield

$$\begin{aligned}\oplus_M(\phi A_1,\ldots,\phi A_m) &= \bigcup_{(\alpha_1,\ldots,\alpha_m)\in M} \sum_{j=1}^m \alpha_j \phi A_j \\ &= \bigcup_{(\alpha_1,\ldots,\alpha_m)\in M} \phi\Big(\sum_{j=1}^m \alpha_j A_j\Big) \\ &= \phi\Big(\bigcup_{(\alpha_1,\ldots,\alpha_m)\in M} \sum_{j=1}^m \alpha_j A_j\Big) \;=\; \phi\big(\oplus_M (A_1,\ldots,A_m)\big). \qquad \square\end{aligned}$$

Theorem 6. *If $M \subset \mathbb{R}^m$, $A_1,\ldots,A_m \subset \mathbb{R}^n$ and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is a diagonal matrix with components $\phi_{jj} = \beta_j$, $j \in \{1,\ldots,m\}$, then*

$$\oplus_{\phi M}(A_1,\ldots,A_m) \;=\; \oplus_M(\beta_1 A_1,\ldots,\beta_m A_m).$$

Proof. Since elements of ϕM have the form $(\beta_1\alpha_1,\ldots,\beta_m\alpha_m)$ for some $(\alpha_1,\ldots,\alpha_m) \in M$,

$$\begin{aligned}\oplus_{\phi M}(A_1,\ldots,A_m) &= \bigcup_{(\alpha_1,\ldots,\alpha_m)\in M} \sum_{j=1}^m \beta_j\alpha_j A_j \\ &= \bigcup_{(\alpha_1,\ldots,\alpha_m)\in M} \sum_{j=1}^m \alpha_j(\beta_j A_j) \;=\; \oplus_M(\beta_1 A_1,\ldots,\beta_m A_m). \qquad \square\end{aligned}$$

Thus in M-addition, dilating the M-set parallel to the jth axis in $\mathbb{R}^m$ is equivalent to scaling the jth summand $A_j \subset \mathbb{R}^n$.

Corollary 7. *If $M \subset \mathbb{R}^m$, $A_1,\ldots,A_m \subset \mathbb{R}^n$ and $\beta \in \mathbb{R}$, then*

$$\oplus_{\beta M}(A_1,\ldots,A_m) \;=\; \oplus_M(\beta A_1,\ldots,\beta A_m).$$

Proof. Since $\beta M = \beta I_m M$, where I_m is the $m \times m$ identity matrix, this is an immediate consequence of Theorem 6. $\square$

Considering βA_j as $\beta I_n A_j$, we note that Corollary 7 and Theorem 5 together yield

$$\oplus_{\beta M}(A_1,\ldots,A_m) \;=\; \oplus_M(\beta A_1,\ldots,\beta A_m) \;=\; \beta \oplus_M (A_1,\ldots,A_m) \tag{19}$$

for any $M \subset \mathbb{R}^m$, $A_1,\ldots,A_m \subset \mathbb{R}^n$ and $\beta \in \mathbb{R}$.

Theorem 6 naturally leads one to wonder if

$$\oplus_{\phi M}(A_1,\ldots,A_m) \;=\; \oplus_M(\phi A_1,\ldots,\phi A_m) \tag{20}$$

for *any* linear $\phi : \mathbb{R}^m \to \mathbb{R}^m$ when M, $A_1,\ldots,A_m$ all reside in $\mathbb{R}^m$. A simple counterexample in $\mathbb{R}^2$, however, shows that this is not the case.

Example 8. Consider the sets $K = M = [-1,1]^2$ and $L = [-1,1] \times [-2,2]$. Then if $N = \partial M \cap [0,\infty)^2$, we have $\widehat{N} = M$, and so by the o-symmetry of K and L and Proposition 3,

$$K \oplus_M L \;=\; K \oplus_{\widehat{N}} L \;=\; K \oplus_N L.$$

We claim that $K \oplus_N L = K + L$. Note that N is the union of two line segments

$$N \;=\; [e_1, e_1 + e_2] \cup [e_2, e_1 + e_2] \;=\; N_1 \cup N_2,$$

say. Since by o-symmetry $\alpha_2 L \subset L$ for $0 \le \alpha_2 \le 1$, we can see that

$$K \oplus_{N_1} L \;=\; \bigcup \big\{\, 1K + \alpha_2 L \;:\; 0 \le \alpha_2 \le 1 \,\big\} \;\subset\; K + L.$$

Since $K + L \subset K \oplus_{N_1} L$ as well, $K \oplus_{N_1} L = K + L$. Similarly, as $(1,1) \in N_2$ and K is o-symmetric,

$$K + L \;\subset\; K \oplus_{N_2} L \;=\; \bigcup \big\{\, \alpha_1 K + L \;:\; 0 \le \alpha_1 \le 1 \,\big\} \;\subset\; K + L,$$

showing that $K \oplus_{N_2} L = K + L$, and thus

$$K \oplus_N L \;=\; K + L \;=\; [-2,2] \times [-3,3].$$

Now take $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ to be the counterclockwise rotation of the plane about the origin by $\pi/2$. Using Theorem 5 and the fact that $\phi M = M$, we have

$$\begin{aligned} \phi K \oplus_M \phi L \;=\; \phi\big(K \oplus_M L\big) \;&=\; [-3,3] \times [-2,2] \\ &\neq\; [-2,2] \times [-3,3] \;=\; K \oplus_M L \;=\; K \oplus_{\phi M} L, \end{aligned}$$

and so (20) does not hold in general.

A moment's reflection shows that this is entirely reasonable. The set M provides the weights in the Minkowski linear combinations

$$\bigcup_{(\alpha_1,\ldots,\alpha_m) \in M} \; \sum_{j=1}^{m} \alpha_j A_j \;=\; \oplus_M (A_1, \ldots, A_m).$$

If we rotate these m-tuples of scalars (or apply any other linear transformation), there is no *a priori* reason why the union should undergo the *same* action.

3.3 The support function

Given the prominence of the support function in the study of convex sets, any consideration of M-addition which neglected it would be incomplete. Gardner, Hug and Weil laid the groundwork in this respect by providing a formula [1, Theorem 6.5] for the support function of an M-sum. In this section we offer a fresh proof of part (i) of their result, which we restate as Proposition 10.

Lemma 9. *Suppose that $K_\alpha \in \mathcal{K}^n$, $\alpha \in I$, and that $\bigcup_{\alpha\in I} K_\alpha \in \mathcal{K}^n$. Then for any $u \in S^{n-1}$,*

$$h_{\bigcup_{\alpha\in I} K_\alpha}(u) \;=\; \max_{\alpha\in I}\{h_{K_\alpha}(u)\}.$$

Proof. By definition,

$$h_{\bigcup_{\alpha\in I} K_\alpha}(u) \;=\; \max_{x\,\in\,\bigcup_{\alpha\in I} K_\alpha}\{x\cdot u\} \;=\; \max_{\alpha\in I}\left\{\max_{x\in K_\alpha}\{x\cdot u\}\right\} \;=\; \max_{\alpha\in I}\{h_{K_\alpha}(u)\}. \qquad \square$$

Proposition 10. *[1, Theorem 6.5(i)] Let $M \in \mathcal{K}^m$ be contained in one of the 2^m closed orthants of $\mathbb{R}^m$. Let $\varepsilon_j = \pm 1$, $j \in \{1,\ldots,m\}$, denote the sign of the jth coordinate of a point in the interior of this orthant and let*

$$M^+ \;=\; \{\,(\varepsilon_1\alpha_1,\ldots,\varepsilon_m\alpha_m) \;:\; (\alpha_1,\ldots,\alpha_m)\in M\} \tag{21}$$

be the reflection of M into $[0,\infty)^m$. If $K_1,\ldots,K_m \in \mathcal{K}^n$, then for all $x\in\mathbb{R}^n$,

$$h_{\oplus_M(K_1,\ldots,K_m)}(x) \;=\; h_{M^+}\big(h_{\varepsilon_1 K_1}(x),\ldots,h_{\varepsilon_m K_m}(x)\big).$$

Proof. Let $x\in\mathbb{R}^n$. From definition (1) of M-addition,

$$h_{\oplus_M(K_1,\ldots,K_m)}(x) \;=\; h_{\cup_{(\alpha_1,\ldots,\alpha_m)\in M}\sum_{j=1}^m \alpha_j K_j}(x).$$

Since any dilatation $\alpha_j K_j$ of a convex set K_j is convex and the Minkowski sum of convex sets is convex, $\sum_{j=1}^m \alpha_j K_j$ is convex for each $(\alpha_1,\ldots,\alpha_m)\in M$. It is also easy to see that this sum is compact, and hence is an element of $\mathcal{K}^n$. By Proposition 2,

$$\oplus_M(K_1,\ldots,K_m) \;=\; \bigcup_{(\alpha_1,\ldots,\alpha_m)\in M}\;\sum_{j=1}^m \alpha_j K_j \;\in\; \mathcal{K}^n,$$

and thus Lemma 9 is applicable. Using it along with property (8) of the support function combined with the fact that $\varepsilon_j\alpha_j \ge 0$ for each j, we have

$$\begin{aligned}
h_{\cup_{(\alpha_1,\ldots,\alpha_m)\in M}\sum_{j=1}^m \alpha_j K_j}(x) \;&=\; \max_{(\alpha_1,\ldots,\alpha_m)\in M}\left\{h_{\sum_{j=1}^m \alpha_j K_j}(x)\right\}\\
&=\; \max_{(\alpha_1,\ldots,\alpha_m)\in M}\left\{\sum_{j=1}^m h_{\alpha_j K_j}(x)\right\}\\
&=\; \max_{(\alpha_1,\ldots,\alpha_m)\in M}\left\{\sum_{j=1}^m h_{\varepsilon_j\alpha_j\varepsilon_j K_j}(x)\right\}\\
&=\; \max_{(\alpha_1,\ldots,\alpha_m)\in M}\left\{\sum_{j=1}^m \varepsilon_j\alpha_j h_{\varepsilon_j K_j}(x)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \max_{(\alpha_1,\dots,\alpha_m)\in M} \left\{ \big(h_{\varepsilon_1 K_1}(x),\dots,h_{\varepsilon_m K_m}(x)\big)\cdot(\varepsilon_1\alpha_1,\dots,\varepsilon_m\alpha_m)\right\} \\
&= \max_{(\beta_1,\dots,\beta_m)\in M^+} \left\{ \big(h_{\varepsilon_1 K_1}(x),\dots,h_{\varepsilon_m K_m}(x)\big)\cdot(\beta_1,\dots,\beta_m)\right\} \\
&= h_{M^+}\big(h_{\varepsilon_1 K_1}(x),\dots,h_{\varepsilon_m K_m}(x)\big),
\end{aligned}
$$

where the last equality follows from (4). □

3.4 Unions and intersections

We next turn our attention to the interplay between M-addition and unions and intersections of sets. Unions are by far the easier of the two cases, primarily because M-addition *is* a union (see definition (1)) and taking unions is a commutative operation.

Theorem 11. *If $M_\beta \subset \mathbb{R}^m$ for $\beta \in I$, and $A_1,\dots,A_m \subset \mathbb{R}^n$, then*

$$\oplus_{\bigcup_{\beta\in I} M_\beta}(A_1,\dots,A_m) \;=\; \bigcup_{\beta\in I} \oplus_{M_\beta}(A_1,\dots,A_m).$$

Proof. Using the union definition (1) of M-addition, we find

$$
\begin{aligned}
\oplus_{\bigcup_{\beta\in I} M_\beta}(A_1,\dots,A_m) \;&=\; \bigcup_{(\alpha_1,\dots,\alpha_m)\in \bigcup_{\beta\in I} M_\beta} \sum_{j=1}^m \alpha_j A_j \\
&= \bigcup_{\beta\in I} \; \bigcup_{(\alpha_1,\dots,\alpha_m)\in M_\beta} \sum_{j=1}^m \alpha_j A_j \;=\; \bigcup_{\beta\in I} \oplus_{M_\beta}(A_1,\dots,A_m).
\end{aligned}
$$
□

Theorem 12. *Let $M \subset \mathbb{R}^m$ and let $A_{\beta_j} \subset \mathbb{R}^n$ for $\beta_j \in I_j$, $j \in \{1,\dots,m\}$. Then*

$$\oplus_M\Bigg(\bigcup_{\beta_1\in I_1} A_{\beta_1},\;\dots,\;\bigcup_{\beta_m\in I_m} A_{\beta_m}\Bigg) \;=\; \bigcup_{\beta_1\in I_1}\cdots\bigcup_{\beta_m\in I_m} \oplus_M(A_{\beta_1},\dots,A_{\beta_m}).$$

Proof. By the pointwise definition (2) of M-addition,

$$
\begin{aligned}
\oplus_M\Bigg(\bigcup_{\beta_1\in I_1} A_{\beta_1},\;\dots,\;\bigcup_{\beta_m\in I_m} A_{\beta_m}\Bigg) \;&=\; \left\{ \sum_{j=1}^m \alpha_j v_j \;:\; (\alpha_1,\dots,\alpha_m)\in M,\; v_j \in \bigcup_{\beta_j\in I_j} A_{\beta_j}\right\} \\
&= \bigcup_{\beta_1\in I_1}\cdots\bigcup_{\beta_m\in I_m} \left\{ \sum_{j=1}^m \alpha_j v_j \;:\; (\alpha_1,\dots,\alpha_m)\in M,\; v_j \in A_{\beta_j}\right\} \\
&= \bigcup_{\beta_1\in I_1}\cdots\bigcup_{\beta_m\in I_m} \oplus_M(A_{\beta_1},\dots,A_{\beta_m}).
\end{aligned}
$$
□

As we will often find occasion to do, we state this result for two important particular instances of M-addition in the following corollary.

Corollary 13. (i) *If $A_{\beta_j} \subset \mathbb{R}^n$ for $\beta_j \in I_j$, $j \in \{1, \dots, m\}$, then*

$$\sum_{j=1}^{m} \bigcup_{\beta_j \in I_j} A_{\beta_j} = \bigcup_{\beta_1 \in I_1} \cdots \bigcup_{\beta_m \in I_m} \sum_{j=1}^{m} A_{\beta_j}.$$

(ii) *Let $K_{\beta_j} \in \mathcal{K}_o^n$ for $\beta_j \in I_j$, $j \in \{1, \dots, m\}$, such that $\bigcup_{\beta_j \in I_j} K_{\beta_j} \in \mathcal{K}_o^n$ for each j. Then for any $1 < p \leq \infty$,*

$$\left(\bigcup_{\beta_1 \in I_1} K_{\beta_1}\right) +_p \cdots +_p \left(\bigcup_{\beta_m \in I_m} K_{\beta_m}\right) = \bigcup_{\beta_1 \in I_1} \cdots \bigcup_{\beta_m \in I_m} (K_{\beta_1} +_p \cdots +_p K_{\beta_m}). \tag{22}$$

Proof. (i) Since the Minkowski sum of m sets is the same as $\oplus_M$ for

$$M = \{(1, 1, \dots, 1)\} \subset \mathbb{R}^m,$$

by Theorem 12

$$\begin{aligned}\sum_{j=1}^{m} \bigcup_{\beta_j \in I_j} A_{\beta_j} &= \oplus_M\Big(\bigcup_{\beta_1 \in I_1} A_{\beta_1}, \dots, \bigcup_{\beta_m \in I_m} A_{\beta_m}\Big) \\ &= \bigcup_{\beta_1 \in I_1} \cdots \bigcup_{\beta_m \in I_m} \oplus_M(A_{\beta_1}, \dots, A_{\beta_m}) = \bigcup_{\beta_1 \in I_1} \cdots \bigcup_{\beta_m \in I_m} \sum_{j=1}^{m} A_{\beta_j}.\end{aligned}$$

(ii) By the assumptions on the unions and the individual K_{β_j}'s, the L_p sums on the left-hand side and on the right-hand side of (22), respectively, make sense. Since the L_p sum of m sets is $\oplus_M$ for the M in (17), by Theorem 12,

$$\begin{aligned}\left(\bigcup_{\beta_1 \in I_1} K_{\beta_1}\right) +_p \cdots +_p \left(\bigcup_{\beta_m \in I_m} K_{\beta_m}\right) &= \oplus_M\Big(\bigcup_{\beta_1 \in I_1} K_{\beta_1}, \dots, \bigcup_{\beta_m \in I_m} K_{\beta_m}\Big) \\ &= \bigcup_{\beta_1 \in I_1} \cdots \bigcup_{\beta_m \in I_m} \oplus_M(K_{\beta_1}, \dots, K_{\beta_m}) \\ &= \bigcup_{\beta_1 \in I_1} \cdots \bigcup_{\beta_m \in I_m} (K_{\beta_1} +_p \cdots +_p K_{\beta_m}).\end{aligned}$$ □

In contrast to unions, intersections relate to M-sums in a much more complex manner, and this will be reflected in the scope of the following results. For the intersection version of Theorem 11, we will only have one direction of set containment instead of equality. And rather than the sweeping conclusion of Theorem 12, the intersection analogue will require a convexity condition which will allow intersections of just *pairs* of sets to move outside an M-sum.

Theorem 14. *Let $M_\beta \subset \mathbb{R}^m$, $\beta \in I$, be such that $\bigcap_{\beta\in I} M_\beta \neq \emptyset$. Then for any $A_1,\dots,A_m \subset \mathbb{R}^n$,*

$$\oplus_{\bigcap_{\beta\in I} M_\beta}(A_1,\dots,A_m) \ \subset\ \bigcap_{\beta\in I} \oplus_{M_\beta}(A_1,\dots,A_m).$$

Proof. If $x \in \oplus_{\bigcap_{\beta\in I} M_\beta}(A_1,\dots,A_m)$, then there exist $(\alpha_1,\dots,\alpha_m) \in \bigcap_{\beta\in I} M_\beta$ and $v_j \in A_j$, $j \in \{1,\dots,m\}$, such that $x = \sum_{j=1}^m \alpha_j v_j$. Since $(\alpha_1,\dots,\alpha_m) \in M_\beta$ for every $\beta \in I$, $\sum_{j=1}^m \alpha_j v_j \in \oplus_{M_\beta}(A_1,\dots,A_m)$ for every $\beta \in I$, and thus $x \in \bigcap_{\beta\in I} \oplus_{M_\beta}(A_1,\dots,A_m)$. $\square$

Theorem 15. *Let $M \subset \mathbb{R}^m$ and suppose $n_1,\dots,n_m \in \mathbb{N}$ are such that for the sets*

$$A_{11}, A_{21},\ \dots,\ A_{n_1 1},\quad A_{12}, A_{22},\dots,\ A_{n_2 2},\quad \dots,\quad A_{1m}, A_{2m},\ \dots,\ A_{n_m m}\ \subset\ \mathbb{R}^n,$$

$\bigcap_{j=1}^{n_1} A_{j1}, \bigcap_{j=1}^{n_2} A_{j2},\dots,\bigcap_{j=1}^{n_m} A_{jm}$ are each nonempty. Then

$$\oplus_M\left(\bigcap_{j=1}^{n_1} A_{j1},\dots,\bigcap_{j=1}^{n_m} A_{jm}\right)\ \subset\ \bigcap_{j_1=1}^{n_1}\cdots\bigcap_{j_m=1}^{n_m} \oplus_M(A_{j_1 1},\dots,A_{j_m m}).$$

Proof. If $x \in \oplus_M\left(\bigcap_{j=1}^{n_1} A_{j1},\dots,\bigcap_{j=1}^{n_m} A_{jm}\right)$, then for some $(\alpha_1,\dots,\alpha_m) \in M$ and $v_k \in \bigcap_{j=1}^{n_k} A_{jk}$, $k \in \{1,\dots,m\}$, we have $x = \sum_{k=1}^m \alpha_k v_k$. Since each $v_k \in A_{jk}$ for every $j \in \{1,\dots,n_k\}$, we see that $x \in \oplus_M(A_{j_1 1},\dots,A_{j_m m})$ for arbitrary $j_1 \in \{1,\dots,n_1\},\dots,j_m \in \{1,\dots,n_m\}$, which is the same as saying

$$x\ \in\ \bigcap_{j_1=1}^{n_1}\cdots\bigcap_{j_m=1}^{n_m} \oplus_M(A_{j_1 1},\dots,A_{j_m m}). \qquad \square$$

Theorem 16. *Let M be convex and contained in one of the 2^m closed orthants of $\mathbb{R}^m$. Let $\ell \in \{1,\dots,m\}$ and suppose that the sets*

$$K_{11}, K_{21},\quad K_{12}, K_{22},\quad \dots,\quad K_{1\ell}, K_{2\ell}$$

are all members of $\mathcal{K}^n$. If $K_{1k} \cup K_{2k}$ is convex for each $k \in \{1,\dots,\ell\}$, then for any convex $C_{\ell+1},\dots,C_m \subset \mathbb{R}^n$,

$$\oplus_M\left(K_{11}\cap K_{21},\dots,K_{1\ell}\cap K_{2\ell}, C_{\ell+1},\dots,C_m\right) = \bigcap_{j_1=1}^{2}\cdots\bigcap_{j_\ell=1}^{2} \oplus_M(K_{j_1 1},\dots,K_{j_\ell \ell}, C_{\ell+1},\dots,C_m).$$

Proof. Let $k \in \{1,\dots,\ell\}$, $v_1 \in K_{1k}$ and $v_2 \in K_{2k}$. Then the closed sets $[v_1,v_2]\cap K_{1k}$ and $[v_1,v_2]\cap K_{2k}$ either have empty intersection and are separated by positive Euclidean distance or have nonempty intersection. By the convexity of the union, $[v_1,v_2] \subset K_{1k}\cup K_{2k}$, and thus the former option is not possible, implying $(1-\lambda)v_1+\lambda v_2 \in K_{1k}\cap K_{2k}$ for some $\lambda \in [0,1]$. In particular, $K_{1k}\cap K_{2k} \neq \emptyset$. By Theorem 15 it therefore suffices to demonstrate

$$\bigcap_{j_1=1}^{2}\cdots\bigcap_{j_\ell=1}^{2} \oplus_M(K_{j_1 1},\dots,K_{j_\ell \ell},C_{\ell+1},\dots,C_m) \subset \oplus_M(K_{11}\cap K_{21},\dots,K_{1\ell}\cap K_{2\ell},C_{\ell+1},\dots,C_m). \tag{23}$$

If $\ell \neq 1$, then let $C_k = K_{1k}\cap K_{2k}$ for $k \in \{2,\dots,\ell\}$. We begin by proving

$$\bigcap_{j=1}^{2} \oplus_M(K_{j1},C_2,\dots,C_m) \subset \oplus_M(K_{11}\cap K_{21},C_2,\dots,C_m). \tag{24}$$

If x is an element of the left-hand side, then for each $j \in \{1,2\}$,

$$x \;=\; \alpha_{1j}v_j + \sum_{i=2}^{m} \alpha_{ij}w_{ij} \tag{25}$$

for some $(\alpha_{1j},\dots,\alpha_{mj}) \in M$, $v_j \in K_{j1}$ and $w_{2j} \in C_2,\dots,w_{mj} \in C_m$. Note that if either $\alpha_{11} = 0$ or $\alpha_{12} = 0$, then we may replace v_1 or v_2, respectively, in the corresponding decomposition (25) of x with an element of $K_{11}\cap K_{21}$ and have $x \in \oplus_M(K_{11}\cap K_{21},C_2,\dots,C_m)$. Hence assume $\alpha_{11} \neq 0$ and $\alpha_{12} \neq 0$. If for each $i \in \{2,\dots,m\}$ either $\alpha_{i1} \neq 0$ or $\alpha_{i2} \neq 0$, then for any $\lambda \in (0,1)$,

$$\begin{aligned}
x \;&=\; (1-\lambda)x + \lambda x \\
&=\; (1-\lambda)\left(\alpha_{11}v_1 + \sum_{i=2}^{m}\alpha_{i1}w_{i1}\right) + \lambda\left(\alpha_{12}v_2 + \sum_{i=2}^{m}\alpha_{i2}w_{i2}\right) \\
&=\; (1-\lambda)\alpha_{11}v_1 + \lambda\alpha_{12}v_2 + \sum_{i=2}^{m}\big((1-\lambda)\alpha_{i1}w_{i1} + \lambda\alpha_{i2}w_{i2}\big) \\
&=\; \big((1-\lambda)\alpha_{11} + \lambda\alpha_{12}\big)\left(\frac{(1-\lambda)\alpha_{11}}{(1-\lambda)\alpha_{11}+\lambda\alpha_{12}}v_1 + \frac{\lambda\alpha_{12}}{(1-\lambda)\alpha_{11}+\lambda\alpha_{12}}v_2\right) \\
&\quad + \sum_{i=2}^{m}\big((1-\lambda)\alpha_{i1} + \lambda\alpha_{i2}\big)\left(\frac{(1-\lambda)\alpha_{i1}}{(1-\lambda)\alpha_{i1}+\lambda\alpha_{i2}}w_{i1} + \frac{\lambda\alpha_{i2}}{(1-\lambda)\alpha_{i1}+\lambda\alpha_{i2}}w_{i2}\right).
\end{aligned} \tag{26}$$

Since by the orthant condition on M both α_{i1} and α_{i2} are nonpositive or nonnegative for each $i \in \{1,\dots,m\}$ and we have assumed at least one is nonzero, all the above denominators are nonzero. Moreover, this also shows that we may write

$$x \;=\; \big((1-\lambda)\alpha_{11} + \lambda\alpha_{12}\big)\left(\frac{(1-\lambda)|\alpha_{11}|}{(1-\lambda)|\alpha_{11}|+\lambda|\alpha_{12}|}v_1 + \frac{\lambda|\alpha_{12}|}{(1-\lambda)|\alpha_{11}|+\lambda|\alpha_{12}|}v_2\right) \tag{27}$$

$$+\sum_{i=2}^{m}\big((1-\lambda)\alpha_{i1}+\lambda\alpha_{i2}\big)\left(\frac{(1-\lambda)|\alpha_{i1}|}{(1-\lambda)|\alpha_{i1}|+\lambda|\alpha_{i2}|}w_{i1}+\frac{\lambda|\alpha_{i2}|}{(1-\lambda)|\alpha_{i1}|+\lambda|\alpha_{i2}|}w_{i2}\right).$$

Note that in the second factor on the right-hand side of (27) we have a convex combination of v_1 and v_2 since the scalars on these vectors are nonnegative and sum to 1. As the same holds for the w_{ij}'s, $i \in \{2, \ldots, m\}$, by the convexity of the C_i's

$$\frac{(1-\lambda)|\alpha_{i1}|}{(1-\lambda)|\alpha_{i1}|+\lambda|\alpha_{i2}|}w_{i1}+\frac{\lambda|\alpha_{i2}|}{(1-\lambda)|\alpha_{i1}|+\lambda|\alpha_{i2}|}w_{i2} \ \in \ C_i$$

for each $i \in \{2, \ldots, m\}$. Because M is likewise convex,

$$\begin{aligned}&\big((1-\lambda)\alpha_{11}+\lambda\alpha_{12}, \ldots, (1-\lambda)\alpha_{m1}+\lambda\alpha_{m2}\big)\\ &\qquad\qquad\qquad = \ (1-\lambda)(\alpha_{11}, \ldots, \alpha_{m1})+\lambda(\alpha_{12}, \ldots, \alpha_{m2}) \in M,\end{aligned}$$

and therefore, for *any* $\lambda \in (0,1)$, we can see from (27) that

$$x \ \in \ \oplus_M\left(\left\{\frac{(1-\lambda)|\alpha_{11}|}{(1-\lambda)|\alpha_{11}|+\lambda|\alpha_{12}|}v_1+\frac{\lambda|\alpha_{12}|}{(1-\lambda)|\alpha_{11}|+\lambda|\alpha_{12}|}v_2\right\}, \ C_2, \ \ldots, \ C_m\right). \quad (28)$$

If $\alpha_{i1} = \alpha_{i2} = 0$ for any $i \in \{2, \ldots, m\}$, then we may replace the linear combination of w_{i1} and w_{i2} in (26) with an arbitrary element of C_i and maintain equality since $(1-\lambda)\alpha_{i1}+\lambda\alpha_{i2} = 0$.

Since $K_{11} \cup K_{21}$ is convex, by the argument in the first paragraph of the proof there exists $\mu \in [0,1]$ such that

$$(1-\mu)v_1+\mu v_2 \ \in \ K_{11} \cap K_{21}. \quad (29)$$

If $\mu = 0$, then $v_1 \in K_{11} \cap K_{21}$ and so by (25) with $j = 1$,

$$x \ = \ \alpha_{11}v_1+\sum_{i=2}^{m}\alpha_{i1}w_{i1} \ \in \ \oplus_M(K_{11} \cap K_{21}, C_2, \ldots, C_m).$$

If $\mu = 1$, then $v_2 \in K_{11} \cap K_{21}$, and so we obtain the same conclusion using $j = 2$. For $\mu \in (0,1)$, if we could select $\lambda \in (0,1)$ such that

$$\frac{(1-\lambda)|\alpha_{11}|}{(1-\lambda)|\alpha_{11}|+\lambda|\alpha_{12}|} \ = \ 1-\mu \qquad \text{and} \qquad \frac{\lambda|\alpha_{12}|}{(1-\lambda)|\alpha_{11}|+\lambda|\alpha_{12}|} \ = \ \mu, \quad (30)$$

then from (28) and (29), x would be an element of $\oplus_M(K_{11} \cap K_{21}, C_2, \ldots, C_m)$, completing this portion of the proof. Indeed, set

$$\lambda \ = \ \frac{\mu|\alpha_{11}|}{(1-\mu)|\alpha_{12}|+\mu|\alpha_{11}|}.$$

(Recall that by assumption $\alpha_{11} \neq 0$ and $\alpha_{12} \neq 0$ since we have already handled the other case.) Then $0 \leq \lambda \leq 1$ and the denominator in (30) is

$$\begin{aligned}
(1-\lambda)|\alpha_{11}| + \lambda|\alpha_{12}| &= \left(1 - \frac{\mu|\alpha_{11}|}{(1-\mu)|\alpha_{12}| + \mu|\alpha_{11}|}\right)|\alpha_{11}| + \frac{\mu|\alpha_{11}|}{(1-\mu)|\alpha_{12}| + \mu|\alpha_{11}|}|\alpha_{12}| \\
&= \frac{(1-\mu)|\alpha_{12}|}{(1-\mu)|\alpha_{12}| + \mu|\alpha_{11}|}|\alpha_{11}| + \frac{\mu|\alpha_{11}\alpha_{12}|}{(1-\mu)|\alpha_{12}| + \mu|\alpha_{11}|} \\
&= \frac{|\alpha_{11}\alpha_{12}|}{(1-\mu)|\alpha_{12}| + \mu|\alpha_{11}|},
\end{aligned}$$

yielding

$$\frac{\lambda|\alpha_{12}|}{(1-\lambda)|\alpha_{11}| + \lambda|\alpha_{12}|} = \frac{\left(\dfrac{\mu|\alpha_{11}\alpha_{12}|}{(1-\mu)|\alpha_{12}| + \mu|\alpha_{11}|}\right)}{\left(\dfrac{|\alpha_{11}\alpha_{12}|}{(1-\mu)|\alpha_{12}| + \mu|\alpha_{11}|}\right)} = \mu,$$

and so automatically

$$1 - \frac{\lambda|\alpha_{12}|}{(1-\lambda)|\alpha_{11}| + \lambda|\alpha_{12}|} = \frac{(1-\lambda)|\alpha_{11}|}{(1-\lambda)|\alpha_{11}| + \lambda|\alpha_{12}|} = 1 - \mu.$$

Thus λ satisfies the needed relationship with respect to μ, and so

$$x \in \oplus_M(K_{11} \cap K_{21}, C_2, \ldots, C_m).$$

Since this proof did not depend on the placement of the intersection among the summands, (23) follows from iterating (24) ℓ times. $\square$

3.5 Valuation

A property of M-addition that combines unions and intersections is *valuation*. As defined by Gardner, Hug and Weil [1, Section 4], a binary map $*$ between sets in a class $\mathcal{S}$ satisfies the valuation property if $(A \cup B) * (A \cap B) = A * B$ whenever $A, B, A \cap B$ and $A \cup B \in \mathcal{S}$. The origins of this property lie in measure theory, where a finitely-additive measure μ satisfies

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B). \tag{31}$$

Geometers have generalized the notion of measure on convex sets by defining *valuations* as maps $\mu : \mathcal{K}^n \to \mathbb{R}$ which satisfy (31) for all $A, B \in \mathcal{K}^n$ such that $A \cup B \in \mathcal{K}^n$ (henceforth abbreviated $A, B, A \cup B \in \mathcal{K}^n$). Examples include volume and surface area. A further generalization is to *Minkowski valuations*, or convex-set-valued maps $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ for which

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi(K) + \Phi(L)$$

for all $K, L, K \cup L \in \mathcal{K}^n$, where now $+$ is Minkowski addition. The identity map is a particular example, since $(K \cup L) + (K \cap L) = K + L$ whenever $K, L, K \cup L \in \mathcal{K}^n$ [4, Lemma 3.1.1, p. 127]. A natural further extension is to *M-valuations*.

Definition 1. A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is an *M-valuation* with respect to a fixed $M \subset \mathbb{R}^2$ if

$$\Phi(K \cup L) \oplus_M \Phi(K \cap L) \;=\; \Phi(K) \oplus_M \Phi(L)$$

for all $K, L, K \cup L \in \mathcal{K}^n$.

The possible Φ here might vary dramatically as M varies. However, in seeking some starting point to the investigation of M-valuations, we might reason that the identity map ought to be an M-valuation for any set M worth considering, since this is case for Minkowski valuations. The present section partially classifies possible sets by providing sufficient conditions on M for when this is the case. Our theorem is preceded by two lemmas, the first of which is stated without proof in Schneider [4, (3.1.1), p. 127].

Lemma 17. *If $A, B, C \subset \mathbb{R}^n$, then*

$$(A + B) \cup (A + C) \;=\; A + (B \cup C).$$

Proof. Note that $x \in (A+B)\cup(A+C)$ if and only if either $x = v+w_1$ or $x = v+w_2$ for some $v \in A$, $w_1 \in B$ or $w_2 \in C$. This in turn happens if and only if $x \in A+(B\cup C)$. $\square$

Lemma 18. *If $K, L \subset \mathbb{R}^n$ are convex and $0 \leq \alpha \leq \beta$, then*

$$(\alpha K + \beta L) \cup (\beta K + \alpha L) \;=\; \alpha(K + L) + (\beta - \alpha)(K \cup L).$$

Proof. We observe

$$\alpha K + \beta L \;=\; \alpha K + ((\beta - \alpha) + \alpha)L,$$

and since L is convex and $\beta - \alpha \geq 0$, by (13),

$$\alpha K + ((\beta - \alpha) + \alpha)L \;=\; \alpha K + (\beta - \alpha)L + \alpha L \;=\; \alpha(K + L) + (\beta - \alpha)L.$$

Interchanging the roles of K and L gives

$$\beta K + \alpha L \;=\; \alpha(K + L) + (\beta - \alpha)K,$$

and thus this combined with Lemma 17 yields

$$\begin{aligned}(\alpha K + \beta L) \cup (\beta K + \alpha L) \;&=\; \big(\alpha(K + L) + (\beta - \alpha)L\big) \cup \big(\alpha(K + L) + (\beta - \alpha)K\big) \\ &=\; \alpha(K + L) + \big(((\beta - \alpha)L) \cup ((\beta - \alpha)K)\big) \\ &=\; \alpha(K + L) + (\beta - \alpha)(K \cup L).\end{aligned}$$

$\square$

Theorem 19. *The identity map is an M-valuation with respect to any $M \subset (-\infty, 0]^2 \cup [0,\infty)^2$ that is symmetric in the line $y = x$. In other words, for any such set M and any $K, L, K \cup L \in \mathcal{K}^n$,*

$$(K \cup L) \oplus_M (K \cap L) \;=\; K \oplus_M L. \tag{32}$$

Proof. Suppose first that $M \subset [0,\infty)^2$ such that M is symmetric in $y = x$. Set $M' = M \cap \{(x, y) \;:\; x \le y\}$. Then by the symmetry of M and Lemma 18 based on the fact that both $K \cup L$ and $K \cap L$ are convex and $\alpha_1 \le \alpha_2$ for $(\alpha_1, \alpha_2) \in M'$,

$$\begin{aligned}
(K \cup L) \oplus_M (K \cap L) \;&=\; \bigcup_{(\alpha_1,\alpha_2)\in M} \alpha_1(K \cup L) + \alpha_2(K \cap L)\\
&= \bigcup_{(\alpha_1,\alpha_2)\in M'} \big(\alpha_1(K \cup L) + \alpha_2(K \cap L)\big) \cup \big(\alpha_2(K \cup L) + \alpha_1(K \cap L)\big)\\
&= \bigcup_{(\alpha_1,\alpha_2)\in M'} \Big(\alpha_1\big((K \cup L) + (K \cap L)\big) + (\alpha_2 - \alpha_1)\big((K \cup L) \cup (K \cap L)\big)\Big)\\
&= \bigcup_{(\alpha_1,\alpha_2)\in M'} \big(\alpha_1(K + L) + (\alpha_2 - \alpha_1)(K \cup L)\big),
\end{aligned}$$

since $(K \cup L) + (K \cap L) = K + L$ (i.e. the identity map is a Minkowski valuation). By another application of Lemma 18,

$$\begin{aligned}
\bigcup_{(\alpha_1,\alpha_2)\in M'} \big(\alpha_1(K + L) + (\alpha_2 - \alpha_1)(K \cup L)\big) \;&=\; \bigcup_{(\alpha_1,\alpha_2)\in M'} \big((\alpha_1 K + \alpha_2 L) \cup (\alpha_2 K + \alpha_1 L)\big)\\
&= \bigcup_{(\alpha_1,\alpha_2)\in M} \alpha_1 K + \alpha_2 L \;=\; K \oplus_M L.
\end{aligned}$$

Thus (32) holds for $M \subset [0,\infty)^2$ symmetric in $y = x$.

If $M \subset (-\infty, 0]^2$ is symmetric in $y = x$, then $-M \subset [0,\infty)^2$ likewise shares this symmetry, and so using Corollary 7, the fact that $-K \cup -L$ is convex and the above, we have

$$\begin{aligned}
(K \cup L) \oplus_M (K \cap L) \;&=\; (K \cup L) \oplus_{-(-M)} (K \cap L)\\
&= \big(-(K \cup L)\big) \oplus_{-M} \big(-(K \cap L)\big)\\
&= (-K \cup -L) \oplus_{-M} (-K \cap -L)\\
&= (-K) \oplus_{-M} (-L)\\
&= K \oplus_{-(-M)} L \;=\; K \oplus_M L,
\end{aligned}$$

showing (32) holds in this case as well.

Finally, if $M \subset (-\infty, 0]^2 \cup [0,\infty)^2$ is symmetric in $y = x$ such that $M_1 = M \cap [0,\infty)^2 \neq \emptyset$ and $M_2 = M \cap (-\infty, 0]^2 \neq \emptyset$, then using Theorem 11 combined with the preceding yields

$$(K \cup L) \oplus_M (K \cap L) \;=\; (K \cup L) \oplus_{M_1 \cup M_2} (K \cap L)$$

$$
\begin{aligned}
&= \big((K \cup L) \oplus_{M_1} (K \cap L)\big) \cup \big((K \cup L) \oplus_{M_2} (K \cap L)\big) \\
&= (K \oplus_{M_1} L) \cup (K \oplus_{M_2} L) \\
&= K \oplus_{M_1 \cup M_2} L \;=\; K \oplus_M L. \qquad \square
\end{aligned}
$$

Note that we did not use the compactness of K, L or $K \cup L$ in the proof.

The following two examples show that we cannot omit Theorem 19's containment condition or symmetry condition on M, respectively, and still have (32) hold for all $K, L, K \cup L \in \mathcal{K}^n$.

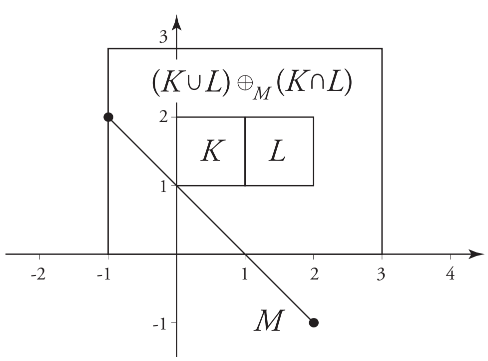
Example 20. First consider $M = [(-1,2),(2,-1)]$, $K = [0,1] \times [1,2]$ and $L = [1,2]^2$, as in Figure 3a. Then $M \not\subset (-\infty,0]^2 \cup [0,\infty)^2$ but is symmetric in $y = x$. Here $K \cup L = [0,2] \times [1,2]$ and $K \cap L = \{1\} \times [1,2]$. To compute $(K \cup L) \oplus_M (K \cap L)$, we decompose M as the union of the three line segments $M_1 = [(-1,2), e_2]$, $M_2 = [e_1, e_2]$ and $M_3 = [e_1, (2,-1)]$ and use Theorem 11 to obtain

$$
\begin{aligned}
&(K \cup L) \oplus_M (K \cap L) \;=\; (K \cup L) \oplus_{M_1 \cup M_2 \cup M_3} (K \cap L) \\
&= \big((K \cup L) \oplus_{M_1} (K \cap L)\big) \cup \big((K \cup L) \oplus_{M_2} (K \cap L)\big) \cup \big((K \cup L) \oplus_{M_3} (K \cap L)\big). \qquad (33)
\end{aligned}
$$

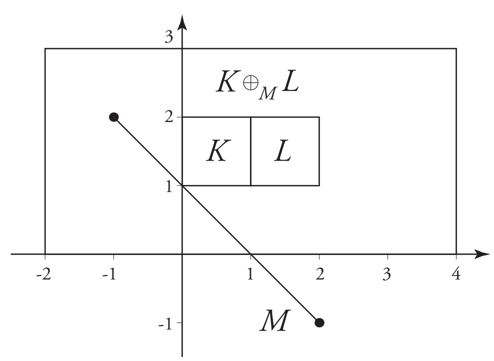
To determine $(K \cup L) \oplus_{M_j} (K \cap L)$, $j \in \{1,2,3\}$, we use Theorem 29 below. Since the line segment M_j is the convex hull of its endpoints x_{1j} and x_{2j}, say, and $K \cup L$ and $K \cap L$ are convex,

$$
\begin{aligned}
(K \cup L) \oplus_{M_j} (K \cap L) &= \operatorname{conv}(K \cup L) \oplus_{\operatorname{conv}(\{x_{1j}\} \cup \{x_{2j}\})} \operatorname{conv}(K \cap L) \\
&= \operatorname{conv}\big((K \cup L) \oplus_{\{x_{1j}\} \cup \{x_{2j}\}} (K \cap L)\big),
\end{aligned}
$$

where the last equality is by Theorem 29(i). Another application of Theorem 11 then yields



(a)



(b)

Figure 3: If M is symmetric about $y = x$ but not contained in $(-\infty,0]^2 \cup [0,\infty)^2$, equality may not hold in (32) for all $K, L, K \cup L \in \mathcal{K}^n$.

$$(K \cup L) \oplus_{M_j} (K \cap L)$$
$$= \operatorname{conv}\Big(\big((K \cup L) \oplus_{\{x_{1j}\}} (K \cap L)\big) \cup \big((K \cup L) \oplus_{\{x_{2j}\}} (K \cap L)\big)\Big). \quad (34)$$

Since the M-sets are now just individual points, each of these M-sums is the Minkowski sum of dilatated copies of $K \cup L$ and $K \cap L$. For M_1 with its endpoints $x_{11} = (-1, 2)$ and $x_{21} = e_2$, we have

$$-(K \cup L) + 2(K \cap L) = [-2, 0] \times [-2, -1] + \{2\} \times [2, 4] = [0, 2] \times [0, 3] \quad \text{and}$$
$$0(K \cup L) + 1(K \cap L) = \{1\} \times [1, 2],$$

and thus by (34),

$$(K \cup L) \oplus_{M_1} (K \cap L) = \operatorname{conv}\big(([0, 2] \times [0, 3]) \cup (\{1\} \times [1, 2])\big) = [0, 2] \times [0, 3].$$

For the endpoints $x_{12} = e_2$ and $x_{22} = e_1$ of M_2, we have

$$0(K \cup L) + 1(K \cap L) = K \cap L \quad \text{and} \quad 1(K \cup L) + 0(K \cap L) = K \cup L,$$

and so by (34) again,

$$(K \cup L) \oplus_{M_2} (K \cap L) = \operatorname{conv}\big((K \cap L) \cup (K \cup L)\big) = K \cup L = [0, 2] \times [1, 2].$$

The endpoints $x_{13} = e_1$ and $x_{23} = (2, -1)$ of M_3 give, respectively, $K \cup L$ and

$$2(K \cup L) - (K \cap L) = [0, 4] \times [2, 4] + \{-1\} \times [-2, -1] = [-1, 3] \times [0, 3],$$

showing that

$$(K \cup L) \oplus_{M_3} (K \cap L) = \operatorname{conv}\big(([0, 2] \times [1, 2]) \cup [-1, 3] \times [0, 3]\big) = [-1, 3] \times [0, 3].$$

Hence from these computations and (33),

$$\begin{aligned}(K \cup L) \oplus_M (K \cap L) &= ([0, 2] \times [0, 3]) \cup ([0, 2] \times [1, 2]) \cup ([-1, 3] \times [0, 3]) \\ &= [-1, 3] \times [0, 3],\end{aligned}$$

as shown in Figure 3a.

Taking the same approach for computing $K \oplus_M L$ yields

$$K \oplus_M L = [-2, 4] \times [0, 3],$$

as illustrated in Figure 3b. We thus see that in this case, $(K \cup L) \oplus_M (K \cap L) \subsetneq K \oplus_M L$.

Example 21. Next, consider $M = [e_2, (e_1 + e_2)/2]$ with K and L as in Example 20, i.e. $K = [0, 1] \times [1, 2]$ and $L = [1, 2]^2$. Then M is *not* symmetric in $y = x$ but is contained in the first quadrant. Calculations similar to those in Example 20 show

$$(K \cup L) \oplus_M (K \cap L) = [1/2, 3/2] \times [1, 2]$$

while

$$K \oplus_M L = [1/2, 1] \times [1, 2],$$

as in Figures 4a and 4b, respectively. Hence in this case $K \oplus_M L \subsetneq (K \cup L) \oplus_M (K \cap L)$.

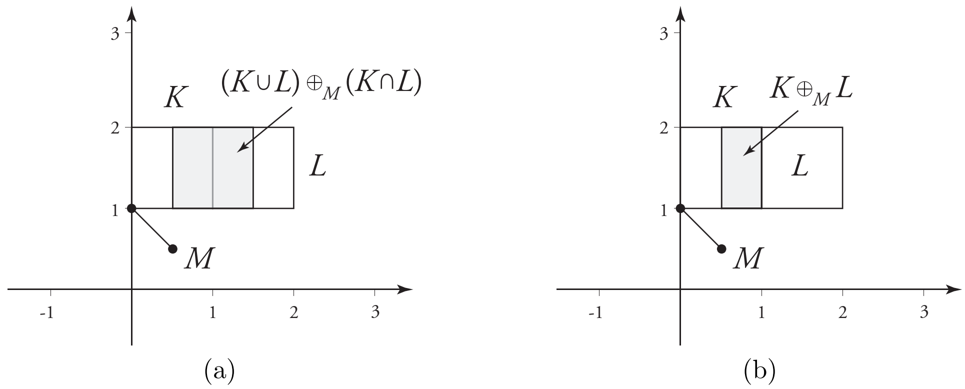


Figure 4: If $M \subset (-\infty, 0]^2 \cup [0, \infty)^2$ is not symmetric about $y = x$, it is also possible that $(K \cup L) \oplus_M (K \cap L) \neq K \oplus_M L$.

3.6 Symmetry

M-addition is built upon the dilatation of sets, and dilatation is always with respect to the origin. We might wonder, then, whether M-addition preserves properties of summands or M-sets that are related to the origin, such as certain varieties of symmetry. This section explores such questions.

Theorem 22. *If either $M \subset \mathbb{R}^m$ is o-symmetric or each $A_1, \ldots, A_m \subset \mathbb{R}^n$ is o-symmetric, then $\oplus_M(A_1, \ldots, A_m)$ is o-symmetric.*

Proof. If M is o-symmetric, then by (19) with $\beta = -1$,

$$- \oplus_M (A_1, \ldots, A_m) = \oplus_{-M}(A_1, \ldots, A_m) = \oplus_M(A_1, \ldots, A_m).$$

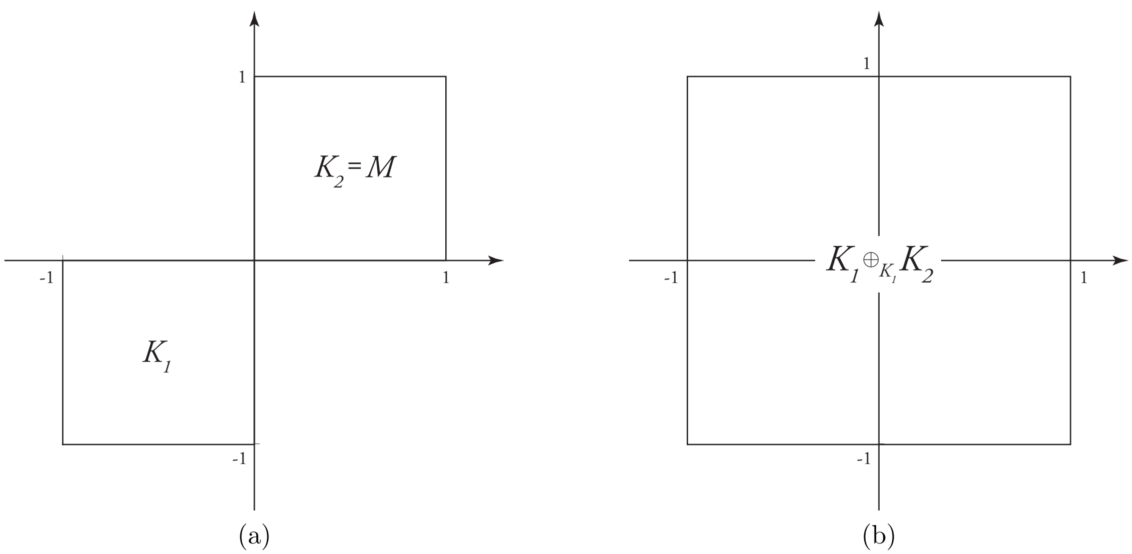


Figure 5: Neither the converse of Theorem 22 nor of Theorem 25 holds.

If the A_j's are o-symmetric, again by (19) we have

$$- \oplus_M (A_1, \ldots, A_m) \;=\; \oplus_M(-A_1, \ldots, -A_m) \;=\; \oplus_M(A_1, \ldots, A_m). \qquad \square$$

Example 23. The sets $K_1 = [-1, 0]^2$ and $K_2 = M = [0, 1]^2$ (see Figure 5a) show that the converse of this theorem does not hold. That is, if $\oplus_M(A_1, \ldots, A_m)$ is o-symmetric, it does not automatically follow that either the M-set or the summands are. For these sets,

$$\begin{aligned} K_1 \oplus_M K_2 \;&=\; \bigcup \left\{ \alpha_1 K_1 + \alpha_2 K_2 \;:\; 0 \le \alpha_1, \alpha_2 \le 1 \right\} \\ &=\; \bigcup \left\{ [-\alpha_1, 0]^2 + [0, \alpha_2]^2 \;:\; 0 \le \alpha_1, \alpha_2 \le 1 \right\} \\ &=\; \bigcup \left\{ [-\alpha_1, \alpha_2]^2 \;:\; 0 \le \alpha_1, \alpha_2 \le 1 \right\} \;=\; [-1, 1]^2, \end{aligned}$$

as illustrated in Figure 5b. Thus this M-sum is o-symmetric while neither M, K_1 or K_2 are.

It also bears mention that M-addition does not generally preserve symmetry about any point other than the origin, as the following example illustrates.

Example 24. Consider $K = M = [1, 2]^2$ and $L = [0, 3]^2$, as illustrated in Figure 6. Then K, L and M are each symmetric about $(1.5, 1.5)$. As we can see from their M-sum, however, this symmetry is not preserved in $K \oplus_M L$, which is in fact not symmetric about any point. Note that the left and right edges of $K \oplus_M L$ are symmetric about $x = 5.5$ while the bottom and top edges are symmetric about $y = 5.5$. Hence if the entire set were centrally symmetric, it would have to be about $(5.5, 5.5)$. But then for the vertex $(10, 2)$ we would need

$$-\big((10, 2) - (5.5, 5.5)\big) + (5.5, 5.5) \;=\; (1, 9)$$

to be in $K \oplus_M L$, which is not the case.

The conditions in which M-addition preserves 1-unconditionality are more restrictive than those in which it preserves o-symmetry.

Theorem 25. *If $M \subset \mathbb{R}^m$ is arbitrary and $A_1, \ldots, A_m \subset \mathbb{R}^n$ are all 1-unconditional, then $\oplus_M(A_1, \ldots, A_m)$ is 1-unconditional.*

Proof. Let $x \in \oplus_M(A_1, \ldots, A_m)$. Then in accord with definition (2), $x = \sum_{j=1}^m \alpha_j v_j$ for some $(\alpha_1, \ldots, \alpha_m) \in M$ and $v_j = (v_{1j}, \ldots, v_{nj}) \in A_j$, $j \in \{1, \ldots, m\}$. Thus

$$x \;=\; \sum_{j=1}^{m} \alpha_j (v_{1j}, \ldots, v_{nj}) \;=\; \left(\sum_{j=1}^{m} \alpha_j v_{1j}, \ldots, \sum_{j=1}^{m} \alpha_j v_{nj} \right),$$

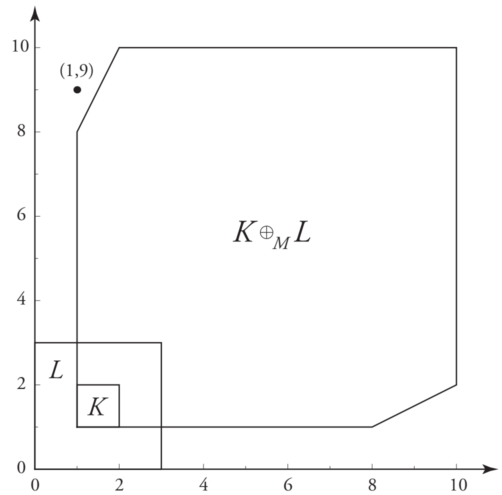


Figure 6: Central symmetry is not always preserved in M-addition.

and so for arbitrary $\varepsilon_1, \ldots, \varepsilon_n$ such that each $\varepsilon_j \in \{-1, 1\}$, we have

$$\begin{aligned}\left(\varepsilon_1 \sum_{j=1}^{m} \alpha_j v_{1j}, \ldots, \varepsilon_n \sum_{j=1}^{m} \alpha_j v_{nj}\right) &= \left(\sum_{j=1}^{m} \alpha_j \varepsilon_1 v_{1j}, \ldots, \sum_{j=1}^{m} \alpha_j \varepsilon_n v_{nj}\right) \\ &= \sum_{j=1}^{m} \alpha_j (\varepsilon_1 v_{1j}, \ldots, \varepsilon_n v_{nj}) \ \in \ \oplus_M (A_1, \ldots, A_m),\end{aligned}$$

since every A_j is 1-unconditional. $\square$

A notable distinction between Theorems 22 and 25 is that the latter does not claim that when the M-set possesses the symmetry in question, then so does any associated M-sum. This is readily seen not to be the case by considering $M = [-1, 1] \subset \mathbb{R}$ and $K = [0, 1]^2 \subset \mathbb{R}^2$. Here M is 1-unconditional, but $\oplus_M (K) = [-1, 0]^2 \cup [0, 1]^2$, which is not.

Example 23 also shows that the converse of Theorem 25 does not hold, since there $K_1 \oplus_M K_2$ is 1-unconditional while neither K_1 nor K_2 is.

3.7 Miscellaneous

We will occasionally wish to compute an M-sum where the summands are all dilatated copies of the same set K. The following theorem shows that, under certain conditions on the scaling factors, the result is just a union of dilatates of K.

Theorem 26. *Let M be contained in one of the 2^m closed orthants of $\mathbb{R}^m$ and let $\varepsilon_j = \pm 1$, $j \in \{1, \ldots, m\}$, denote the sign of the jth coordinate of a point in the interior* *of this orthant. Then if* $r_1, \ldots, r_m \geq 0$ *and* $K \in \mathcal{K}^n$*,*

$$\begin{aligned}\oplus_M(\varepsilon_1 r_1 K, \ldots, \varepsilon_m r_m K) &= \bigcup_{(\alpha_1,\ldots,\alpha_m)\in M} (\alpha_1\varepsilon_1 r_1 + \cdots + \alpha_m \varepsilon_m r_m)K \\ &= \oplus_{\oplus_M(\{\varepsilon_1 r_1\},\ldots,\{\varepsilon_m r_m\})}(K).\end{aligned}$$

Proof. Using definition (1) of M-addition, the fact that $\alpha_j \varepsilon_j r_j \geq 0$ for every $j \in \{1, \ldots, m\}$ and each $(\alpha_1, \ldots, \alpha_m) \in M$, the convexity of K and the consequent generalization of property (13) for m scalars, we have

$$\begin{aligned}\oplus_M(\varepsilon_1 r_1 K, \ldots, \varepsilon_m r_m K) &= \bigcup_{(\alpha_1,\ldots,\alpha_m)\in M} \alpha_1\varepsilon_1 r_1 K + \cdots + \alpha_m \varepsilon_m r_m K \\ &= \bigcup_{(\alpha_1,\ldots,\alpha_m)\in M} (\alpha_1\epsilon_1 r_1 + \cdots + \alpha_m \epsilon_m r_m)K \\ &= \bigcup_{\beta\in\oplus_M(\{\epsilon_1 r_1\},\ldots,\{\epsilon_m r_m\})} \beta K \\ &= \oplus_{\oplus_M(\{\epsilon_1 r_1\},\ldots,\{\epsilon_m r_m\})}(K).\end{aligned}$$

□

4 Interaction with operations on individual sets

Having laid the groundwork with the proceeding fundamental properties, we proceed to consider how M-addition interacts with other operations in geometry. We are motivated by the often far-reaching consequences of relationships between operations in mathematics. We learn in grade school, for example, that $a(b+c) = ab + ac$, or, that multiplication *distributes* over addition. While this property is both elementary and simple, it is one of the most crucial ways in which real numbers interact with each other. It is foundational for any algebraic manipulation of expressions and is also the reason for the particulars of the binomial theorem, to cite just two consequences.

In Theorem 5 we saw that a linear transformation ϕ distributes over M-addition in the sense that $\phi\big(\oplus_M (A_1, \ldots, A_m)\big) = \oplus_M(\phi A_1, \ldots, \phi A_m)$. In the more general case of a map $A \mapsto *A$ on individual sets, we might also wonder whether

$$* \oplus_M (A_1, \ldots, A_m) = \oplus_M(*A_1, \ldots, *A_m) \tag{35}$$

or whether a different relationship holds, such as

$$* \oplus_M (A_1, \ldots, A_m) = \oplus_{*M}(*A_1, \ldots, *A_m) \tag{36}$$

or

$$* \oplus_M (A_1, \ldots, A_m) = \oplus_{*M}(A_1, \ldots, A_m) \tag{37}$$

or

$$\oplus_{*M}(A_1,\dots,A_m) \;=\; \oplus_M(*A_1,\dots,*A_m). \tag{38}$$

We have already seen an instance of (35), (37) and (38) in (19), which showed how one can pass scalars to different portions of an M-sum. The main conclusion of this section is that (36) holds under certain conditions when $*$ is the convex hull operation.

To reach this result, we first observe that for any $A_1,\dots,A_m \subset \mathbb{R}^n$,

$$\operatorname{conv}\left(\sum_{j=1}^m A_j\right) \;=\; \sum_{j=1}^m \operatorname{conv} A_j \tag{39}$$

(see [4, Theorem 1.1.2] for the case $m=2$, from which the above immediately follows through induction). Since M-addition is *weighted* Minkowski summation, we also need to consider how the convex hull relates to set dilatation.

Lemma 27. *For any $A \subset \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,*

$$\operatorname{conv}(\alpha A) \;=\; \alpha \operatorname{conv} A.$$

Proof. Using definition (3) of the convex hull, we find

$$\begin{aligned}
\operatorname{conv}(\alpha A) \;&=\; \left\{\sum_{j=1}^N \lambda_j \alpha w_j \;:\; N\in\mathbb{N},\ \lambda_1,\dots,\lambda_N \geq 0,\ \sum_{j=1}^N \lambda_j = 1,\ w_j \in A\right\}\\
&=\; \left\{\alpha\sum_{j=1}^N \lambda_j w_j \;:\; N\in\mathbb{N},\ \lambda_1,\dots,\lambda_N \geq 0,\ \sum_{j=1}^N \lambda_j = 1,\ w_j \in A\right\}\\
&=\; \alpha \operatorname{conv} A.
\end{aligned}$$

□

Combining (39) with Lemma 27 shows that the convex hull has a property analogous to linearity, in that for any $\alpha_1,\dots,\alpha_m \in \mathbb{R}$ and $A_1,\dots,A_m \subset \mathbb{R}^n$,

$$\operatorname{conv}\left(\sum_{j=1}^m \alpha_j A_j\right) \;=\; \sum_{j=1}^m \alpha_j \operatorname{conv} A_j. \tag{40}$$

Theorem 28. *If $M \subset \mathbb{R}^m$ and $A_1,\dots,A_m \subset \mathbb{R}^n$, then*

$$\oplus_{\operatorname{conv} M}\big(\operatorname{conv} A_1,\dots,\operatorname{conv} A_m\big) \;\subset\; \operatorname{conv}\big(\oplus_M(A_1,\dots,A_m)\big).$$

Proof. By definition (1),

$$\oplus_{\operatorname{conv} M}\big(\operatorname{conv} A_1,\dots,\operatorname{conv} A_m\big) \;=\; \bigcup_{(\beta_1,\dots,\beta_m)\in\operatorname{conv} M} \sum_{j=1}^m \beta_j \operatorname{conv} A_j.$$

Elements of $\operatorname{conv} M$ have the form

$$(\beta_1, \dots, \beta_m) \;=\; \sum_{k=1}^{N} \lambda_k (\alpha_{1k},\; \dots,\; \alpha_{mk}) \;=\; \left(\sum_{k=1}^{N} \lambda_k \alpha_{1k},\; \dots,\; \sum_{k=1}^{N} \lambda_k \alpha_{mk} \right) \tag{41}$$

for some $N \in \mathbb{N}$, convex combination scalars $\lambda_1, \dots, \lambda_N$ and $(\alpha_{1k}, \dots, \alpha_{mk}) \in M$, $k \in \{1, \dots, N\}$. If we denote the collection of all sets of convex combination scalars as

$$\Lambda \;=\; \left\{ \{\lambda_1, \dots, \lambda_N\} \;:\; N \in \mathbb{N},\; \lambda_j \geq 0 \text{ for every } j \in \{1, \dots, N\},\; \sum_{j=1}^{N} \lambda_j = 1 \right\},$$

then using (41) along with (10), (40) and Corollary 13(i), we have

$$\begin{aligned}
&\oplus_{\operatorname{conv} M}\big(\operatorname{conv} A_1, \dots, \operatorname{conv} A_m\big) \\
&\quad= \bigcup_{\{\lambda_1,\dots,\lambda_N\}\in\Lambda} \; \bigcup_{(\alpha_{11},\dots,\alpha_{m1})\in M} \cdots \bigcup_{(\alpha_{1N},\dots,\alpha_{mN})\in M} \sum_{j=1}^{m} \left(\sum_{k=1}^{N} \lambda_k \alpha_{jk} \right) \operatorname{conv} A_j \\
&\quad\subset \bigcup_{\{\lambda_1,\dots,\lambda_N\}\in\Lambda} \; \bigcup_{(\alpha_{11},\dots,\alpha_{m1})\in M} \cdots \bigcup_{(\alpha_{1N},\dots,\alpha_{mN})\in M} \sum_{j=1}^{m} \sum_{k=1}^{N} \lambda_k \alpha_{jk} \operatorname{conv} A_j \\
&\quad= \bigcup_{\{\lambda_1,\dots,\lambda_N\}\in\Lambda} \; \bigcup_{(\alpha_{11},\dots,\alpha_{m1})\in M} \cdots \bigcup_{(\alpha_{1N},\dots,\alpha_{mN})\in M} \sum_{k=1}^{N} \lambda_k \sum_{j=1}^{m} \alpha_{jk} \operatorname{conv} A_j \\
&\quad= \bigcup_{\{\lambda_1,\dots,\lambda_N\}\in\Lambda} \; \bigcup_{(\alpha_{11},\dots,\alpha_{m1})\in M} \cdots \bigcup_{(\alpha_{1N},\dots,\alpha_{mN})\in M} \sum_{k=1}^{N} \lambda_k \operatorname{conv} \left(\sum_{j=1}^{m} \alpha_{jk} A_j \right) \\
&\quad= \bigcup_{\{\lambda_1,\dots,\lambda_N\}\in\Lambda} \; \sum_{k=1}^{N} \lambda_k \bigcup_{(\alpha_{1k},\dots,\alpha_{mk})\in M} \operatorname{conv} \left(\sum_{j=1}^{m} \alpha_{jk} A_j \right) \\
&\quad\subset \bigcup_{\{\lambda_1,\dots,\lambda_N\}\in\Lambda} \; \sum_{k=1}^{N} \lambda_k \operatorname{conv}\big(\oplus_M (A_1, \dots, A_m)\big) \;=\; \operatorname{conv}\big(\oplus_M (A_1, \dots, A_m)\big).
\end{aligned}$$

Note that the subcontainment of the last line is the best we can do in general, since the union of the convex hulls is not always identical to the convex hull of the union. $\square$

Theorem 29. (i) *If M is contained in one of the 2^m closed orthants of $\mathbb{R}^m$ and $A_1, \dots, A_m \subset \mathbb{R}^n$, then*

$$\operatorname{conv}\big(\oplus_M (A_1, \dots, A_m)\big) \;=\; \oplus_{\operatorname{conv} M}\big(\operatorname{conv} A_1, \dots, \operatorname{conv} A_m\big). \tag{42}$$

(ii) *Equation* (42) *also holds if $M \subset \mathbb{R}^m$ has positive or negative symmetry and each of $A_1, \dots, A_m \subset \mathbb{R}^n$ is o-symmetric.*

Proof. (i) If $x \in \operatorname{conv}\big(\oplus_M(A_1,\ldots,A_m)\big)$, then there exist $v_1,\ldots,v_N \in \oplus_M(A_1,\ldots,A_m)$ and convex combination scalars $\lambda_1,\ldots,\lambda_N$ such that $x = \sum_{i=1}^N \lambda_i v_i$. Each $v_i = \sum_{j=1}^m \alpha_{ji} w_{ji}$ for some $(\alpha_{1i},\ldots,\alpha_{mi}) \in M$ and $w_{ji} \in A_j$, $j \in \{1,\ldots,m\}$. Therefore

$$\begin{aligned} x \;=\; \sum_{i=1}^N \lambda_i \sum_{j=1}^m \alpha_{ji} w_{ji} \;&=\; \sum_{j=1}^m \sum_{i=1}^N \lambda_i \alpha_{ji} w_{ji} && (43)\\ &=\; \sum_{j=1}^m \left(\sum_{k=1}^N \lambda_k \alpha_{jk}\right) \sum_{i=1}^N \frac{\lambda_i \alpha_{ji}}{\sum_{k=1}^N \lambda_k \alpha_{jk}}\, w_{ji} \\ &=\; \sum_{j=1}^m \left(\sum_{k=1}^N \lambda_k \alpha_{jk}\right) \sum_{i=1}^N \frac{\lambda_i |\alpha_{ji}|}{\sum_{k=1}^N \lambda_k |\alpha_{jk}|}\, w_{ji}, && (44) \end{aligned}$$

since the orthant condition on M implies that the sign of α_{ji} is always the same for a fixed j (when $\alpha_{ji} \neq 0$). Writing this factorization, of course, assumes that every $\sum_{k=1}^N \lambda_k \alpha_{jk} \neq 0$. If for some $j \in \{1,\ldots,m\}$ this is not the case, then since the sign of each $\lambda_k \alpha_{jk}$ is the same, $\lambda_k \alpha_{jk} = 0$ for every $k \in \{1,\ldots,N\}$. Thus every term in the corresponding sum $\sum_{i=1}^N \lambda_i \alpha_{ji} w_{ji}$ in (43) is 0, and so we may replace the sum over index i in (44) with an arbitrary element of A_j since it is multiplied by $0 = \sum_{k=1}^N \lambda_k \alpha_{jk}$. Hence

$$x \;=\; \sum_{j=1}^m \left(\sum_{k=1}^N \lambda_k \alpha_{jk}\right) x_j, \tag{45}$$

where

$$x_j \;=\; \begin{cases} \displaystyle\sum_{i=1}^N \frac{\lambda_i |\alpha_{ji}|}{\sum_{k=1}^N \lambda_k |\alpha_{jk}|}\, w_{ji} & \text{if } \sum_{k=1}^N \lambda_k \alpha_{jk} \neq 0, \\ \text{some } w_j \in A_j & \text{if } \sum_{k=1}^N \lambda_k \alpha_{jk} = 0. \end{cases}$$

When $\sum_{k=1}^N \lambda_k \alpha_{jk} \neq 0$, we have

$$\frac{\lambda_i |\alpha_{ji}|}{\sum_{k=1}^N \lambda_k |\alpha_{jk}|} \;\geq\; 0 \quad \text{for each } i \in \{1,\ldots,N\} \quad \text{and} \quad \sum_{i=1}^N \frac{\lambda_i |\alpha_{ji}|}{\sum_{k=1}^N \lambda_k |\alpha_{jk}|} \;=\; 1,$$

and so the weights on the w_{ji} are convex combination scalars, implying $x_j \in \operatorname{conv} A_j$. This likewise holds if $\sum_{k=1}^N \lambda_k \alpha_{jk} = 0$, since $w_j \in A_j \subset \operatorname{conv} A_j$. Furthermore, since

$$\left(\sum_{k=1}^N \lambda_k \alpha_{1k},\ \ldots,\ \sum_{k=1}^N \lambda_k \alpha_{mk}\right) \;=\; \sum_{k=1}^N \lambda_k (\alpha_{1k},\ \ldots,\ \alpha_{mk}) \;\in\; \operatorname{conv} M,$$

the expression in (45) shows that $x \in \oplus_{\operatorname{conv} M}\big(\operatorname{conv} A_1, \dots, \operatorname{conv} A_m\big)$, and therefore

$$\operatorname{conv}\big(\oplus_M (A_1, \dots, A_m)\big) \;\subset\; \oplus_{\operatorname{conv} M}\big(\operatorname{conv} A_1, \dots, \operatorname{conv} A_m\big).$$

The other direction of inclusion is Theorem 28.

(ii) Only small adjustments need to be made to handle the case when M has positive or negative symmetry and $A_1, \dots, A_m$ are o-symmetric. We proceed as above until (43), where we write

$$x \;=\; \sum_{j=1}^{m}\sum_{i=1}^{N} \lambda_i \alpha_{ji} w_{ji} \;=\; \sum_{j=1}^{m}\sum_{i=1}^{N} \lambda_i \varepsilon |\alpha_{ji}| (\pm w_{ji}).$$

Here $\varepsilon = 1$ or -1, depending on whether M has positive or negative symmetry, respectively, and the sign on w_{ji} is chosen so that $\lambda_i \alpha_{ji} w_{ji} = \lambda_i \varepsilon |\alpha_{ji}| (\pm w_{ji})$ for each i and j. Since every $\varepsilon|\alpha_{ji}|$ has the same sign (when it is nonzero) and $\pm w_{ji} \in A_j$ by o-symmetry, we may finish the proof using exactly the same steps as in (i). $\square$

Before proceeding to some significant corollaries, we offer the following examples to show that several plausible variations of (42) do not hold.

Example 30. Firstly, it is not generally true that

$$\operatorname{conv}\big(\oplus_M (A_1, \dots, A_m)\big) \;=\; \oplus_M(\operatorname{conv} A_1, \dots, \operatorname{conv} A_m).$$

For consider $M = [e_1, e_1 + e_2] \cup [e_2, e_1 + e_2] \subset \mathbb{R}^2$ along with $A_1 = \{(1, 0)\}$ and $A_2 = \{(0, 1)\}$. Then since

$$\oplus_M(A_1, A_2) \;=\; \bigcup_{(\alpha_1,\alpha_2)\in M} \alpha_1(1, 0) + \alpha_2(0, 1) \;=\; \bigcup_{(\alpha_1,\alpha_2)\in M} (\alpha_1, \alpha_2) \;=\; M,$$

we find

$$\operatorname{conv}\big(\oplus_M (A_1, A_2)\big) \;=\; \operatorname{conv} M \;\supsetneq\; M \;=\; \oplus_M(A_1, A_2) \;=\; \oplus_M(\operatorname{conv} A_1, \operatorname{conv} A_2).$$

This is an illustration of a more general fact, since for an M contained in an orthant of $\mathbb{R}^m$, by Theorem 29(i) and the inclusion $\operatorname{conv} M \supset M$,

$$\begin{aligned}&\operatorname{conv}\big(\oplus_M (A_1, \dots, A_m)\big)\\ &\qquad = \oplus_{\operatorname{conv} M}\big(\operatorname{conv} A_1, \dots, \operatorname{conv} A_m\big) \;\supset\; \oplus_M\big(\operatorname{conv} A_1, \dots, \operatorname{conv} A_m\big).\end{aligned}$$

In words, when M is contained in an orthant of $\mathbb{R}^m$, distributing the convex hull to just the summands generally yields a smaller set. A similar statement holds for M with positive or negative symmetry when the A_j's are all o-symmetric.

Example 31. A simple example also shows that

$$\operatorname{conv}\big(\oplus_M(A_1,\ldots,A_m)\big) \neq \oplus_{\operatorname{conv} M}(A_1,\ldots,A_m),$$

generally speaking. Consider M from Example 30 and take $N=\{1\}\subset\mathbb{R}$. Then

$$\operatorname{conv}\big(\oplus_N(M)\big) = \operatorname{conv} M \supsetneq M = \oplus_N(M) = \oplus_{\operatorname{conv} N}(M).$$

More generally, whenever M is contained in an orthant of $\mathbb{R}^m$, by Theorem 29(i) and the fact that $\operatorname{conv} A_j \supset A_j$,

$$\operatorname{conv}\big(\oplus_M(A_1,\ldots,A_m)\big) = \oplus_{\operatorname{conv} M}(\operatorname{conv} A_1,\ldots,\operatorname{conv} A_m) \supset \oplus_{\operatorname{conv} M}(A_1,\ldots,A_m).$$

Hence as in the case of the summands, distributing the convex hull to just the M-set generally yields a smaller set.

A related question is whether or not

$$\oplus_M(\operatorname{conv} A_1,\ldots,\operatorname{conv} A_m) = \oplus_{\operatorname{conv} M}(A_1,\ldots,A_m).$$

Example 30, however, shows that this is not the case, since there

$$\oplus_M(\operatorname{conv} A_1,\operatorname{conv} A_2) = \oplus_M(A_1,A_2) = M \subsetneq \operatorname{conv} M = \oplus_{\operatorname{conv} M}(A_1,A_2).$$

Moving the convex hull from the summands to the M-set does not always yield this inclusion, though, since in Example 31,

$$\oplus_N(\operatorname{conv} M) = \operatorname{conv} M \supsetneq M = \oplus_N(M) = \oplus_{\operatorname{conv} N}(M).$$

We can mine a rich vein of corollaries from Theorem 29 regarding what may initially appear to be disparate topics. The following subsections discuss conclusions we can draw regarding the nature of an M-sum of convex polytopes, the necessary conditions for $\oplus_M$ to map $(\mathcal{K}^n)^m$ to $\mathcal{K}^n$ in the case $n<m$, and how Lutwak, Yang and Zhang's pointwise definition (16) of L_p addition extends to the case $p=\infty$.

4.1 M-sums of convex polytopes

Corollary 32. (i) *Let M be a convex polytope contained in one of the 2^m closed orthants of $\mathbb{R}^m$. If $P_1,\ldots,P_m\subset\mathbb{R}^n$ are each convex polytopes, then $\oplus_M(P_1,\ldots,P_m)$ is a convex polytope.*

(ii) *Let $M\subset\mathbb{R}^m$ be a convex polytope with positive or negative symmetry. If $P_1,\ldots,P_m\subset\mathbb{R}^n$ are each o-symmetric convex polytopes, then $\oplus_M(P_1,\ldots,P_m)$ is a convex polytope.*

Proof. (i) If M and $P_1, \ldots, P_m$ are each convex polytopes, then $M = \operatorname{conv} M'$, $P_1 = \operatorname{conv} P'_1, \ldots,\ P_m = \operatorname{conv} P'_m$ for some finite sets $M' \subset \mathbb{R}^m$, $P'_1, \ldots, P'_m \subset \mathbb{R}^n$. Since $\operatorname{conv} M'$ is contained in one of the 2^m closed orthants of $\mathbb{R}^m$, so is the smaller set M', and thus by Theorem 29(i),

$$\oplus_M(P_1, \ldots, P_m) \;=\; \oplus_{\operatorname{conv} M'}(\operatorname{conv} P'_1, \ldots, \operatorname{conv} P'_m) \;=\; \operatorname{conv}\big(\oplus_{M'}(P'_1, \ldots, P'_m)\big). \tag{46}$$

As each of $M', P'_1, \ldots, P'_m$ is finite, $\oplus_{M'}(P'_1, \ldots, P'_m)$ is finite, and thus this is a convex polytope.

The proof for (ii) uses Theorem 29(ii) analogously. $\square$

These results furnish us with an algorithm for computing $\oplus_M(P_1, \ldots, P_m)$ when $M = \operatorname{conv} M'$ is a convex polytope contained in an orthant of $\mathbb{R}^m$ and $P_1 = \operatorname{conv} P'_1, \ldots, P_m = \operatorname{conv} P'_m \subset \mathbb{R}^n$ are also convex polytopes (or when M has positive or negative symmetry and $P_1, \ldots, P_m$ are o-symmetric). If

$$M \;=\; \operatorname{conv} M' \;=\; \operatorname{conv}\{x_1, \ldots, x_\ell\}$$

for some $x_1, \ldots, x_\ell \in \mathbb{R}^m$, then by (46) and Theorem 11,

$$\begin{aligned}
\oplus_M(P_1, \ldots, P_m) \;&=\; \operatorname{conv}\big(\oplus_{M'}(P'_1, \ldots, P'_m)\big) \\
&=\; \operatorname{conv}\big(\oplus_{\bigcup_{k=1}^{\ell}\{x_k\}}(P'_1, \ldots, P'_m)\big) \\
&=\; \operatorname{conv}\left(\bigcup_{k=1}^{\ell} \oplus_{\{x_k\}}(P'_1, \ldots, P'_m)\right).
\end{aligned} \tag{47}$$

Writing $x_k = (\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{mk})$, we have by (1) that

$$\oplus_{\{x_k\}}(P'_1, \ldots, P'_m) \;=\; \sum_{j=1}^{m} \alpha_{jk} P'_j. \tag{48}$$

Equations (47) and (48) therefore provide the following three-step algorithm to compute $\oplus_M(P_1, \ldots, P_m)$:

1. For each $k \in \{1, \ldots, \ell\}$, compute $\sum_{j=1}^{m} \alpha_{jk} P'_j$.
2. Union these ℓ sets.
3. Take the convex hull.

This process essentially reduces determining $\oplus_M(P_1, \ldots, P_m)$ to finding a finite number of Minkowski sums $\sum_{j=1}^{m} \alpha_{jk} P'_j$ in which each summand is a finite set.

4.2 Necessary conditions for $\oplus_M : (\mathcal{K}^n)^m \to \mathcal{K}^n$ in the case $n < m$

A foundational result in M-addition theory is Proposition 2, which details necessary and sufficient conditions on M for $\oplus_M$ to map $(\mathcal{K}^n)^m$ into $\mathcal{K}^n$. Although the proposition starts by assuming $2 \le m \le n$, the concluding parenthetical remark clarifies that the sufficient conditions actually hold for any $m, n \ge 2$. The only necessary condition that carries over from the case $2 \le m \le n$ to when $2 \le n < m$, however, is that M must be contained in a closed orthant of $\mathbb{R}^m$. In particular, there is no necessary conclusion about the compactness and convexity of M. In this section we prove that if $\oplus_M : (\mathcal{K}^n)^m \to \mathcal{K}^n$ and $2 \le n < m$, then M must be bounded and certain projections of it must be convex. Then, with the assistance of Theorem 29, we offer an example of a set M where $\oplus_M : (\mathcal{K}^n)^m \to \mathcal{K}^n$ with $n < m$ but where M is not convex, thus showing that our convexity conclusions essentially cannot be strengthened.

Theorem 33. *If $2 \le n < m$ and $\oplus_M : (\mathcal{K}^n)^m \to \mathcal{K}^n$, then M is contained in one of the 2^m closed orthants of $\mathbb{R}^m$, is bounded, and every orthogonal projection of M onto a coordinate subspace of $\mathbb{R}^m$ of dimension n or less is convex.*

Proof. As mentioned above, Proposition 2 ensures that M is contained in a closed orthant of $\mathbb{R}^m$. Let $\{e_1, \ldots, e_n\}$ be the collection of standard basis vectors in $\mathbb{R}^n$ and let $\{\tilde{e}_1, \ldots, \tilde{e}_n, \ldots, \tilde{e}_m\}$ be those of $\mathbb{R}^m$. We will show that the projection of M onto $\operatorname{span}\{\tilde{e}_1, \ldots, \tilde{e}_n\} \subsetneq \mathbb{R}^m$ is compact and convex; the proofs for other projections involving the same amount or fewer of the basis vectors are similar. We observe that

$$\begin{aligned}\oplus_M \big(\{e_1\}, \ldots, \{e_n\}, \{o\}, \ldots, \{o\}\big) \\ = \{ (\alpha_1, \ldots, \alpha_n) : (\alpha_1, \ldots, \alpha_n, \ldots, \alpha_m) \in M \} \in \mathcal{K}^n,\end{aligned}$$

by assumption. Since this is an embedding of $M \big| \operatorname{span}\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ into $\mathbb{R}^n$, we see that $M \big| \operatorname{span}\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ is itself compact and convex.

As this similarly holds for any subspace spanned by n or fewer of $\tilde{e}_1, \ldots, \tilde{e}_m$, M must be bounded. $\square$

The following example shows that when $n < m$, there are non-convex sets M such that $\oplus_M : (\mathcal{K}^n)^m \to \mathcal{K}^n$.

Example 34. Let $M = \partial([0,1]^3)$, a manifestly non-convex set. We claim that $\oplus_M : (\mathcal{K}^2)^3 \to \mathcal{K}^2$ nevertheless. To see this, select $K_1, K_2, K_3 \in \mathcal{K}^2$ arbitrarily. Then since each of M, K_1, K_2 and K_3 is compact, so is $\oplus_M(K_1, K_2, K_3)$, and to show this M-sum is convex we will prove

$$\operatorname{conv}\big(\oplus_M (K_1, K_2, K_3)\big) = \oplus_M(K_1, K_2, K_3).$$

For this it suffices to show $\operatorname{conv}\big(\oplus_M (K_1, K_2, K_3)\big) \subset \oplus_M(K_1, K_2, K_3)$, as the reverse inclusion always holds. Since M is contained in a single octant of $\mathbb{R}^3$, by Theorem 29(i),

$$\operatorname{conv}\big(\oplus_M (K_1, K_2, K_3)\big)$$
$$= \oplus_{\operatorname{conv} M}(\operatorname{conv} K_1, \operatorname{conv} K_2, \operatorname{conv} K_3) = \oplus_{\operatorname{conv} M}(K_1, K_2, K_3).$$

Since $\operatorname{conv} M = [0,1]^3$, $x \in \oplus_{\operatorname{conv} M}(K_1, K_2, K_3)$ implies that $x = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ for some $v_j = (v_{1j}, v_{2j}) \in K_j$ and $0 \le \alpha_j \le 1$, $j \in \{1,2,3\}$. Embed the v_j's in $\mathbb{R}^3$ by setting

$$w_1 = (v_{11}, v_{21}, 0), \qquad w_2 = (v_{12}, v_{22}, 0) \qquad \text{and} \qquad w_3 = (v_{13}, v_{23}, 1).$$

Then the boundary of the (possibly degenerate) parallelepiped $\sum_{j=1}^{3}[0, w_j]$ generated by w_1, w_2 and w_3 is precisely $\oplus_M\big(\{w_1\}, \{w_2\}, \{w_3\}\big)$. Since $\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 = y$ is contained within the parallelepiped, upon orthogonally projecting to the xy-plane we have $y \,|\, e_3^{\perp} \in \oplus_M\big(\{w_1\}, \{w_2\}, \{w_3\}\big) \,\big|\, e_3^{\perp}$. Identifying the xy-plane with $\mathbb{R}^2$ and using the covariance of M-addition with respect to linear transformations (Theorem 5), we have

$$\begin{aligned} x &= \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \alpha_1 w_1 \,|\, e_3^{\perp} + \alpha_2 w_2 \,|\, e_3^{\perp} + \alpha_3 w_3 \,|\, e_3^{\perp} = y \,|\, e_3^{\perp} \\ &\in \oplus_M\big(\{w_1\}, \{w_2\}, \{w_3\}\big) \,\big|\, e_3^{\perp} \\ &= \oplus_M\big(\{w_1\} \,|\, e_3^{\perp}, \{w_2\} \,|\, e_3^{\perp}, \{w_3\} \,|\, e_3^{\perp}\big) \\ &= \oplus_M\big(\{v_1\}, \{v_2\}, \{v_3\}\big) \subset \oplus_M(K_1, K_2, K_3). \end{aligned}$$

4.3 A pointwise formula for L_∞ addition

As alluded to in Section 2, Lutwak, Yang and Zhang [2] have recently provided the pointwise formula (16) for L_p addition. Beyond the light this sheds on how L_p addition combines sets compared with the former implicit definition (14), (16) further allows this operation to be extended to arbitrary subsets of $\mathbb{R}^n$. Since the authors restrict their attention to real p, however, it is not clear whether or not a similar pointwise formula holds when $p = \infty$. We can easily address this question using Theorem 29(i) and show that the natural extension of (16) to this situation does, in fact, hold. Since the dual exponent of $p = \infty$ is $q = 1$, this extension is the formula in the following theorem.

Theorem 35. *If $K, L \in \mathcal{K}^n$, then*

$$K +_\infty L = \{\, (1-\lambda)v + \lambda w \ : \ 0 \le \lambda \le 1,\ v \in K,\ w \in L \,\}.$$

Proof. Let $K, L \in \mathcal{K}^n$. One can see from the support function definition (15) of $K +_\infty L$ that $K +_\infty L = \operatorname{conv}(K \cup L)$. By the pointwise definition of M-addition (2), the convexity of K and L, Theorem 29(i) and Theorem 11, we have

$$\begin{aligned} \{\, (1-\lambda)v + \lambda w \ : \ 0 \le \lambda \le 1,\ v \in K,\ w \in L \,\} &= K \oplus_{[e_1, e_2]} L \\ &= \operatorname{conv} K \oplus_{\operatorname{conv}(\{e_1\} \cup \{e_2\})} \operatorname{conv} L \\ &= \operatorname{conv}\big(K \oplus_{\{e_1\} \cup \{e_2\}} L\big) \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{conv}\big((K \oplus_{\{e_1\}} L) \cup (K \oplus_{\{e_2\}} L)\big) \\
&= \operatorname{conv}(K \cup L),
\end{aligned}
$$

and thus the claimed result holds. □

5 Interaction with operations on many sets

Having seen instances of operations which map individual sets to individual sets "distribute" over M-addition in the sense of equations (35) to (38), we turn our attention to binary maps between sets. Here we wish to focus on the more narrow inquiry as to whether there are mappings $*$ such that

$$\oplus_{M_1 * M_2}(A_1, \ldots, A_m) \;=\; \oplus_{M_1}(A_1, \ldots, A_m) * \oplus_{M_2}(A_1, \ldots, A_m),$$

or where, more generally,

$$
\begin{aligned}
&\oplus_{M_1 * M_2 * \cdots * M_n}(A_1, \ldots, A_m) \\
&\qquad = \oplus_{M_1}(A_1, \ldots, A_m) * \oplus_{M_2}(A_1, \ldots, A_m) * \cdots * \oplus_{M_n}(A_1, \ldots, A_m).
\end{aligned}
$$

A reasonable starting point is to consider the case when $*$ is M-addition itself. After all, familiar operations such as Minkowski addition and L_p addition are merely M-addition for a specific M. Whatever conclusions we can draw for M-addition in general, then, will cover these particular instances. We thus begin by asking under what circumstances

$$\oplus_{\oplus_N(M_1,\ldots,M_n)}(A_1, \ldots, A_m) \;=\; \oplus_N\big(\oplus_{M_1}(A_1, \ldots, A_m), \ldots, \oplus_{M_n}(A_1, \ldots, A_m)\big),$$

which is to say, when does M-addition distribute over itself?

The short answer is hardly surprising: not always. But one direction of containment does always hold.

Theorem 36. *For any $M_1, \ldots, M_n \subset \mathbb{R}^m$, $N \subset \mathbb{R}^n$ and $A_1, \ldots, A_m \subset \mathbb{R}^p$,*

$$\oplus_{\oplus_N(M_1,\ldots,M_n)}(A_1, \ldots, A_m) \;\subset\; \oplus_N\big(\oplus_{M_1}(A_1, \ldots, A_m), \ldots, \oplus_{M_n}(A_1, \ldots, A_m)\big). \tag{49}$$

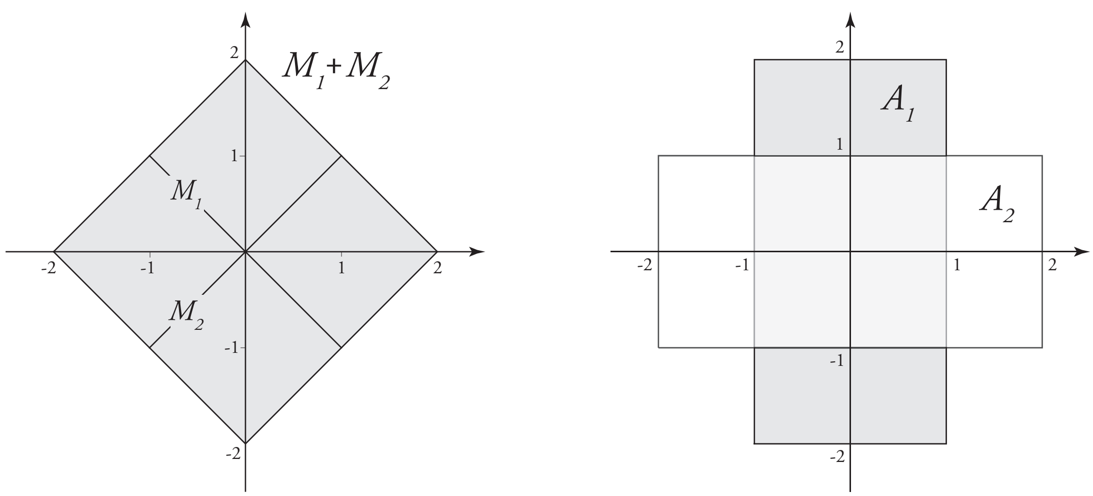
Proof. First note that if $x \in \oplus_N(M_1, \ldots, M_n)$, then for some $(\beta_1, \ldots, \beta_n) \in N$ and $v_k = (\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{mk}) \in M_k$, $k \in \{1, \ldots, n\}$,

$$
\begin{aligned}
x &= \sum_{k=1}^{n} \beta_k v_k \\
&\qquad = \sum_{k=1}^{n} \beta_k(\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{mk}) \;=\; \Big(\sum_{k=1}^{n} \beta_k \alpha_{1k},\, \sum_{k=1}^{n} \beta_k \alpha_{2k},\, \ldots,\, \sum_{k=1}^{n} \beta_k \alpha_{mk}\Big).
\end{aligned}
$$

Using this along with (1), (10) and Corollary 13(i) therefore yields

$$
\begin{aligned}
&\oplus_{\oplus_N(M_1,\dots,M_n)}(A_1,\dots,A_m) && (50)\\
&\quad = \bigcup_{(\gamma_1,\dots,\gamma_m)\in\oplus_N(M_1,\dots,M_n)} \sum_{j=1}^{m} \gamma_j A_j \\
&\quad = \bigcup_{(\beta_1,\dots,\beta_n)\in N} \bigcup_{(\alpha_{11},\dots,\alpha_{m1})\in M} \cdots \bigcup_{(\alpha_{1n},\dots,\alpha_{mn})\in M} \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \beta_k \alpha_{jk} \right) A_j && (51)\\
&\quad \subset \bigcup_{(\beta_1,\dots,\beta_n)\in N} \bigcup_{(\alpha_{11},\dots,\alpha_{m1})\in M} \cdots \bigcup_{(\alpha_{1n},\dots,\alpha_{mn})\in M} \sum_{j=1}^{m} \sum_{k=1}^{n} \beta_k \alpha_{jk} A_j && (52)\\
&\quad = \bigcup_{(\beta_1,\dots,\beta_n)\in N} \bigcup_{(\alpha_{11},\dots,\alpha_{m1})\in M} \cdots \bigcup_{(\alpha_{1n},\dots,\alpha_{mn})\in M} \sum_{k=1}^{n} \beta_k \sum_{j=1}^{m} \alpha_{jk} A_j \\
&\quad = \bigcup_{(\beta_1,\dots,\beta_n)\in N} \sum_{k=1}^{n} \beta_k \bigcup_{(\alpha_{1k},\alpha_{2k},\dots,\alpha_{mk})\in M_k} \sum_{j=1}^{m} \alpha_{jk} A_j \\
&\quad = \bigcup_{(\beta_1,\dots,\beta_n)\in N} \sum_{k=1}^{n} \beta_k \oplus_{M_k} (A_1,\dots,A_m) \\
&\quad = \oplus_N\big(\oplus_{M_1} (A_1,\dots,A_m),\dots,\oplus_{M_n}(A_1,\dots,A_m)\big). && \square
\end{aligned}
$$

Example 37. To show that the containment in (49) can be strict, we consider a case where $m = n = p = 2$ and take $N = \{(1,1)\}$, $M_1 = [-e_1 + e_2, e_1 - e_2]$, $M_2 =$



(a) (b)

Figure 7: An example which shows the containment of (49) can be strict. The diagonal line segments in 7a are the sets M_1 and M_2 while $M_1 + M_2$ is the shaded parallelogram.

$[-e_1-e_2, e_1+e_2]$ (as in Figure 7a), $A_1 = [-1,1]\times[-2,2]$ and $A_2 = [-2,2]\times[-1,1]$ (see Figure 7b). Since $\oplus_N$ is just Minkowski addition, showing the desired containment amounts to proving

$$\oplus_{M_1+M_2}(A_1,A_2) \subsetneq \oplus_{M_1}(A_1,A_2)+\oplus_{M_2}(A_1,A_2).$$

Here $M_1+M_2 = \mathrm{conv}\,\{2e_1, 2e_2, -2e_1, -2e_2\}$, as also in Figure 7a. Since $A_1, A_2 \in \mathcal{K}^2_s$, $\oplus_M(A_1,A_2) = \oplus_{\widehat{M}}(A_1,A_2)$ for any $M\subset\mathbb{R}^2$ by Proposition 3, and as M_1+M_2 is 1-unconditional, the 1-unconditional hull of $S=(M_1+M_2)\cap[0,\infty)^2$ is M_1+M_2. It follows that

$$\oplus_{M_1+M_2}(A_1,A_2) = \oplus_{\widehat{S}}(A_1,A_2) = \oplus_S(A_1,A_2) = \oplus_{(M_1+M_2)\cap[0,\infty)^2}(A_1,A_2).$$

As $S=\mathrm{conv}\{o,2e_1,2e_2\}$ and A_1 and A_2 are convex, by Theorems 29(i) and 11 we have

$$\begin{aligned}\oplus_{(M_1+M_2)\cap[0,\infty)^2}(A_1,A_2) &= \oplus_{\mathrm{conv}\{o,2e_1,2e_2\}}(\mathrm{conv}\,A_1, \mathrm{conv}\,A_2)\\ &= \mathrm{conv}\left(\oplus_{\{o,2e_1,2e_2\}}(A_1,A_2)\right)\\ &= \mathrm{conv}\left(\oplus_{\{o\}\cup\{2e_1\}\cup\{2e_2\}}(A_1,A_2)\right)\\ &= \mathrm{conv}\left(\oplus_{\{o\}}(A_1,A_2)\cup\oplus_{\{2e_1\}}(A_1,A_2)\cup\oplus_{\{2e_2\}}(A_1,A_2)\right)\\ &= \mathrm{conv}\left(\{o\}\cup 2A_1\cup 2A_2\right) = 2\,\mathrm{conv}(A_1\cup A_2),\end{aligned}$$

the larger octagon of Figure 8. On the other hand, since $\widehat{M_1}=\widehat{M_2}$,

$$\begin{aligned}\oplus_{M_1}(A_1,A_2) &= \oplus_{M_2}(A_1,A_2) = \oplus_{\widehat{M_2}}(A_1,A_2)\\ &= \oplus_{[-1,1]^2}(A_1,A_2) = \oplus_{\widehat{[0,1]^2}}(A_1,A_2) = \oplus_{[0,1]^2}(A_1,A_2).\end{aligned}$$

Furthermore, since

$$\oplus_{[0,1]^2}(A_1,A_2) = \bigcup\left\{\alpha_1A_1+\alpha_2A_2 : 0\le\alpha_1,\alpha_2\le 1\right\}$$

and $\alpha_1A_1+\alpha_2A_2\subset A_1+A_2$ for all $(\alpha_1,\alpha_2)\in[0,1]^2$, $\oplus_{[0,1]^2}(A_1,A_2)=A_1+A_2=[-3,3]^2$, and so

$$\oplus_{M_1}(A_1,A_2)+\oplus_{M_2}(A_1,A_2) = [-3,3]^2+[-3,3]^2 = [-6,6]^2,$$

a strict superset of $\oplus_{M_1+M_2}(A_1,A_2)$, as seen in Figure 8.

The proof of Theorem 36 shows that if equality could be preserved in moving from (51) to (52), then containment could be replaced with equality in (49). This would be accomplished if

$$\left(\sum_{k=1}^n \beta_k\alpha_{jk}\right)A_j = \sum_{k=1}^n \beta_k\alpha_{jk}A_j \tag{53}$$

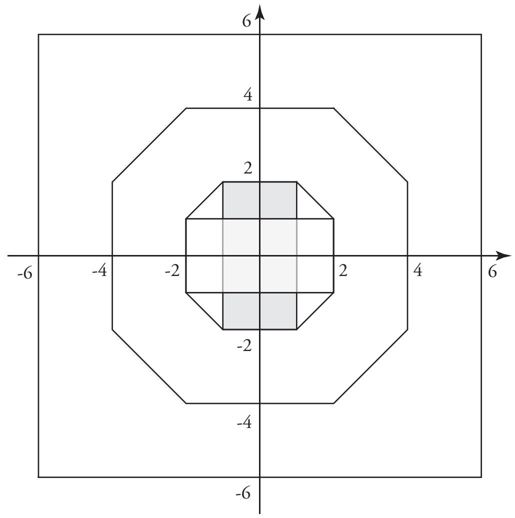


Figure 8: The inner octagon is $\operatorname{conv}(A_1 \cup A_2)$, while the larger one is $2\operatorname{conv}(A_1 \cup A_2) = \oplus_{M_1+M_2}(A_1, A_2)$. The square is $\oplus_{M_1}(A_1, A_2) + \oplus_{M_2}(A_1, A_2)$, showing $\oplus_{M_1+M_2}(A_1, A_2) \subsetneq \oplus_{M_1}(A_1, A_2) + \oplus_{M_2}(A_1, A_2)$.

for all $(\alpha_{1k}, \ldots, \alpha_{mk}) \in M_k$, $k \in \{1, \ldots, n\}$, $(\beta_1, \ldots, \beta_n) \in N$ and $A_1, \ldots, A_m$. We considered the version of this equation with just two scalars in (13), and the natural extension to n scalars tells us that (53) holds if $\beta_1\alpha_{j1}, \ldots, \beta_n\alpha_{jn}$ are all non-positive or all non-negative and the A_j's are convex. These conditions motivate the following theorem.

Theorem 38. *Let $M_1, \ldots, M_n \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$. If for any $(\beta_1, \ldots, \beta_n) \in N$ the sets $\beta_1 M_1, \ldots, \beta_n M_n$ all lie within the same closed orthant of $\mathbb{R}^m$, then for any convex $C_1, \ldots, C_m \subset \mathbb{R}^p$,*

$$\oplus_{\oplus_N(M_1,\ldots,M_n)}(C_1, \ldots, C_m) = \oplus_N\big(\oplus_{M_1}(C_1, \ldots, C_m), \ldots, \oplus_{M_n}(C_1, \ldots, C_m)\big). \tag{54}$$

Proof. With A_j replaced by C_j, the proof proceeds as in Theorem 36 until (51). Notice that in the inner sum there index j is fixed, and hence the terms $\beta_1\alpha_{j1}, \beta_2\alpha_{j2}, \ldots, \beta_n\alpha_{jn}$ are the jth components of vectors

$$(\beta_1\alpha_{11}, \beta_1\alpha_{21}, \ldots, \beta_1\alpha_{m1}) \in \beta_1 M_1, \ \ldots, \ (\beta_n\alpha_{1n}, \beta_n\alpha_{2n}, \ldots, \beta_n\alpha_{mn}) \in \beta_n M_n.$$

Since $\beta_1 M_1, \ldots, \beta_n M_n$ all reside in the same closed orthant, each of these terms has the same sign when it is not zero, and so by the convexity of the C_j's, (53) holds with A_j replaced with C_j. Thus in this case we move from (51) to (52) while maintaining equality, and so (54) follows. $\square$

Example 37 is also instructive in light of this result. There $N = \{(1, 1)\}$, $1 \cdot M_1$ and $1 \cdot M_2$ do not both reside in the same quadrant, and we saw

$$\oplus_{\oplus_N(M_1,M_2)}(A_1,A_2) \;=\; \oplus_{M_1+M_2}(A_1,A_2)$$
$$\subsetneq\; \oplus_{M_1}(A_1,A_2)+\oplus_{M_2}(A_1,A_2) \;=\; \oplus_N\big(\oplus_{M_1}(A_1,A_2),\oplus_{M_2}(A_1,A_2)\big).$$

Hence it is possible that (54) does not hold if the $\beta_j M_j$ do not all reside in the same orthant.

Note that the condition on the orthants of $\beta_1 M_1,\dots,\beta_n M_n$ in Theorem 38 implies that each non-dummy summand M_j in $\oplus_N(M_1,\dots,M_n)$ resides within a *single* orthant of $\mathbb{R}^m$ (although the orthant need not be the same for each summand).

Corollary 39. (i) *Let $M_1,\dots,M_n$ be contained in the same closed orthant of $\mathbb{R}^m$. Then for any convex $C_1,\dots,C_m\subset\mathbb{R}^p$,*

$$\oplus_{\sum_{j=1}^n M_j}(C_1,\dots,C_m) \;=\; \sum_{j=1}^n \oplus_{M_j}(C_1,\dots,C_m).$$

(ii) *Let $M_1,\dots,M_n\in\mathcal{K}^m_o$ be contained in the same closed orthant of $\mathbb{R}^m$. Then for any $K_1,\dots,K_m\in\mathcal{K}^n$ and $1<p\le\infty$,*

$$\oplus_{M_1+_p\cdots+_pM_n}(K_1,\dots,K_m) \;=\; \oplus_{M_1}(K_1,\dots,K_m)+_p\cdots+_p\oplus_{M_n}(K_1,\dots,K_m). \tag{55}$$

Proof. (i) The Minkowski sum $\sum_{j=1}^n M_j$ is $\oplus_N(M_1,\dots,M_n)$ for $N=\{(1,1,\dots,1)\}\subset\mathbb{R}^n$. Since by assumption $1\cdot M_1,\dots,1\cdot M_n$ all belong to the same closed orthant of $\mathbb{R}^m$, by Theorem 38,

$$\begin{aligned}\oplus_{\sum_{j=1}^n M_j}(C_1,\dots,C_m) &= \oplus_{\oplus_N(M_1,\dots,M_n)}(C_1,\dots,C_m)\\ &= \oplus_N\big(\oplus_{M_1}(C_1,\dots,C_m),\dots,\oplus_{M_n}(C_1,\dots,C_m)\big)\\ &= \sum_{j=1}^m \oplus_{M_j}(C_1,\dots,C_m).\end{aligned}$$

(ii) Since each $M_j\in\mathcal{K}^m_o$ is contained in an orthant of $\mathbb{R}^m$, $\oplus_{M_j}(K_1,\dots,K_m)\in\mathcal{K}^n_o$ for every $j\in\{1,\dots,k\}$ by Proposition 2. Hence the L_p sum on the right-hand side of (55) makes sense. Furthermore,

$$M_1+_p+\cdots+_pM_n \;=\; \oplus_N(M_1,\dots,M_n)$$

where N is the set M in (17) with dimension m replaced by n. Since $N\subset[0,\infty)^n$, the fact that $M_1,\dots,M_n$ are all in the same orthant ensures that $\beta_1M_1,\dots,\beta_nM_n$ also are for any $(\beta_1,\dots,\beta_n)\in N$, implying

$$\begin{aligned}\oplus_{M_1+_p\cdots+_pM_n}(K_1,\dots,K_m) &= \oplus_{\oplus_N(M_1,\dots,M_n)}(K_1,\dots,K_m)\\ &= \oplus_N\big(\oplus_{M_1}(K_1,\dots,K_m),\dots,\oplus_{M_n}(K_1,\dots,K_m)\big)\\ &= \oplus_{M_1}(K_1,\dots,K_m)+_p\cdots+_p\oplus_{M_n}(K_1,\dots,K_m). \qquad\square\end{aligned}$$

The following is the version of Theorem 38 which covers the case of sets with o-symmetry.

Theorem 40. *Let $M_1, \ldots, M_n \subset \mathbb{R}^m$ be 1-unconditional. Then for any $N \subset \mathbb{R}^n$ and any o-symmetric convex $C_1, \ldots, C_m \subset \mathbb{R}^p$,*

$$\oplus_{\oplus_N(M_1,\ldots,M_n)}(C_1, \ldots, C_m) \;=\; \oplus_N\big(\oplus_{M_1} (C_1, \ldots, C_m), \ldots, \oplus_{M_n}(C_1, \ldots, C_m)\big).$$

Proof. Using the union definition of M-addition and the ability to interchange a union and a sum (Corollary 13), we have

$$\begin{aligned}
&\oplus_N\big(\oplus_{M_1} (C_1, \ldots, C_m), \ldots, \oplus_{M_n} (C_1, \ldots, C_m)\big)\\
&\quad= \bigcup_{(\beta_1,\ldots,\beta_n)\in N} \sum_{k=1}^{n} \beta_k \oplus_{M_k} (C_1, \ldots, C_m)\\
&\quad= \bigcup_{(\beta_1,\ldots,\beta_n)\in N} \sum_{k=1}^{n} \beta_k \bigcup_{(\alpha_{1k},\ldots,\alpha_{mk})\in M_k} \sum_{j=1}^{m} \alpha_{jk} C_j\\
&\quad= \bigcup_{(\beta_1,\ldots,\beta_n)\in N} \bigcup_{\substack{(\alpha_{1k},\ldots,\alpha_{mk})\in M_k\\ k\in\{1,\ldots,n\}}} \sum_{k=1}^{n} \sum_{j=1}^{m} \beta_k \alpha_{jk} C_j\\
&\quad= \bigcup_{(\beta_1,\ldots,\beta_n)\in N} \bigcup_{\substack{(\alpha_{1k},\ldots,\alpha_{mk})\in M_k\\ k\in\{1,\ldots,n\}}} \sum_{j=1}^{m} \sum_{k=1}^{n} \beta_k \alpha_{jk} C_j, \qquad (56)\\
&\quad= \bigcup_{(\beta_1,\ldots,\beta_n)\in N} \bigcup_{\substack{(\alpha_{1k},\ldots,\alpha_{mk})\in M_k\\ k\in\{1,\ldots,n\}}} \sum_{j=1}^{m} \sum_{k=1}^{n} |\beta_k \alpha_{jk}| C_j,
\end{aligned}$$

since each C_j is o-symmetric. Define

$$\varepsilon(\beta_k, \alpha_{jk}) \;=\; \begin{cases} 1 & \text{if } \beta_k \text{ and } \alpha_{jk} \text{ have the same sign,}\\ -1 & \text{otherwise.}\end{cases}$$

Then $|\beta_k\alpha_{jk}| = \varepsilon(\beta_k, \alpha_{jk})\beta_k\alpha_{jk}$, and since for a fixed j the scalars $\varepsilon(\beta_k, \alpha_{jk})\beta_k\alpha_{jk}$ are all nonnegative and C_j is convex, the generalization of (13) to n scalars with K replaced by C_j holds, implying

$$\begin{aligned}
&\bigcup_{(\beta_1,\ldots,\beta_n)\in N} \bigcup_{\substack{(\alpha_{1k},\ldots,\alpha_{mk})\in M_k\\ k\in\{1,\ldots,n\}}} \sum_{j=1}^{m} \sum_{k=1}^{n} \varepsilon(\beta_k, \alpha_{jk})\beta_k\alpha_{jk} C_j\\
&\quad= \bigcup_{(\beta_1,\ldots,\beta_n)\in N} \bigcup_{\substack{(\alpha_{1k},\ldots,\alpha_{mk})\in M_k\\ k\in\{1,\ldots,n\}}} \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \varepsilon(\beta_k, \alpha_{jk})\beta_k\alpha_{jk}\right) C_j. \qquad (57)
\end{aligned}$$

Furthermore, for any $(\beta_1, \dots, \beta_n) \in N$ and $(\alpha_{1k}, \dots, \alpha_{mk}) \in M_k$, $k \in \{1, \dots, n\}$,

$$\Big(\sum_{k=1}^{n} \varepsilon(\beta_k, \alpha_{1k})\beta_k\alpha_{1k}, \; \dots, \; \sum_{k=1}^{n} \varepsilon(\beta_k, \alpha_{mk})\beta_k\alpha_{mk} \Big)$$

$$= \; \sum_{k=1}^{n} \beta_k \big(\varepsilon(\beta_k, \alpha_{1k})\alpha_{1k}, \dots, \varepsilon(\beta_k, \alpha_{mk})\alpha_{mk} \big) \; \in \; \oplus_N(M_1, \dots, M_n),$$

since by 1-unconditionality $\big(\varepsilon(\beta_k, \alpha_{1k})\alpha_{1k}, \dots, \varepsilon(\beta_k, \alpha_{mk})\alpha_{mk}\big) \in M_k$. It follows that the set in (57) is contained in $\oplus_{\oplus_N(M_1,\dots,M_n)}(C_1, \dots, C_m)$, which is to say,

$$\oplus_N\big(\oplus_{M_1} (C_1, \dots, C_m), \dots, \oplus_{M_n}(C_1, \dots, C_m)\big) \; \subset \; \oplus_{\oplus_N(M_1,\dots,M_n)}(C_1, \dots, C_m).$$

Combined with Theorem 36, this completes the proof. □

If N is contained within the positive and negative orthants of $\mathbb{R}^n$, then weaker conditions on the M_j's suffice for obtaining the result in Theorem 40.

Corollary 41. *Let $M_1, \dots, M_n \subset \mathbb{R}^m$ be such that every M_j has positive symmetry or every M_j has negative symmetry. Then for any $N \subset (-\infty, 0]^n \cup [0, \infty)^n$ and any o-symmetric convex $C_1, \dots, C_m \subset \mathbb{R}^p$,*

$$\oplus_{\oplus_N(M_1,\dots,M_n)}(C_1, \dots, C_m) \; = \; \oplus_N\big(\oplus_{M_1} (C_1, \dots, C_m), \dots, \oplus_{M_n}(C_1, \dots, C_m)\big).$$

Proof. We proceed exactly as in the proof of Theorem 40 through (56). To be able to factor out the C_j as in (57), we need all the $\beta_k\alpha_{jk}$'s, $k \in \{1, \dots, n\}$, to be of the same sign. By the containment condition on N, every component of $(\beta_1, \dots, \beta_n) \in N$ is either nonpositive or nonnegative, and thus to obtain scalars of the same sign it suffices to take $\beta_k\varepsilon|\alpha_{jk}|$, where

$$\varepsilon \; = \; \begin{cases} 1 & \text{if each } M_j \text{ has positive symmetry,} \\ -1 & \text{if each } M_j \text{ has negative symmetry.} \end{cases}$$

Hence following (56) and using the o-symmetry and convexity of the C_j's, we now have

$$\bigcup_{(\beta_1,\dots,\beta_n)\in N} \; \bigcup_{\substack{(\alpha_{1k},\dots,\alpha_{mk})\in M_k \\ k\in\{1,\dots,n\}}} \; \sum_{j=1}^{m}\sum_{k=1}^{n} \beta_k\varepsilon|\alpha_{jk}|C_j$$

$$= \; \bigcup_{(\beta_1,\dots,\beta_n)\in N} \; \bigcup_{\substack{(\alpha_{1k},\dots,\alpha_{mk})\in M_k \\ k\in\{1,\dots,n\}}} \; \sum_{j=1}^{m} \Big(\sum_{k=1}^{n} \beta_k\varepsilon|\alpha_{jk}| \Big) C_j,$$

where now

$$\Big(\sum_{k=1}^{n} \beta_k\varepsilon|\alpha_{1k}|, \; \dots, \; \sum_{k=1}^{n} \beta_k\varepsilon|\alpha_{mk}| \Big) \; = \; \sum_{k=1}^{n} \beta_k \big(\varepsilon|\alpha_{1k}|, \dots, \varepsilon|\alpha_{mk}| \big) \; \in \; \oplus_N(M_1, \dots, M_n)$$

by the positive or negative symmetry of each M_k. The argument closes as in Theorem 40. $\square$

The relevance of this corollary is what it allows us to deduce about Minkowski addition and L_p addition, which forms a parallel statement to Corollary 39.

Corollary 42. (i) *Let $M_1, \ldots, M_n \subset \mathbb{R}^m$ be such that every M_j has positive symmetry or every M_j has negative symmetry. Then for any o-symmetric convex $C_1, \ldots, C_m \subset \mathbb{R}^p$,*

$$\oplus_{\sum_{j=1}^n M_j}(C_1, \ldots, C_m) \;=\; \sum_{j=1}^n \oplus_{M_j}(C_1, \ldots, C_m).$$

(ii) *Let $M_1, \ldots, M_n \in \mathcal{K}_o^m$ be such that every $\widehat{M_j}$ is convex and either every M_j has positive symmetry or every M_j has negative symmetry. Then for any $K_1, \ldots, K_m \in \mathcal{K}_s^n$ and $1 < p \leq \infty$,*

$$\oplus_{M_1 +_p \cdots +_p M_n}(K_1, \ldots, K_m) \;=\; \oplus_{M_1}(K_1, \ldots, K_m) +_p \cdots +_p \oplus_{M_n}(K_1, \ldots, K_m). \tag{58}$$

Proof. (i) Since $\sum_{j=1}^n M_j = \oplus_N(M_1, \ldots, M_n)$ for $N = \{(1, \ldots, 1)\}$, this is an immediate consequence of Corollary 41.

(ii) By Proposition 4, the convexity condition on the $\widehat{M_j}$'s ensures that each

$$\oplus_{M_j}(K_1, \ldots, K_m) \;\in\; \mathcal{K}^n,$$

and since $o \in M_j$, $\oplus_{M_j}(K_1, \ldots, K_m) \in \mathcal{K}_o^n$. Thus the L_p sum on the right-hand side of (58) is legitimate. As L_p addition is $\oplus_N(M_1, \ldots, M_n)$ for $N = M$ in (17) with m replaced by n, our conclusion follows directly from Corollary 41. $\square$

6 Convexity and M-addition

The Shapley-Folkman Lemma, a well-known result in mathematical economics, forges a significant connection between Minkowski addition and convex sets.

Proposition 43 (Shapley-Folkman Lemma)**.** *Let $A_1, \ldots, A_p \subset \mathbb{R}^n$. If*

$$x \;\in\; \sum_{j=1}^p \operatorname{conv} A_j,$$

then there exists $I \subset \{1, \ldots, p\}$ with $|I| \leq n$ such that

$$x \;\in\; \sum_{j \in I} \operatorname{conv} A_j + \sum_{j \notin I} A_j.$$

See [4, p. 128] for a proof. As Schneider suggests on the same page, we may interpret this as saying that Minkowski addition is, in some sense, a convexifying operation. We shall be more precise about this momentarily. Given the intimate bond between Minkowski addition and M-addition, it is reasonable to inquire whether or not the latter is likewise convexifying. The following generalization of the Shapley-Folkman Lemma provides an affirmative answer.

Theorem 44. *Let $M \subset \mathbb{R}^p$ and $A_1, \ldots, A_p \subset \mathbb{R}^n$. If $x \in \oplus_M(\operatorname{conv} A_1, \ldots, \operatorname{conv} A_p)$, then there exists $I \subset \{1, \ldots, p\}$ with $|I| \le n$ such that*

$$x \in \oplus_M(B_1, \ldots, B_p) \qquad \textit{where} \qquad B_j \;=\; \begin{cases} \operatorname{conv} A_j & j \in I, \\ A_j & j \notin I. \end{cases} \tag{59}$$

Proof. If $x \in \oplus_M(\operatorname{conv} A_1, \ldots, \operatorname{conv} A_p)$, then for some $(\alpha_1, \ldots, \alpha_p) \in M$,

$$x \;\in\; \sum_{j=1}^{p} \alpha_j \operatorname{conv} A_j \;=\; \sum_{j=1}^{p} \operatorname{conv}(\alpha_j A_j)$$

by Lemma 27. Thus from the Shapley-Folkman Lemma there exists $I \subset \{1, \ldots, p\}$ with $|I| \le n$ such that

$$x \;\in\; \sum_{j \in I} \operatorname{conv}(\alpha_j A_j) + \sum_{j \notin I} \alpha_j A_j \;=\; \sum_{j \in I} \alpha_j \operatorname{conv} A_j + \sum_{j \notin I} \alpha_j A_j \;=\; \sum_{j=1}^{p} \alpha_j B_j,$$

where the B_j's are defined as in (59). Since $\sum_{j=1}^{p} \alpha_j B_j \subset \oplus_M(B_1, \ldots, B_p)$, the desired result follows. $\square$

In light of Theorem 29, where we saw a convex hull removed from both the M-set and the summands, it is natural to ask whether

$$x \;\in\; \oplus_{\operatorname{conv} M}(\operatorname{conv} A_1, \ldots, \operatorname{conv} A_m)$$

implies that $x \in \oplus_M(B_1, \ldots, B_m)$ for some B_j as in (59). That is, in addition to dropping the convex hull on some of the A_j's, can we also remove it from M? The following example shows that the answer is generally no.

Example 45. Let $M' = \{o, (1,0), (1,1), (0,1)\}$ and take

$$M \;=\; K \;=\; [0,1]^2 \;=\; \operatorname{conv} M' \qquad \text{and} \qquad L \;=\; [1,2]^2,$$

as illustrated in Figure 9. Then using Theorems 29(i) and 11, we see

$$\begin{aligned} K \oplus_M L \;&=\; \operatorname{conv} K \oplus_{\operatorname{conv} M'} \operatorname{conv} L \\ &=\; \operatorname{conv}\big(K \oplus_{M'} L\big) \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{conv}\big((0K+0L)\cup(1K+0L)\cup(1K+1L)+(0K+1L)\big)\\
&= \operatorname{conv}\big(K\cup(K+L)\cup L\big),
\end{aligned}
$$

which is the outer polygon in Figure 9. Since $(2,1/2)\notin K\cup(K+L)\cup L$, we have

$$(2,1/2)\ \in\ \operatorname{conv} K \oplus_{\operatorname{conv} M'} \operatorname{conv} L\ =\ K\oplus_M L$$

while

$$(2,1/2)\ \notin\ \operatorname{conv} K \oplus_{M'} \operatorname{conv} L\ =\ K\oplus_{M'} L.$$

Since we therefore cannot always drop the convex hull on the M-set while retaining it on all the summands, much less can we do so when we remove it from some of the summands.

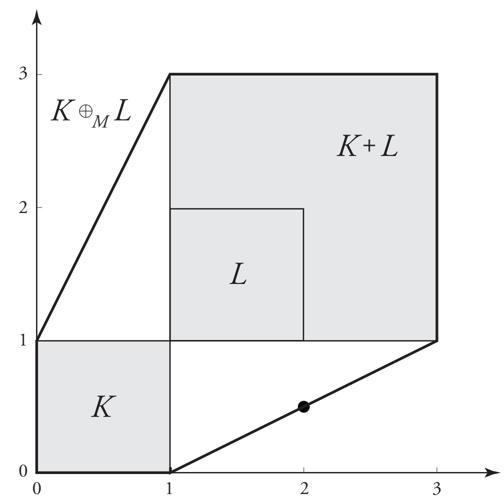


Figure 9: In general, $\operatorname{conv} K \oplus_{\operatorname{conv} M'} \operatorname{conv} L \neq \operatorname{conv} K \oplus_{M'} \operatorname{conv} L$. In this case, $\operatorname{conv} K \oplus_{\operatorname{conv} M'} \operatorname{conv} L = K\oplus_M L$ and has the thick border. The shaded set strictly contained within is $\operatorname{conv} K \oplus_{M'} \operatorname{conv} L$.

A consequence of the Shapley-Folkman Lemma for Minkowski addition is the following upper bound on the Hausdorff distance δ from a Minkowski sum of compact sets to its convex hull.

Proposition 46. *If $A_1,\ldots,A_p\in\mathcal{C}^n$ and $\beta\in\mathbb{R}$ are such that $A_j\subset\beta B^n$ for every $j\in\{1,\ldots,p\}$, then*

$$\delta\bigg(\sum_{j=1}^{p} A_j,\ \operatorname{conv}\Big(\sum_{j=1}^{p} A_j\Big)\bigg)\ \le\ \beta n.$$

See [4, Corollary 3.1.3] for a proof. Using this bound allows us to establish an analogous inequality for M-sums.

Theorem 47. *Let $M \in \mathcal{K}^p$ be contained in a closed orthant of $\mathbb{R}^p$ and let*

$$\beta_1 \;=\; \max\left\{\, |\alpha_j| \;:\; (\alpha_1, \dots, \alpha_j, \dots, \alpha_p) \in M,\, j \in \{1, \dots, p\} \,\right\}.$$

If $A_1, \dots, A_p \in \mathcal{C}^n$ and $\beta_2 \in \mathbb{R}$ are such that $A_j \subset \beta_2 B^n$ for every $j \in \{1, \dots, p\}$, then

$$\delta\big(\oplus_M (A_1, \dots, A_p), \operatorname{conv}\big(\oplus_M (A_1, \dots, A_p)\big)\big) \;\le\; \beta_1 \beta_2 n. \tag{60}$$

Proof. First note that $\oplus_M(A_1, \dots, A_p)$ is compact since M and the A_j's are, and thus the distance in question make sense. Since $\oplus_M(A_1, \dots, A_p) \subset \operatorname{conv}\big(\oplus_M (A_1, \dots, A_p)\big)$, we see from (18) that

$$\begin{aligned}
&\delta\big(\oplus_M (A_1, \dots, A_p), \operatorname{conv}\big(\oplus_M (A_1, \dots, A_p)\big)\big) \\
&\qquad\qquad = \; \min\{\, \lambda \ge 0 \;:\; \operatorname{conv}\big(\oplus_M (A_1, \dots, A_p)\big) \subset \oplus_M(A_1, \dots, A_p) + \lambda B^n \,\}.
\end{aligned}$$

If $x \in \operatorname{conv}\big(\oplus_M (A_1, \dots, A_p)\big)$, then by Theorem 29(i) and the convexity of M,

$$x \;\in\; \oplus_{\operatorname{conv} M}(\operatorname{conv} A_1, \dots, \operatorname{conv} A_p) \;=\; \oplus_M(\operatorname{conv} A_1, \dots, \operatorname{conv} A_p),$$

and thus for some $(\alpha_1, \dots, \alpha_p) \in M$,

$$x \;\in\; \sum_{j=1}^{p} \alpha_j \operatorname{conv} A_j \;=\; \operatorname{conv}\left(\sum_{j=1}^{p} \alpha_j A_j\right)$$

by (40). Since $|\alpha_j| \le \beta_1$ for every $j \in \{1, \dots, p\}$ and each $A_j \subset \beta_2 B^n$, $\alpha_j A_j \subset \beta_1 \beta_2 B^n$ for every j. Hence by Proposition 46 there exists $y \in \sum_{j=1}^{p} \alpha_j A_j$ such that $|y - x| \le \beta_1 \beta_2 n$. Since $y \in \oplus_M(A_1, \dots, A_m)$, we see that

$$\operatorname{conv}\big(\oplus_M (A_1, \dots, A_p)\big) \;\subset\; \oplus_M(A_1, \dots, A_p) + \beta_1 \beta_2 n B^n,$$

proving (60). □

Note that as in Proposition 46, the bound in (60) does not depend on the number of summands, but rather only on the dimension of the space housing the summands. As we will momentarily prove, this implies that the average $\frac{1}{p} \oplus_M (A_1, \dots, A_p)$ of an M-sum of many summands is essentially convex. Or to use the words of Schneider for describing the parallel phenomenon for Minkowski addition [4, Remark 3.1.4], we can say the averaging of M-sums is *asymptotically convexifying*.

Corollary 48. *Let $\{M_k\}_{k\in\mathbb{N}}$ be a sequence of sets such that each $M_k \in \mathcal{K}^k$ is contained in a closed orthant of $\mathbb{R}^k$ and where there exists $\beta_1 \in \mathbb{R}$ such that*

$$\max\left\{ |\alpha_j| \;:\; (\alpha_1,\ldots,\alpha_j,\ldots,\alpha_k)\in M_k,\ j\in\{1,\ldots,k\},\ k\in\mathbb{N} \right\} \;\le\; \beta_1.$$

Then if $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{C}^n$ and $\beta_2 \in \mathbb{R}$ are such that $A_k \subset \beta_2 B^n$ for every $k \in \mathbb{N}$,

$$\lim_{k\to\infty} \delta\left(\frac{1}{k} \oplus_{M_k} (A_1,\ldots,A_k),\, \mathrm{conv}\left(\frac{1}{k}\oplus_{M_k}(A_1,\ldots,A_k)\right)\right) \;=\; 0.$$

Proof. By the homogeneity of M-addition (19),

$$\frac{1}{k}\oplus_{M_k}(A_1,\ldots,A_k) \;=\; \oplus_{M_k}\left(\frac{1}{k}A_1,\ldots,\frac{1}{k}A_k\right),$$

and each $A_j/k \subset (\beta_2/k)B^n$. Therefore by Theorem 47 with a fixed $k\in\mathbb{N}$,

$$\delta\left(\frac{1}{k} \oplus_{M_k} (A_1,\ldots,A_k),\, \mathrm{conv}\left(\frac{1}{k}\oplus_{M_k}(A_1,\ldots,A_k)\right)\right) \;\le\; \frac{\beta_1\beta_2}{k}\,B^n,$$

and letting k tend to infinity gives the desired result. □

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