Spring 2012

The Four Color Theorem

Patrick Turner
Western Washington University

HONORS THESIS

Signature ____________________

Senior Project

Patrick Turner

The Four Color Theorem

The history of mathematics is pervaded by problems which can be stated simply, but are difficult and in some cases impossible to prove. The pursuit of solutions to these problems has been an important catalyst in mathematics, aiding the development of many disparate fields. While Fermat's Last theorem, which states $x^n + y^n = z^n$ has no integer solutions for $n > 2$ and $x, y, z \neq 0$[12], is perhaps the most famous of these problems, the Four Color Theorem proved a challenge to some of the greatest mathematical minds from its conception 1852 until its eventual proof in 1976.

The Four Color Theorem was first stated in 1852 by a young English mathematician, Francis Guthrie, who noticed that he could color a map of the counties of England using at most four colors such that no two counties of the same color were touching along a measurable border and from this observation postulated that he could color all maps this way [7]. In 1852 there was no formalized field of mathematics which could be drawn on to study the four color theorem, so the first mathematicians to study the four color theorem used two dimensional maps of regions that resemble geographical maps. Using this language a statement of the four color theorem is as follows: for every two dimensional map of regions, the regions of the map can be colored by at most four colors such that no two regions of the same color share a measurable border.

Later, as problems similar to the four color theorem began to be explored, the field of graph theory developed into a formal branch of mathematics. As part of this formalization, the regions of the geographical maps originally used to describe the problem were replaced by vertices, which are simply points and the finite borders were replaced by lines called edges. More specifically the geographical map was replaced by the concept of an graph which is defined as a set of vertices, edges and incidence relationships between each edge and two vertices. This is the terminology that will be used for the remainder of this paper. There are two issues to be quickly addressed before stating the four color theorem in terms of modern graph theory. First, edges of a graph can potentially cross, but on a map one finite border connecting two regions cannot be overlapped by another finite border connecting two other regions. Second, an edge in a graph can be a loop where both ends of the same edge are connected to the same vertex while a region of a map cannot share a border with itself. To avoid these issues we restrict ourselves to graphs with no crossing edges and no loops. Such graphs are called simple graphs.

Even though the four color theorem became very well known within the mathematical community and to an extent within mainstream culture, with Martin Gardner featuring it in an April Fools joke in a 1975 edition of Scientific American[11], there was a large stretch of time following its introduction that few people showed any interest in finding its proof.

After Francis's brother Frank informed his professor Augustus De Morgan of University College, London, about the four color theorem, De Morgan excitedly informed some of his contemporaries about the theorem. However, interest in the problem did not gain traction until the late 1870s when during a meeting of the London Mathematical Society Arthur Cayley enquired as to whether it had been proved. Finding that no meaningful progress had been made Cayley investigated the problem and wrote a paper outlining the difficulty of finding a proof. In this paper Cayley noted that if an area is added to a graph it by no means follows that we can *without altering the original colouring* color (the new area) with one of the four colors[1]. With this logic it is possible that no finite number, let alone four, would be sufficient to color all maps. By now, peoples interest was piqued, and the search for a proof began in earnest.

In 1879, soon after Cayley helped bring the question to the fore, Alfred Bray Kempe claimed to have found a proof. While we won't fully explain Kempe's attempted proof here we need to define the concept of a path before we show where he was wrong. A *path* can be viewed a thread of edges which is interspersed with vertices and doesn't branch. The definition of a path is a subgraph where each vertex is incident with one or two edges and if a vertex in the path is incident with only one edge then there is exactly one other vertex incident with one edge. Kempe's proof involved the coloring of vertices that surround a single vertex of unknown color. If a vertex of unknown color is incident with each vertex in a path of length two or three then there will always be some fourth color available which the central vertex can be colored, so these small graphs are always four colorable. Kemp showed that if the vertex of unknown color is only incident with the vertices of a path containing four vertices then the surrounding path can be colored with at most three colors leaving the unknown vertex to be colored by the fourth color.

A sketch of the proof showing that the vertices in a path of length four surrounding a central vertex can be colored with at most 3 colors is as follows: Let the vertices around the path be v_1, v_2, v_3, and v_4, such that v_1 is not incident with v_3 and v_2 is not incident with v_4. If the vertices are colored blue, yellow, green and red respectively, consider the subgraph formed by taking the blue vertex v_1, all of the green vertices incident with v_1, all of the blue vertices incident with these green vertices, all of the green vertices incident with all these blue vertices, and so on. If this subgraph includes v_3, then there is a path called a *Kempe Chain* connecting v_1 and v_3. However, if a Kempe Chain does not connect v_1 and v_3, as in Figure 1, then all of the blue vertices and green vertices in the aforementioned subset containing v_1 can be recolored green and blue respectively, resulting in the v_1 being recolored green. Hence, the four vertices surrounding the central vertex would now be colored by three colors. Now, observe that if there exists a Kempe Chain containing v_1 and v_3 there cannot be a Kempe Chain containing both v_2 and v_4 and the reverse would be true as well. Therefore, the four surrounding vertices can always be colored by three colors, leaving the center vertex to be colored by the fourth color.

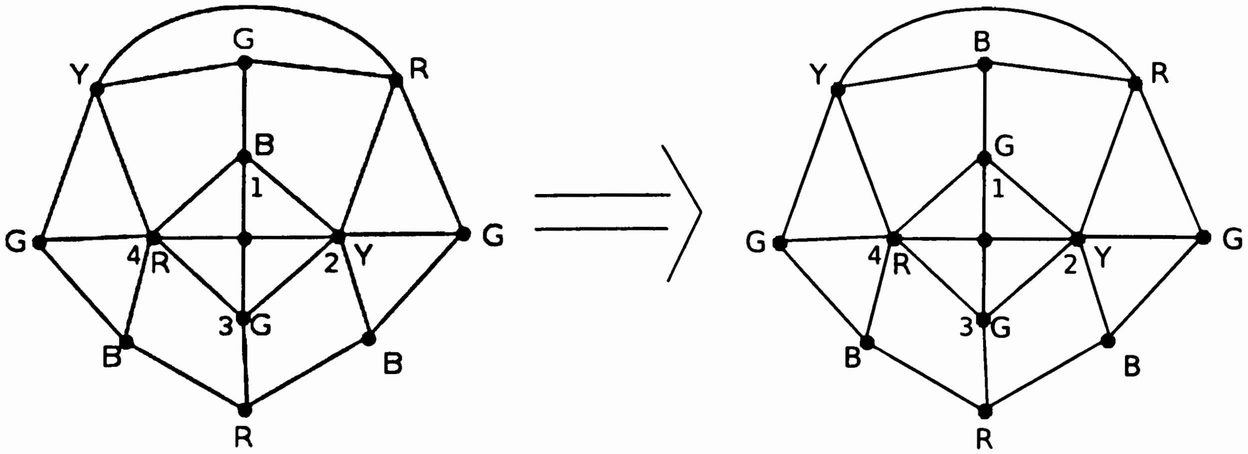

Figure 1

The problem with Kempe's proof arises when his methodology is extended to situations where a single vertex is surrounded by more than four vertices. However, this mistake was overlooked for ten years and Kempe received great general acclaim for his proof, including membership in the Royal Society in 1881[2]. The even though the four color theorem was believed to have been solved, mathematicians continued to work on the problem coming up with proofs based on Kempe's and alternate formulations of the four color theorem. The most notable of the mathematicians to work on the four color theorem during this period was Peter Gutherie Tait, who was one of the first to develop an alternate formulation when he showed that coloring the edges of a graph, where all the vertices have a degree of three, with three colors such that no two edges of the same color are incident with the same vertex is equivalent to the four color theorem. Later more equivalences were found, for example, Louis Kauffman developed an equivalent statement in linear algebra [2]. To explore Kauffman's formulation two definitions are needed. First, $\times$ is the vector cross product in $\mathbf{R}^3$. Secondly, an association is an expression composed of set of k vectors, $u_1, u_2, ..., u_k$, $k-1$ cross products and $k-2$ sets of parentheses pairs where the parentheses are positioned such that the expression can be evaluated in $\mathbf{R}^3$. If $k = 4$ two possible associations are as follows $(u_1 \times u_2) \times (u_3 \times u_4)$ and $(u_1 \times (u_2 \times u_3)) \times u_4$. As an example of an alternate formulation of the four color theorem Kauffman's theorem will now be proved in part. The statement of his theorem is as follows:

Kauffman's theorem: Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the usual unit vector basis of $\mathbf{R}^3$. If two associations of $u_1 \times u_2 \times ... \times u_k$ are given, there exists an assignment of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to $u_1 \times u_2 \times ... \times u_k$ such that the evaluations of the two associations are equal and nonzero.

To prove that Kauffman's theorem is equivalent to the four color theorem we must show that Kauffman's theorem implies that the four color theorem is true and that the four color theorem implies that Kauffman's theorem is true. Proving that the Kauffman's theorem implies the four color theorem is difficult so it will not be shown here, but we will prove that the four color theorem implies Kauffman's theorem.

Proof that the four color theorem implies Kauffman's theorem: Assume that the four color theorem is true. Now by Tait's reformulation the edges of a graph where all vertices have a degree of three are three colorable. Let A_1 and A_2 be two associations of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ assigned to the vectors $u_1, u_2, ..., u_k$. Next express these associations as a graph G of two trees connected by edges at the roots $u_1, u_2, ..., u_k$ and by the top of each tree as in H in Figure 2, where $A_1 = (u_1 \times u_2) \times (u_3 \times u_4)$ and $A_2 = (u_1 \times (u_2 \times u_3)) \times u_4$. Now, a root in one tree is equal to the root it is incident with in the other tree so all of the vertices in the graph with a degree of two can be suppressed as in H' in the Figure 2. Next, every remaining vertex in the graph has a degree of 3 so the edges of the graph are three colorable. Now, the resulting cross product of any two vertices is in i, j, k so the graph is three colorable by i, j, k. This means that the edge connecting A_1 and A_2 is in $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Therefore, A_1 and A_2 are equivalent and nonzero.

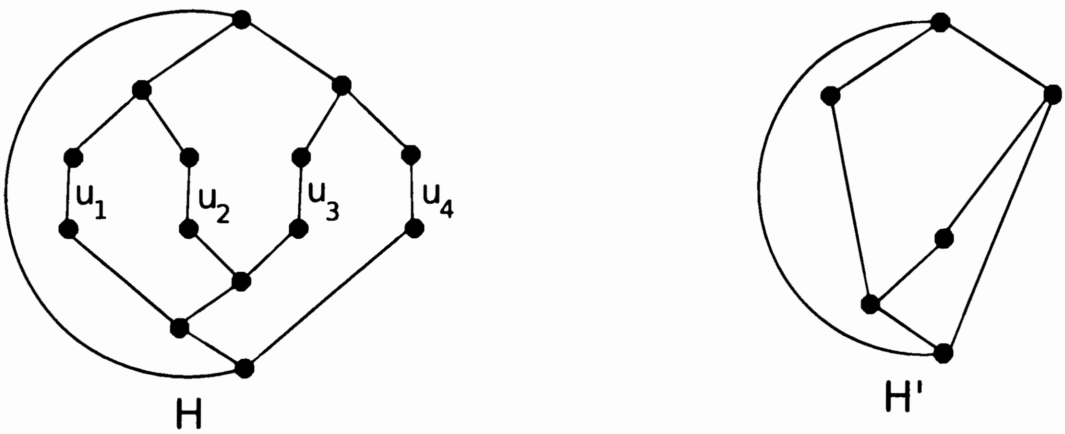

Figure 2

In 1889 Percy John Heawood identified that Kempe's method does not apply to a vertex incident with five other vertices and wrote a paper on the subject. Heawood's method for showing the falsehood of Kempe's proof was to apply the Kempe Chains method to a graph incident with five other vertices like the graph shown in Figure 3. In this case, which is not the example presented by Heawood in his disproof, vertices 2 and 4 are in the same Green/Yellow Kempe Chain and vertices 2 and 5 are in the same Green/Blue Kempe chain, so vertices 1 and 3 which are both colored red cannot be in the same Kempe chain. Now, by using the Kempe chain method the Red/Yellow Kempe chain containing vertex 1 is alternated so that vertex 1 is now colored yellow and the Blue/Yellow Kempe chain containing vertex 3 is alternated so that vertex 3 is now blue. However, there are now two red vertices incident with one another. So, by using Heawoods argument it is shown that the Kempe Chains method does not prove the Four Color theorem for all graphs.

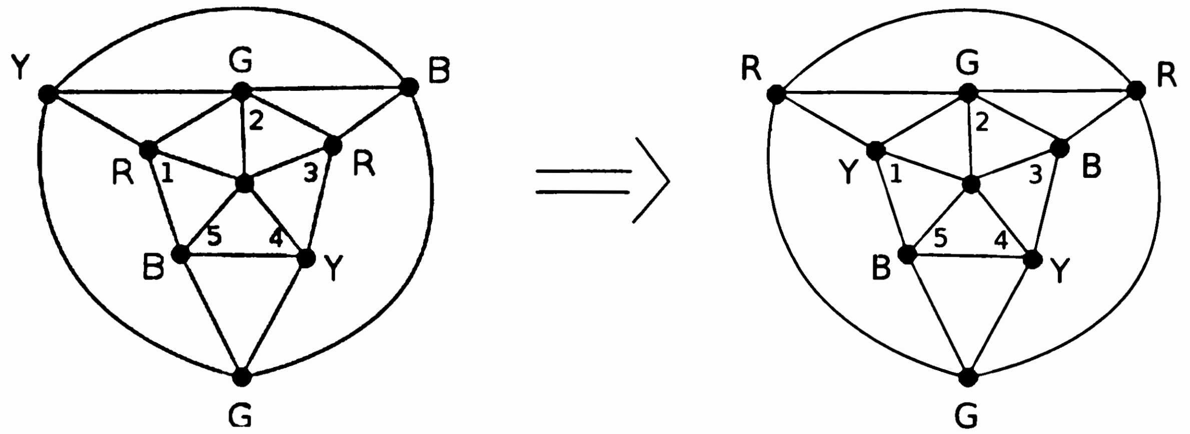

Figure 3

Heawood's proof wasn't only creative destruction. In dismantling the core of Kempe's argument Heawood found a proof for the five color theorem. A sketch of the proof of the five color theorem presented by Béla Bollobás's [9] will now be given.

Lemma 1: In a graph there exists a vertex with a degree of at most 5.

Proof of Lemma 1:
Let G be a simple loopless graph. By combining Euler's formula for the characteristic, $|V(G)| - |E(G)| + |R(G)| = 1$ and the inequality $2|E(G)| \geq 3|R(G)|$ we find $|E(G)| \leq 3|V(G)| - 3$. Next, each edge is incident with two vertices so the sum of the degrees for all of the vertices in G is twice the number of edges in G. Hence, $2|E(G)| = \Sigma deg(v_i)$ for all the vertices v_i in G. From this we have $\Sigma deg(v_i) \leq 6|V(G)| - 6$. Therefore in every graph there exists a vertex with a degree of at most five.

Proof of Five Color Thereom:
Let G be a simple loopless graph. Assume G is a minimal counterexample to the five color theorem, meaning that all graphs G' where $|V(G')| + |E(G')| < |V(G)| + |E(G)|$ G' are 4 colorable. From Lemma 1 we know that G must contain a vertex, v, with a degree of at most 5. If v has a degree of 4 or less and is removed the remaining graph will be 5 colorable, then v can be reinserted and recolored one of the colors not used by its 4 neighbors. Hence, a counterexample to the five color theorem must have a vertex of degree 5. Now, if two of v's neighbors share a color, v can be removed and reinserted so that G must be five colorable. From this we can assume that the neighbors of v are all colored differently.
Without loss of generality consider the case illustrated in Figure 4. Let the vertices incident with v be v, v_1, v_2, v_3, v_4 and v_5. If c is a mapping from the colors Red, Green, Blue, Yellow and Orange to the vertices in G define the coloring of the neighbors of v as $c(v_1)$ =Red, $c(v_2)$ =Green, $c(v_3)$ =Blue, $c(v_4)$ =Yellow and $c(v_5)$ =Orange. Let the vertices be ordered such that v_1 is incident with v_2 and v_5 and the rest of the vertices are similarly ordered. Consider the red/blue Kempe Chain from v_1 to v_3. If there isn't a path in the Kempe Chain between v_1 and v_3, v_1's component in the Kempe Chain can be recolored such that all red vertices are colored blue and all blue vertices are colored red, then v can be colored red so that G is five colorable. However, if there is a path between v_1 and v_3 in the Kempe Chain

for v_1 and v_3 then the green/yellow Kempe Chain between v_2 and v_4 cannot be connected so the component of that Kempe Chain containing v_2 can be recolored such that all green vertices are colored yellow and all yellow vertices are colored green and v can be colored green so that G is once again five colorable. Therefore, any minimal six colorable graph will have a vertex v that can always be recolored so that the minimally six colorable graph is five colorable. By this contradiction there cannot exist a minimally six colorable graph. Therefore all graphs are five colorable.

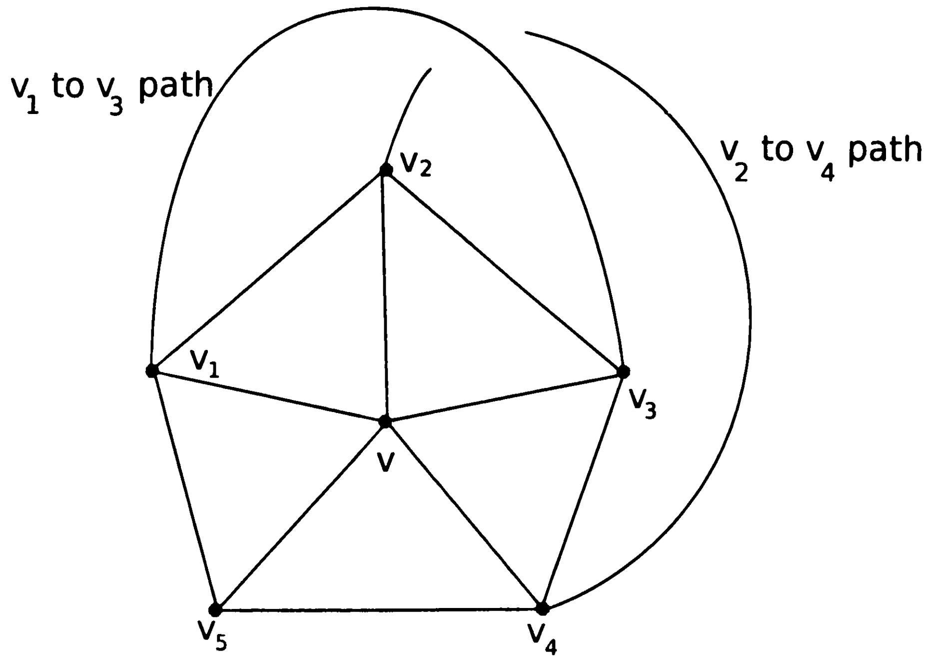

Figure 4

In contrast to the admiration Kempe garnered for his false proof Heawood received few accolades for his achievement which had eluded the mathematical community for 10 years. The response to Heawood's work was so subdued that when a journal published the conjecture in 1894 asking if it had been proved, several individuals wrote to the journal stating that Kempe had proved the theorem [1]. Even though Kempe's proof was false he continued to benefit from his membership in the Royal Society which he gained as a consequence of his 1879 paper by being elevated to the Council of the Royal Society in 1897 and eventually granted a knighthood in recognition of his administrative contributions to the Royal Society in 1912 [2].

Early in the study of the four color theorem mathematicians abandoned attempts to directly find a proof and instead focused their efforts on constructing a proof by contradiction. If the four color theorem were false, there would exist some graph colorable by five colors but not by four. Furthermore there would exist a minimal five colorable graph G, such that any smaller graph, G' is four colorable. Now, the smallness mentioned previously is expressed by the following statement $|V(G')| + |E(G')| < |V(G)| + |E(G)|$ where $|V(G)|$ and $|E(G)|$ are the number of vertices and edges in the graph G respectively. Such a graph is known as a minimal counterexample to the four color theorem and like the proof of the five color theorem presented above, a proof of the four color theorem would show that a minimal counterexample to the four color theorem cannot exist. However, in a tremendous understatement, proving this for four colors is significantly more difficult than for five.

The contradiction of the Four Color Theorem is achieved by proving three theorems. The definitions of terms in these theorems will supplied as each theorem is discussed.

Theorem 1: All minimal counterexamples are internally 6-connected configurations.
Theorem 2: No minimal counterexamples contain good configurations.
Theorem 3: Every internally 6-connected configuration contains a good configuration.

If Theorem 1 is true then Theorem 2 and Theorem 3 cannot both be true. However, as shown below all three of the theorems are true resulting in the contradiction and proving the truth of the four color theorem. A proof of Theorem 1 is simple and will be shown, however Theorems 2 and 3 require a tremendous amount of manual and computerized proof.

To further restrict the type of graphs that need to be analyzed, we will prove that it is sufficient to prove the four color theorem for triangulations, where a *triangulation* is a graph whose regions are each bounded by three edges. Also note that a *near-triangulation* is a triangulation with the exception that one region may or may not be incident with three vertices. This region is the *infinite region.*

Lemma 2: It is sufficient to prove the four color theorem for triangulations.

Proof of Lemma 2: Let G be a five colorable, simple and loopless graph. There are three cases for this proof; G contains a finite region bounded by four vertices, G contains a finite region bounded by 5 vertices and G contains a finite region bounded by 6 or more vertices. Let c be the coloring from $V(G)$ to red, green, blue, yellow and orange.
Assume that G contains a region bounded by n vertices where $n \geq 4$. Let i be in the set $\{1, 2, \ldots, n\}$. Without loss of generality let the vertices be ordered around the region in a clockwise path such that if v_i is located at the top of the circuit v_i is incident with v_{i-1} to the left and v_{i+1} to the right. There are two cases for this proof: $c(v_i)$ is equal to $c(v_{i+2})$ and $c(v_i)$ is not equal to $c(v_{i+2})$.
Case 1: If v_i and v_{i+2} share the same color then a new graph G' can be created by collapsing v_i and v_{i+2} into a single vertex where this vertex is substituted into all of the incidence relationships involving v_i and v_{i+1} in G. Then $V(G') < V(G)$ and $E(G') = E(G)$ so $|V(G')| + |E(G')| < |V(G)| + |E(G)|$. Thus G wouldn't be a minimal counterexample.
Case 2: If v_i and v_{i+2} are not the same color and the Kempe Chain for v_i and v_{i+2} is not connected, then the component containing v_{i+2} can be recolored such that $c(v_i)$ and $c(v_{i+2})$ are replaced. Consequently, v_i and v_{i+2} will have the same coloring and can be collapsed into a single vertex as they are in Case 1. If the Kempe chain formed by v_i and v_{i+2} is connected, then the Kempe chain formed by v_{i+1} and v_{i+3} is not connected and the component containing v_{i+1} can be recolored so that all vertices in the component colored $c(v_{i+1})$ are recolored $c(v_{i+3})$ and all the vertices colored $c(v_{i+3})$ are recolored $c(v_{i+1})$. After this recoloring, $c(v_{i+1})$ will equal $c(v_{i+3})$ so these vertices can be collapsed as in Case 1.

Consequently, G is not a minimal counterexample.
Therefore all minimal counterexamples are triangulations.

In *Graph Theory 1736-1936* Biggs, Lloyd and Wilson note that around the turn of the 20th century the history of four color theorem reflects the growing importance of the United States in academia when much of the important work on the four color theorem was done at American institutions through the 1930s[1]. The first two Americans to make contributions were Oswald Veblen and George David Birkhoff. Both Veblen and Birkhoff began their study of the four color theorem by reformulating it in terms of various algebraic forms. It wasn't until 1913, in a paper titled *The reducibility of maps*, that Birkhoff began to investigate reducibility, "the line of enquiry which ... was eventually to lead to the solution of the problem, in 1976." The premise of reducibility was to leverage the idea of minimal counterexamples to the four color theorem to either find a minimal counterexample, or by investigating the necessary requirements for a minimal counterexample show that it cannot exist. The first restriction is provided by Theorem 1. The definition of an *internally six-connected triangulation* is a triangulation where all of the vertices in the graph have a degree of at least five. We will now prove this theorem in *Theorem 1.*

Theorem 1: All minimal counterexamples to the four color theorem are internally 6-connected triangulations.
Proof of Theorem 1: Let G be a minimal counterexample which is not an internally six-connected triangulation. Then there would exist a vertex in G, v, such that the degree of v would be four or less. Now there are two cases. Either v has a degree of less than four or v has a degree of four.
Case 1: Assume v has a degree of less than four. Now G/v, which is defined as the graph of G without the vertex v or the edges incident with v, would have a four coloring since G is a minimal counterexample. If v is reinserted into G/v such that it has the same incidence relationships as it did in G, v can be colored by at least one of the four colors that colors G/v. Thus, if G contains a vertex of degree less than four, G cannot be a minimal counterexample.
Case 2: Assume v has a degree of four. If the four vertices around v are v_1, v_2, v_3 and v_4 in a clockwise direction then at most either the Kempe Chain containing v_1 and v_3 is connected or the Kempe Chain containing v_2 and v_4 is connected. Without loss of generality assume that the Kempe Chain connecting v_1 and v_3 is not connected. Then we can recolor the component containing v_3 such that it $c'(v_3)$ equals $c(v_1)$. Now, we can collapse the three vertices v, v_1 and v_3 into a single point creating a new graph G' that is by necessity four colorable. Now if we reintroduce v, v_1 and v_3, v_1, v_2, v_3 and v_4 will be colored with at most three colors. Thus v can be colored with the fourth color and G is in fact four colorable. Hence, G is not a minimal counterexample to the four color theorem.

The requirement that every minimal counterexample to the four color theorem must be an internally 6-connected triangulation was first identified by Birkhoff. Between himself and Veblen a large portion of the mathematics of reducibility was developed by the 1930s. In his work on reducibility Birkhoff identified the *ring* as an important structure in the study of configurations. A ring is a special type of *circuit* which is a graph where each vertex has a degree of 2, as in Figure 5. If a circuit R encapsulates a configuration $K = (G, \gamma)$, such that all of the vertices in G are only incident with vertices in either R or G, and

R and G are mutually exclusive subgraphs, then R is a *ring* of G. The size of this ring relates to the configuration with the inequality shown in point three of the definition of configurations below. Despite his contributions towards the eventual proof of the four color theorem Birkhoff came to regret spending so much time working on the problem [1]. This dissatisfaction was not unique to Birkhoff as many mathematicians applied themselves to this problem and believed they had solved it only to find out later that they had not.

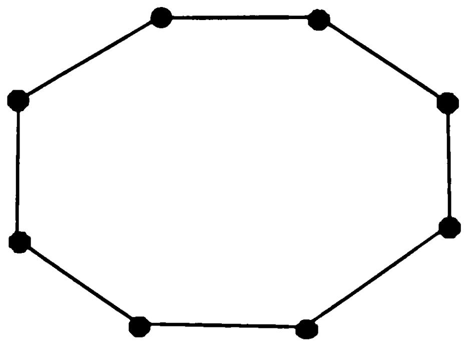

Figure 5

Now, both Theorem 2 and Theorem 3 refer to a mathematical object called a good configuration. First, in general terms a configuration is a pairing of a graph and set of integers mapped to the vertices of the graph. A configuration is good if it is part of a set of reducible configurations where every internally 6-connected triangulation contains a configuration from the set of good configurations. While Birkhoff and Veblen developed most of the theory behind reducibility Kempe began the study of graphs that must necessarily be contained in a given graph when he showed that every graph with more than four vertices must contain one of the following subgraphs: a vertex incident with two other vertices, a vertex incident with three other vertices, a vertex incident with four other vertices or a vertex incident with five other vertices. The final case was the one shown by Heawood to be false [4]. Now, the definition of a configuration will be given and later a restriction will be added that defines the subset of configurations called good configurations. A *configuration* K is a non null pairing (G, γ) where G is a near-triangulation and γ is a mapping from $V(G)$ to the integers with the following properties:
1. For every vertex v of G, if v is not incident with the infinite region of G, then $\gamma(v)$ equals $deg(v)$, the degree of v, and otherwise $\gamma(v) > deg(v)$, thus $\gamma(v) \geq 5$.
2. For every vertex v of G, $G \backslash v$ has at most two components, and if there are two then the degree of v in G is $\gamma(v) - 2$.
3. K has a ring-size of at least 2, where the *ring-size* of k is defined to be $\sum(\gamma(v) - deg(v) - 1)$, summed over all vertices v incident with the infinite region of G such that $G \backslash v$ is connected.

Now that we have a definition of a configuration we can examine the concept of reducibility, which is necessary to prove that no minimal counterexample contains a good configuration. A reducible configuration is a configuration with the property that if it is contained by a graph then vertices may be removed from the configuration, creating a new graph which can be recolored and the vertices reintroduced so that this new coloring is not effected. This means that if the original graph is a minimal counterexample and it contains a reducible configuration then the graph can be reduced to a four colorable graph before having the removed vertices returned so that the graph is the same as the original graph but is instead four colored[8]. Now, S is a *free completion* of K with ring R if $V(S)$ equals the union of $V(R)$ and $V(G)$ and for every vertex v in $V(G)$ the degree of that vertex in S equals $\gamma(v)$.

A near-triangulation S is a free completion of a configuration $K = (G, \gamma)$ with ring R, where R is a ring that bounds the infinite region of S where R equals $S\backslash G$ with all of the edges incident with G removed from $S\backslash G$ and every vertex, v, in S that is also in G has a degree $\gamma(v)$. With this we can see that R and G are mutually exclusive and that the γ value of each vertex in the graph of a configuration is equal to the degree of that vertex when the graph of the configuration is a subgraph of a free completion. Since each region is a triangulation, the degree of each vertex is set and edges cannot cross, without loss of generality there is one free completion for a given configuration.

With the definition of a free completion we can examine *reducers*. Let C' be the set of colorings of the ring R and let C be the set of colorings of S restricted to the vertices of R. Since there are more vertices in S than in R and R is contained by S, C must be contained in C' and the number of colorings in C' must be greater than the number of colorings in C. Now, if $C'\backslash C$ is null then the configuration K is *D-reducible* while if $C'\backslash C$ is not null and there exists a graph S' derived from S by replacing G with a smaller graph such that no coloring in $C'\backslash C$ is in the set of coloring of S' restricted to R then the graph K is *C-reducible.* If this process does not effect a triangulation of which S is a subgraph, then the reducer is *safe.* With these definitions Robertson, Sanders, Seymour and Thomas (who will later be referred to as Robertson et al.) proved that "each of the good 633 configurations (found in their proofs) is either D-reducible or C-reducible with a safe reducer" [8] and in doing so proved that every good configuration is reducible. Verifying that each of the 633 good configurations has a safe reducer and is either D-reducible or C-reducible is one of the two parts of the four color theorem that requires a digital computer to prove.

The third theorem shows that good configurations are unavoidable, which is to say every that minimal counterexample to the four color theorem must contain at least one good configuration. The means of proving this were developed by a German mathematician, Heinrich Heesch. Intrigued by a friends failed attempt to prove the four color theorem Heesch began to work on the problem and realized that he would be able to complete the proof if he could identify a finite set of reducible configurations. In a 1969 paper Heesch explained a process called *discharging* which can be used to show that a configuration is good. To begin, Heesch gave each vertex, v, a charge equal to $10(6 - deg(v))$, so that vertices of degree 5 would have a charge of 10, while vertices of degree 6 would have a charge of 0 and vertices of degree 7 would have a charge of -10. The graph is a triangulation so $3R = 2E$ where R is the number of regions and E is the number of edges, and from Euler's formula we have $V - E + R = 2$ where V is the number of vertices in a graph, so $E = 3V - 6$. Also, note $2E = \Sigma(deg(v))$ over each vertex, v, in the graph. Now, the sum of the charges of all the vertices in the graph is $\Sigma 10(6 - deg(v)) = (10(6V - (6V - 12))) = 120$, so the sum of the charges is positive. Next, to discharge the charges of vertices are shifted around the graph in such a way that good configurations are identified[8]. This shows that every internally 6-connected triangulation contains at least one good configuration.

While the theory necessary to solve the four color theorem had been developed, the final proofs of theorems 2 and 3 required computing power that was not widely available in the middle of the twentieth century. In a lecture Heesch postulated that it might be necessary for an unavoidable set to contain more than ten thousand good configurations. Besides the large number of configurations that have to be verified, the process of checking a single configuration is computationally intensive. Indeed, of the 633 configurations in the unavoidable set used by Robertson et al., 137 each required 20,000,000 separate cases to be shown to be reducible [10]. However, in the 1950s and 1960s there were few if any computers in the world powerful enough to handle the computational demands of the four color theorem and living in post World War Two Germany, Heesch did not have access to them and he wasn't able to attempt a final proof to the theorem.

Finally in the early 1970s, two mathematicians from the University of Illinois, Wolfgang Haken and Kenneth Appel, began the final push to prove the four color theorem. Haken, who was born in Berlin, had first been drawn to the four color theorem after listening to a lecture given by Heesch in 1949. Appel, an accomplished computer programmer, born in Queens, New York, had researched cryptography in the early sixties and developed the programming skills they would need. In 1969 Appel was made a professor at the University of Illinois where Haken had worked since 1962. Here they had access to the computational power needed to finally solve the four color theorem. Aware of their role in finally proving the theorem Haken said that Appel and himself deserved 20 percent of the credit for solving the proof because they were the "fifth generation of people working on the problem" [5]. Published in 1976, the paper detailing Appel and Haken's proof was immensely controversial. As was necessary Appel and Haken used a significant amount of brute force in their proof, verifying ten thousand cases manually before requiring twelve hundred hours of computer time to run the program that completed the reducibility and unavoidability portions of the proof [4]. Besides the volume of work done by Appel and Haken, the programs used were written in assembly language making them incomprehensible to people seeking to verify their work.

Following the publishing of the proof by Appel and Haken the mathematical community had significant doubts regarding the validity of Appel and Haken's proof. At filming of a documentary about the Four Color Theorem in New York, an unnamed professor "would not even let his students talk with Haken and Appel for fear of the students minds 'being contaminated'." [5] Many of these fears were not allayed until other mathematicians replicated their work including Robertson et al. in 1997. Initially their goal was to validate the work by Appel and Haken. However, the hand checkable portion of the Appel and Haken proof was so difficult and the code so indecipherable that no one had independently verified this portion of the proof and when Robertson et al. went to verify Appel's and Haken's work they "decided that it would be more profitable to work out (their) own proof." [3] While Robertson et al. used a similar method as Appel and Haken, the new proof was simpler, reducing the number of necessary good configurations from 1476 to 633. In order to offset the potential for flawed logic in a computer program Robertson et al. created "two independent programs" to verify their results. If there had been an error in one program it is unlikely that there would have been the same errors in the second program.

The proofs by the two teams Appel and Haken, and Robertson et al. shared a weakness; the programs used in their proofs were developed specifically for the solving the four color theorem and the accuracy of these programs was not verifiable by proving other theorems. In the years after Appel and Haken's proof the mathematical community began to embrace computational proof and general proof assistance programs, which could be used to solve multiple different proofs, began to be developed. One general proof assistance programs is Coq, which Georges Gonthier and Benjamin Werner used to prove the four color theorem in 2004. While the proofs of the four color theorem had been accepted as true by this time the proof by Gonthier and Werner addressed some of the doubt surrounding the previous proofs by showing that the theorem can be proved using software not specifically designed to solve the four color theorem.

After doubts questioning the validity of the computationally derived proof the next most important issue facing proofs of the four color theorem is that the vast scale of computation required for the four color proof prevents a human from understanding the entirety of the proof. Even though mathematicians who develop the programs needed to solve the four color theorem understood each step of the algorithm, it is impossible for a human to examine every step of the eventual computation. Digital computers have not been proved to be flawless, so using a computer to aid proof introduces an element of uncertainty into the process of proof where each step is supposed to be certain. However, as shown by Kempe's false proof of the four color theorem, purely human derived proof is fallible. While computers are tremendous at repeating tasks over and over again, as was required to solve the four color theorem, the potential for computerized systems to fail is real, and redundancies like the development of different programs by Robertson et al, and Gonthier and Werner are necessary to avoid these errors. Also computerized proof adds an extra layer of redundancy to proofs which have already been proved by hand.

Besides disliking the methods used to solve the four color theorem some critics have taken issue with the nature of the proof. First and most importantly the methods developed to solve the four color theorem are very focused and are only applicable to solving the Four-Color Theorem. This is in contrast to the proof of Fermat's Last Theorem which contributed to the development of a number of fields of mathematics such as algebraic number theory. While proofs of the four color theorem have not contributed significantly to the knowledge base of mathematics Appel and Haken introduced the computer as a new tool for achieving proof. This advancement and the debate regarding the validity of such proofs has secured computerized proof a role in mathematics. Beyond this the four color theorem is a good example of expressing a nonmathematical problem in mathematical terms so that a solution to the problem can be found.

Citations:

1. Biggs, Norman, E. Keith Lloyd, and Robin Wilson. *Graph Theory 1736-1936.* 2nd ed. New York: Oxford University Press, 1976. 90-108 and 158-186. Print.
2. O'Connor, J J and Robertson, E F. "Alfred Bray Kempe". <http://www-history.mcs.st-andrews.ac.uk/Biographies/Kempe.html>
3. Robertson, Neil, Daniel P. Sanders, Paul Seymour, and Robin Thomas. "The Four Color Theorem." Georgia Tech School of Mathematics. Georgia Tech School of Mathematics, 9

Nov 2007. <http://people.math.gatech.edu/ thomas/FC/fourcolor.html>.
4. MacKenzie, Donald. *Mechanizing Proof: Computing, Risk, and Trust.* Massachusetts Institute of Technology, 2001. 112-120.
5. Peterson, Doug. "The Color of Controversy." *College of Liberal Arts and Sciences News*, April 2009. <http://www.las.illinois.edu/news/2009/math/>
6. Bundy, Alan. "Automated Theorem Provers: A Practial Tool for the Working Mathematician?" <http://libres.uncg.edu/ir/uncg/f/umi-uncg-1589.pdf>
7. Thomas, Robin. "An Update on the Four-Color Theorem." *Notices of the AMS.* Volume 45 .Number 7 (1998): 848-859.
8. Robertson, Neil, Sanders, Daniel P., Seymour, Paul, and Thomas, Robin. "A New Proof of the Four-Color Theorem." *Electron Research Announcements of the American Mathematical Soceity.* Volume 2, Number 1 (1996): 17-25.
9. Béla Bollobás. *Graph Theory: An Introductory Course.* Springer Science+Businesss Media, 1979. 76.
10. Gonthier, Georges. "Formal Proof-The Four Color Theorem." *Notices of the AMS.* Volume 55, Number 11. 1382-1393.
11. Weisstein, Eric W. "Four-Color Theorem." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Four-ColorTheorem.html
12. Weisstein, Eric W. "Fermat's Last Theorem." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/FermatsLastTheorem.html