Winter 2020

Conformal Geometry of Polygons

Michael Albert
Western Washington University

Conformal Geometry of Polygons

Michael Albert

April 2019

Abstract

Conformal maps are functions from subsets of the complex plane to the complex plane that locally preserve angles. Our goal is to understand conformal maps that pass to and from polygonal domains. In order to do so, we derive some of the basic theory of harmonic functions on simply connected domains. In particular, our goal with the first few sections is to prove the Schwarz Reflection principle. Using this, as well as other tools from complex analysis, we give an in-depth explanation of Tao's proof of the Schwarz-Christoffel formula. This is a differential equation that allows one to compute a conformal map from either a half plane or a disk into the interior of a polygonal domain. We apply the result to some basic examples in the analysis of fluid flow.

1 Introduction

The subject of this note is conformal mapping, which is a topic in complex analysis that has many applications to problems in physics and other areas of mathematics. While we will give a formal definition of a conformal map later in this paper, for now we consider the following idea. Let $f : D \to \mathbb{R}^2$ be a function with $D \subset \mathbb{R}^2$ and consider some point $z_0 \in D$. If the angle between any two curves passing through z_0 is the same as the angle between the image curves at $f(z_0)$, then f is said to be conformal at z_0. If f has this property for every $z_0 \in D$, then f is called a conformal map from D to $f(D)$. A classical problem in analysis is to consider some region of interest say $V \subset \mathbb{R}^2$ and to find a conformal map $f : D \to f(D)$ with $f(D) = V$. Typically we would like the domain D to be 'nice' or 'canonical'. Whenever possible, D is usually the open unit disk or the open upper half plane.

We now turn our attention away from conformal mapping in order to review and derive some material from complex analysis that will be important in the study of conformal mapping later on.

2 Preliminaries 1: Precalculus Functions

Note that for all intervals $(a,b) \subset \mathbb{R}$, with $b-a = 2\pi$, the map $\Theta : (a,b) \to \mathbb{C}$ defined by $\theta \mapsto e^{i\theta}$ is injective and continuous. The range of Θ is $S^1 \setminus \{e^{ib} = e^{ia}\}$. We can then form the continuous inverse, called $\arg_a$, which maps $z \in S^1 \setminus \{e^{ia} = e^{ia}\}$ to the unique $\theta \in (a,b)$ such that $e^{i\theta} = z$.

In most places, the canonical choice of interval is $(-\pi, \pi)$. In this case we write $\mathrm{Arg} := \arg_{-\pi}$. By scaling any non-zero complex number by its modulus, we can abuse notation and write $\mathrm{Arg}(z) = \mathrm{Arg}(\frac{z}{|z|})$ even if $|z| \neq 1$. The ray $\mathbb{C} \setminus \{re^{ia} : r \geq 0\}$ is called the *branch cut* of the function $\arg_a$. The functions in the family $\{\arg_a : a \in \mathbb{R}\}$ are called *arguments*, and Arg is called the principle argument. The entire family is sometimes called a multi-valued function, and its members are called branches. For any $z \in \mathbb{C} \setminus \{0\}$ such that both $\arg_{a_1}(z)$ and $\arg_{a_2}(z)$ are defined, (i.e $z \neq re^{ia_1}, re^{ia_2}$ for any $r > 0$), then $\arg_{a_1}(z) + 2\pi n = \arg_{a_2}(z)$ for some $n \in \mathbb{Z}$. Note that we could have chosen our interval to be closed on one side, say $[a,b)$ or $(a,b]$, in which case the inverse, while still well-defined, would not be continuous at $z = e^{ia} = e^{ib}$.

Note that the function $\mathrm{Log}_a : \mathbb{C} \setminus \{re^{ia} : r \geq 0\} \to \mathbb{C}$ defined by $\mathrm{Log}_a(z) = \ln(|z|) + i \arg_a(z)$, where ln is the familar real valued natural logarithm, has the property that $\mathrm{Log}_a \circ \exp = \mathrm{id}_{\mathbb{C} \setminus \{re^{ia} : r \geq 0\}}$. Since exp is an entire function, it follows that Log_a is analytic on $\mathbb{C} \setminus \{re^{ia} : r \geq 0\}$. We assume knowledge of the basic identities of the complex logarithm.

The final function that we will discuss in this preliminary section is the complex power function. For some $a, \beta \in \mathbb{R}$, define the function $P_{a,\beta} : \mathbb{C} \setminus \{re^{ia} : r > 0\} \to \mathbb{C}$ by

$$P_{a,\beta}(z) = \begin{cases} e^{\beta \mathrm{Log}_a(z)} & z \neq 0 \\ 0 & z = 0 \end{cases} . \tag{1}$$

Note that $P_{a,\frac{1}{\beta}}(P_{a,\beta}(z)) = z$ and $P_{a,\beta}$ has branch cut on $\{re^{ia} : r \geq 0\}$. In this paper, we will specify the branch cut $\{re^{ia} : r \geq 0\}$ and then simply write $P_{a,\beta}(z) = z^\beta$. Note that this coincides with the familiar maps $z^n, z^{1/n}$ for $n \in \mathbb{N}$. From our definition, $P_{a,\beta}$ is the composition of analytic functions and is therefore analytic. Note also that $\frac{d}{dz} P_{a,\beta}(z) = \beta P_{a,\beta-1}(z)$ or as will be familar to most calculus students, $\frac{d}{dz} z^\beta = \beta z^{\beta-1}$.

3 Preliminaries 2: Harmonic Functions

Definition 3.1. *Let $D \subset \mathbb{C}$ be a non-empty open subset.If D is path connected, it is called a **domain**. A domain $D \subset \mathbb{C}$ is called **simply connected** if for every loop, that is, every continuous curve $\sigma : [0,1] \longrightarrow D$ with $\sigma(0) = \sigma(1)$, there exist a continuous function $H : [0,1] \times [0,1] \to D$ and a point $z_0 \in D$ with $H(0,t) = \sigma(t)$ and $H(1,t) = z_0$ for all $t \in [0,1]$. Such an H is called a **homotopy**, and loops σ with this property are called **contractible** to a point.*

Simply connected domains have particularly nice properties in complex analysis. Many of these will be quoted without proof.

Definition 3.2. *Recall that the Laplacian operator $\Delta : C^2(D) \to C^0(D)$ is defined by by $\Delta(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. Let $u : D \longrightarrow \mathbb{R}$ be C^2 on the domain $D \subset \mathbb{C}$. The function u is called **harmonic** on D if $\Delta u = 0$.*

It is a simple calculation to verify that the Laplacian is an $\mathbb{R}$-linear operator. We summarize some basic results about harmonic functions with the following theorem.

Theorem 3.1. *Let $u(z) = u(x+iy) = u(x,y)$ be a real valued function defined on a simply connected domain D and $A = (x_0, y_0) \in D$ be given.*

(i) If $f : D \to \mathbb{C}$ given by $f = u + iv$ is analytic, then u and v are harmonic.

(ii) If u is harmonic, then there exists a harmonic function $v : D \to \mathbb{R}$ such that the complex valued function given by $f = u + iv$ is analytic D. Furthermore, it satisfies

$$v(B) = v(x,y) = \int_A^B -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy + v(A)$$

*for a choice of $v(x_0, y_0) = v(A)$. The function v is called the **harmonic conjugate** of u.*

(iii) Let U be a simply connected domain. If $g : U \longrightarrow D$ is a conformal map, then $u \circ g$ is harmonic on D.

We briefly outline how one direction of the proof of ii) goes. We assume v is defined as the line integral below:

$$v(B) = \int_A^B -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy + v(A)$$

Here $v(A)$ is yet to be determined. Since u is harmonic, the differential $-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$ is closed[1]. Since D is a simply connected domain, it is also an exact differential. Therefore, there is a C^1 function $\tilde{v} : D \to \mathbb{R}$ such that $d\tilde{v} = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$. Furthermore, since $-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$ is exact, the integral $\int_A^B -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = \int_A^B d\tilde{v}$ is independent of the path chosen from A to B. Since D is a domain, there is at least one path from A to B. Thus $v = v(B)$ is a well defined function (up to a choice of $v(A)$). By the fundamental theorem of calculus for line integrals we have

$$v(B) = \tilde{v}(B) - \tilde{v}(A) + v(A)$$

Set $v(A) = \tilde{v}(A)$. Thus $v(B) = \tilde{v}(B)$. Therefore v is at least C^1. Furthermore, from the equality $d\tilde{v} = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$, we conclude that $f = u + iv$ satisfies the Cauchy Riemann equations. Clearly f is continuous since u, v are, so f is analytic.

[1] For this definition and an introduction to line integrals, see [1] III.1

4 The Mean Value Property

Let $u : D \to \mathbb{R}$ be continuous on a simply connected domain D. Assume the disk of radius $\rho > 0$ centered at z_0, denoted by $D(z_0,\rho)$, is such that $D(z_0,\rho) \subset D$. Define the **average value** of u on $\partial D(z_0,r)$ for $0 < r < \rho$ by

$$A(r) = \int_0^{2\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi}. \tag{2}$$

Note that $A(0) = u(z_0)$. Since u is continuous and A is given as an integral expression of u, $A : [0,\rho) \to \mathbb{R}$ is differentiable on the open interval $(0,\rho)$. Furthermore, if u is C^1, then we have the equality

$$A'(r) = \int_0^{2\pi} \frac{\partial}{\partial r} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi}, \tag{3}$$

for all $0 < r < \rho$. Furthermore, $A(r) \to u(z_0)$ as $r \to 0$. This follows from a uniform continuity estimate of $|u(z_0) - u(z_0 + re^{i\theta})|$ on the compact set $\overline{D(z_0,s)}$ for some $0 < s < \rho$ and the fact that $\int_0^{2\pi} \frac{d\theta}{2\pi} = 1$. Thus $A(r)$ is continuous on $[0,\rho)$.

Theorem 4.1. *Let D be a simply connected domain. If $u : D \to \mathbb{R}$ is harmonic, then for all $z_0 \in D$ and all $\rho = \rho(z_0)$ such that $D(z_0,\rho) \subset U$, we have that $A(r) = u(z_0)$ for all $0 < r < \rho$.*

Proof. Let $z_0 = x_0 + iy_0$, $\rho > 0$ as in the theorem statement, and $0 < r < \rho$ be given. Let $v : D \to \mathbb{R}$ be a harmonic conjugate of u. Parameterize the circle $\partial D(z_0,r)$ as $z(\theta) = x(\theta) + iy(\theta)$ with $x(\theta) = x_0 + r\cos\theta$ and $y(\theta) = y_0 + r\sin\theta$ for $0 \le \theta \le 2\pi$. Since v is a C^2 function, we have that $\frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = 0$. By Green's Theorem,

$$\begin{aligned}
0 &= \int\int_{D(z_0,r)} \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} dA = \int_{\partial D(z_0,r)} dv = \int_{\partial D(z_0,r)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \\
&= \int_0^{2\pi} -\frac{\partial u}{\partial y}(x(\theta),y(\theta))\frac{dx}{d\theta}d\theta + \int_0^{2\pi} \frac{\partial u}{\partial x}(x(\theta),y(\theta))\frac{dy}{d\theta}d\theta \\
&= \int_0^{2\pi} \frac{\partial u}{\partial y}(x(\theta),y(\theta))r\sin\theta + \frac{\partial u}{\partial x}(x(\theta),y(\theta))r\cos\theta \, d\theta \\
&= r\int_0^{2\pi} \frac{\partial}{\partial r} u(z_0 + re^{i\theta})d\theta = 2\pi r A'(r).
\end{aligned}$$

Since $r > 0$ we have $A'(r) = 0$, so $A(r)$ is contstant on $(0,\rho)$. Since $A(0) = u(z_0)$ and A is continuous on $[0,\rho)$, we have $A(r) = u(z_0)$ for all $0 \le r < \rho$. This is called the **mean value property** of harmonic functions. □

Definition 4.1. *Let $u : D \to \mathbb{R}$ be a continuous function. The function u is said to have the **mean value property** if for all $z_0 \in D$ there exists $\rho_0 > 0$ so that $A(r) = u(z_0)$ for every $0 < r < \rho_0$.*

Theorem 4.2. *A function is harmonic on a domain if and only if it is continuous and has the mean value property.*

We state this theorem without yet proving it at this moment. The following results are studied with the end goal of proving Theorem 3.2.

Theorem 4.3. *Let D be an open set and $u : D \to \mathbb{R}$ have the mean value property. Suppose there is $D(z_0,\rho) \subset D$ be such that there exists $M \in \mathbb{R}$ such that $u(z) \le M$ for all $z \in D(z_0,\rho)$. If there exists $z_1 \in D(z_0,\rho)$ such that $u(z_1) = M$, then u is the constant function $u(z) = M$ on for all $z \in D(z_0,\rho)$.*

Proof. Let $z_1 \in D(z_0,\rho)$ be given such that $u(z_1) = M$. Set $s = \frac{\rho - |z_1 - z_0|}{2}$, so $D(z_1,s) \subset D(z_0,\rho) \subset D$. Thus, we can apply the mean value property on $D(z_1,s)$ to obtain

$$0 = u(z_1) - A_{z_1}(r) = \int_0^{2\pi} u(z_1) - u(z_1 + re^{i\theta})\frac{d\theta}{2\pi}$$

where A_{z_1} indicates that the average is taken about a circle centered at z_1. This equality holds for all $0 < r < s$. Further, $u(z_1) = M \ge u(z_1 + re^{i\theta})$. Thus, since the integral on the right-hand side vanishes and the integrand is positive, the integrand must vanish. Thus $M = u(z_1) = u(z_1 + \tilde{r}e^{i\theta})$ for all $0 \le \theta \le 2\pi$ and all $0 < r < s$. We see that $u(z) = M$ for all $z \in D(z_1,s)$.

We have shown that z_1 is an interior point of $U := \{z \in D(z_0,\rho) : u(z) = M\}$. Since $z_1 \in U$ was arbitrary, U is an open set in $D(z_0,\rho)$. However, since $u|_{D(z_0,\rho)}$ is continuous and $\{M\}$ is a closed set, we use the set equality $U = u|_{D(z_0,\rho)}^{-1}(M)$, to see that U is also closed in $D(z_0,\rho)$. Thus U is closed, open and non-empty in the connected set $D(z_0,\rho)$. We conclude that $U = D(z_0,\rho)$. □

Note that we have not made the assumption that D is a domain. Here we were able to reason locally with only the assumption that D was open. We state a very useful corollary of this result.

Theorem 4.4. *Let D be a domain and $u : D \to \mathbb{R}$. Let $r > 0$ be such that u is continuous on $D(z_0,r) \subset D$ and u has the mean value property on $D(z_0,r)$. If there exists $z_1 \in D(z_0,r)$ with $u(z_1) = \max_{\overline{D(z_0,r)}} u$, then u is constant on $\overline{D(z_0,r)}$.*

Proof. By compactness, u attains a maximum value at some point in $\overline{D(z_0,r)}$. By Theorem 4.3, if this point is in $D(z_0,r)$, then u is constant in $D(z_0,r)$. By continuity, u must then be constant on $\partial D(z_0,r)$ as well, and so u is constant on $\overline{D(z_0,r)}$. □

It follows that a non-constant continuous function $u : \overline{D(z_0,\rho)} \to \mathbb{R}$ with the mean value property on $D(z_0,\rho)$ attains its maximum on $\partial D(z_0,\rho)$. This is called the **maximum principle**. We will frequently use the fact if a continuous function with the mean value property on some disk vanishes on the boundary of that disk, then it vanishes on the closed disk.

5 The Dirichlet Problem

Write $\mathbb{D} = D(0,1)$ and let $u : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function. The Dirichlet Problem is to find a harmonic function $\tilde{u} : \mathbb{D} \to \mathbb{R}$ such that the limit $\lim_{z\to e^{i\theta}, z\in\mathbb{D}} \tilde{u} = u(e^{i\theta})$ holds for all $[-\pi,\pi]$. We indicate how to solve this problem in what follows.

Define the function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{1+z}{1-z}. \tag{4}$$

We see that p is analytic on $\mathbb{D}$ and so the function $P := \mathrm{Re}(p)$ is harmonic on $\mathbb{D}$. This function is called the **Poisson kernel**. Define the **Poisson integral** of u to be the function

$$\tilde{u}(z) = \tilde{u}(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\psi})P(r,\theta-\psi)d\psi. \tag{5}$$

We will show that $\tilde{u}$ solves the Dirichlet problem. To do this, we need to show three properties of the Poisson kernel.

Theorem 5.1. *The following properties hold:*

(i) For all $0 \le r < 1$ and all $-\pi \le \theta \le \pi$, we have $P(r,\theta) > 0$

(ii) For all $0 \le r < 1$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\psi)d\psi = 1$.

(iii) Fix $0 < \delta < \pi$. Then $\max\{P(r,\theta) : \delta \le |\theta| \le \pi\} \to 0$ as $r \to 1$.

Proof. (i) Write

$$P(r,\theta) = Re\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1}{2}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}} + \overline{\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)}\right) = \frac{1}{2}\left(\frac{2-2r^2}{1-r(e^{i\theta}+e^{-i\theta})+r^2}\right) = \frac{1-r^2}{1+r^2-2r\cos\theta}.$$

Since $0 \le r < 1$, the numerator and denominator are both clearly positive.

(ii) Applying Cauchy's integral formula to the function $f(z) = \frac{1}{1-rz}$ at the point $r \in \mathbb{D}$ we have,

$$\begin{aligned}\int_{-\pi}^{\pi} P(r,\psi)\frac{d\psi}{2\pi} &= \int_{-\pi}^{\pi} \frac{1-r^2}{(1-re^{i\psi})(1-re^{-i\psi})}\frac{d\psi}{2\pi} = \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{d\psi}{(1-re^{i\psi})(1-re^{-i\psi})} \\ &= \frac{1-r^2}{2\pi}\int_{\partial\mathbb{D}}\frac{dz}{(1-rz)(1-r\overline{z})}\frac{1}{iz} = \frac{1-r^2}{2\pi i}\int_{\partial\mathbb{D}}\frac{dz}{(1-rz)(z-r)} = \frac{1-r^2}{2\pi i}2\pi i\frac{1}{1-r^2} \\ &= 1\end{aligned}$$

(iii) Let $0 < \delta < \pi$ be fixed and $1 > \varepsilon > 0$ be given. Let $1 > r > r(\varepsilon) := \sqrt{1-\varepsilon(1-\cos^2\delta)}$. Then, for any $|\theta| \in [\delta, \pi]$

$$\begin{aligned}\varepsilon &> \frac{1-r^2}{1-\cos^2\delta} \geq \frac{1-r^2}{(r-\cos\delta)^2+1-\cos^2\delta} = \frac{1-r^2}{r^2-2r\cos\delta+1} \geq \frac{1-r^2}{r^2-2r\cos\theta+1} \\ &= P(r,\theta)\end{aligned}$$

Taking the max over all $|\theta| \in [\delta, \pi]$ gives the desired result.

□

Theorem 5.2. *Let $u : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function. Then $\tilde{u}$ defined in* (5) *is a harmonic function on $\mathbb{D}$ with the property that*

$$\lim_{z\to e^{i\theta}, z\in\mathbb{D}} \tilde{u}(z) = u(e^{i\theta})$$

for every $\theta \in [-\pi, \pi]$.

Proof. First we check harmonicity. To do so, we need to differentiate under the integral sign in polar coordinates. This is fine as long as the integrand is C^2. Recall that the Laplacian in polar coordinates is $\Delta_P := \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$. Since $u(e^{i\psi})P(r,\theta-\psi)$ is a C^2 function of θ and r, we may write

$$\begin{aligned}\Delta_P\tilde{u}(r,\theta) =& \frac{\partial^2}{\partial r^2}\int_{-\pi}^{\pi} u(e^{i\psi})P(r,\theta-\psi)\frac{d\psi}{2\pi} + \frac{1}{r}\frac{\partial}{\partial r}\int_{-\pi}^{\pi} u(e^{i\psi})P(r,\theta-\psi)\frac{d\psi}{2\pi} \\ &+ \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\int_{-\pi}^{\pi} u(e^{i\psi})P(r,\theta-\psi)\frac{d\psi}{2\pi} \\ =& \int_{-\pi}^{\pi} u(e^{i\psi})\left(\frac{\partial^2}{\partial r^2}P(r,\theta-\psi) + \frac{1}{r}\frac{\partial}{\partial r}P(r,\theta-\psi) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}P(r,\theta-\psi)\right)\frac{d\psi}{2\pi} \\ =& \int_{-\pi}^{\pi} u(e^{i\psi})\Delta_P P(r,\theta-\psi)\frac{d\psi}{2\pi} = 0.\end{aligned}$$

To justify the last line, note that $P(r,\theta)$ is still harmonic up to the change of variable $(r,\theta) \mapsto (r, \theta-\psi)$ for each ψ.

Regarding the limit, let $e^{i\theta} \in \partial\mathbb{D}$ and $\varepsilon > 0$ be given. We must show that there exists a $d = d(\varepsilon) > 0$ such that for all $z \in \mathbb{D}$ with $|z - e^{i\theta}| < d$, we have $|\tilde{u}(z) - u(e^{i\theta})| < \varepsilon$. Let $\delta = \delta(\varepsilon) > 0$ be such that for all $\phi, \psi \in [-\pi, \pi]$ we have $|\phi - \psi| < \delta$ implies $|u(e^{i\phi}) - u(e^{i\psi})| < \frac{\varepsilon}{2}$. This is fine since the map $\psi \mapsto e^{i\psi} \mapsto u(e^{i\psi})$ is uniformly continuous on $[-\pi, \pi]$. Furthermore, in light of fact iii) from 5.1, let $r(\varepsilon) < 1$ be large enough that $r(\varepsilon) < r < 1$ implies $\max\{P(r,\psi) : \frac{\delta}{2} \leq |\psi| \leq \pi\} < \frac{\varepsilon}{4M}$, where M is a real number such that $|u| \leq M$ on $\partial\mathbb{D}$. Now pick $d = \min\{1 - r(\varepsilon), 2\sin(\frac{\delta}{4})\}$ and let $z \in \mathbb{D}$ be such that $|z - e^{i\theta}| < d$. Now write $z = re^{i\eta}$ and note that $2\sin(\frac{\delta}{4})$ is the formula for the length of a chord on the unit circle with central angle measure $\frac{\delta}{2}$. Now since z can not have distance from $e^{i\theta}$ larger than the length of this chord, we see that $|\eta - \theta| < \frac{\delta}{2}$. Furthermore, we see that $r > r(\varepsilon)$. Now, we write using fact ii) from 5.1

$$|\tilde{u}(re^{i\eta}) - u(e^{i\theta})| = \left|\int_{-\pi}^{\pi} u(e^{i\psi})P(r,\eta-\psi)\frac{d\psi}{2\pi} - \int_{-\pi}^{\pi} u(e^{i\theta})P(r,\psi)\frac{d\psi}{2\pi}\right|. \tag{6}$$

Since $u(e^{i\cdot})$ and $P(r,\cdot)$ are periodic functions of period 2π, we have the equalities

$$\int_{-\pi}^{\pi} u(e^{i\psi})P(r,\eta-\psi)\frac{d\psi}{2\pi} = -\int_{\eta+\pi}^{\eta-\pi} u(e^{i(\eta-\psi)})P(r,\psi)\frac{d\psi}{2\pi} = \int_{-\pi}^{\pi} u(e^{i(\eta-\psi)})P(r,\psi)\frac{d\psi}{2\pi}. \tag{7}$$

Thus, returning to (6), we have

$$\begin{aligned}&\left|\int_{-\pi}^{\pi} u(e^{i\psi})P(r,\eta-\psi)\frac{d\psi}{2\pi} - \int_{-\pi}^{\pi} u(e^{i\theta})P(r,\psi)\frac{d\psi}{2\pi}\right| = \left|\int_{-\pi}^{\pi} u(e^{i(\eta-\psi)})P(r,\psi)\frac{d\psi}{2\pi} - \int_{-\pi}^{\pi} u(e^{i\theta})P(r,\psi)\frac{d\psi}{2\pi}\right| \\ =&\left|\int_{-\pi}^{\pi} (u(e^{i(\eta-\psi)}) - u(e^{i\theta}))P(r,\psi)\frac{d\psi}{2\pi}\right| \leq \int_{-\pi}^{\pi} |u(e^{i(\eta-\psi)}) - u(e^{i\theta})|P(r,\psi)\frac{d\psi}{2\pi} \\ =& \int_{-\delta/2}^{\delta/2} |u(e^{i(\eta-\psi)}) - u(e^{i\theta})|P(r,\psi)\frac{d\psi}{2\pi} + \int_{|\psi|\in[\delta/2,\pi]} |u(e^{i(\eta-\psi)}) - u(e^{i\theta})|P(r,\psi)\frac{d\psi}{2\pi}.\end{aligned} \tag{8}$$

Now, noting that for $|\psi| < \delta/2$, we have $|\theta - \eta + \psi| \le |\theta - \eta| + |\psi| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$, we have the estimate

$$\begin{aligned}
&\int_{-\delta/2}^{\delta/2} |u(e^{i(\eta-\psi)}) - u(e^{i\theta})| P(r,\psi)\frac{d\psi}{2\pi} + \int_{|\psi|\in[\delta/2,\pi]} |u(e^{i(\eta-\psi)}) - u(e^{i\theta})| P(r,\psi)\frac{d\psi}{2\pi} \\
&< \frac{\varepsilon}{2}\int_{-\delta/2}^{\delta/2} P(r,\psi)\frac{d\psi}{2\pi} + \int_{|\psi|\in[\delta/2,\pi]} 2M \max\{P(r,\psi) : \frac{\delta}{2} \le |\psi| \le \pi\}\frac{d\psi}{2\pi} \\
&< \frac{\varepsilon}{2} + \int_{|\psi|\in[\delta/2,\pi]} M\frac{\varepsilon}{4M}\frac{d\psi}{\pi} \\
&< \varepsilon.
\end{aligned} \tag{9}$$

Since $\varepsilon > 0$ was arbitrary, we conclude the desired limit statement. □

Note that the Dirichlet problem can be solved on any disk $D(z_0,\rho)$ for some continuous function $u : \partial D(z_0,\rho) \to \mathbb{R}$. We consider the auxillary function $v : \partial\mathbb{D} \to \mathbb{R}$, defined by $v(e^{i\theta}) = u(\rho e^{i\theta} + z_0)$. We see that v is a continuous function on the unit circle and from 5.2 there is a function $\tilde{v} : \mathbb{D} \to \mathbb{R}$ that solves the Dirichlet problem for v. It follows that $\tilde{u}(z) := \tilde{v}(\frac{1}{\rho}(z - z_0))$ solves the Dirichlet problem for u.

6 Schwarz's Reflection Principle for Harmonic Functions

We use the solvability of the Dirichlet problem to derive a powerful result for harmonic functions on symmetric domains called Schwarz's Reflection Principle.

Theorem 6.1. *Let $u : D \to \mathbb{R}$ be a continuous function on a domain D satisfying the mean value property. Then u is a harmonic function.*

Proof. Let $z_0 \in D$ be given and $\overline{D(z_0,\rho)} \subset D$ such that u satisfies the mean value property on $D(z_0,\rho)$. By the solvability of the Dirichlet problem, there exists a harmonic function $\tilde{u}$ on $D(z_0,\rho)$ that extends continuously to $\overline{D(z_0,\rho)}$ and coincides with u on $\partial D(z_0,\rho)$. Both $\tilde{u}$ and u satisfy the mean value property on $D(z_0,\rho)$, therefore by the linearity of the integral, so does $\tilde{u} - u$. Since $\tilde{u} - u = 0$ on $\partial D(z_0,\rho)$, by the maximum principle for functions satisfying the mean value property, $\tilde{u} - u = 0$ in $D(z_0,\rho)$. Hence $\tilde{u} = u$ in $D(z_0,\rho)$ and we see that u is indeed harmonic. □

Lemma 6.2. *The zeroes of a non-constant analytic function are isolated.*

Proof. Let $f : D \to \mathbb{C}$ be a non-constant analytic function on the domain D. Consider first the set $A = \{z \in D : f^{(k)}(z) = 0 : \text{for all } k \ge 0\}$. Assume by contradiction $A \ne \emptyset$ and let $z_0 \in A$.

Since f is analytic at z_0, there exists $r > 0$ so that $f(z) = \sum_{k=0}^{\infty} f^{(k)}(z_0)(z - z_0)^k$ for all $|z - z_0| < r$. In particular, we then have that $f(z) = 0$ for all $|z - z_0| < r$ and thus, $f^{(k)}(z) = 0$ for all $k \ge 0$, $z \in D(z_0, r)$. Therefore if $|z - z_0| < r$, then $z \in A$, hence A is an open set. However, note that $A = \bigcap_{k=0}^{\infty}(f^{(k)})^{-1}\{0\}$ which is the intersection of closed sets, since $f^{(k)}$ is analytic for all $k \ge 0$ and $\{0\}$ is closed. Thus, since A is both open and closed, and D is connected, $A = \emptyset$ or D. If $A = D$ then $f(z) = 0$ for all $z \in D$, which cannot be the case since f is non-constant. Thus $A = \emptyset$ and so there are no points $z \in D$ with $f(z) = 0$ such that $f^k(z) = 0$ for all $k \ge 0$.

Suppose now $f(z_0) = 0$ and let $N = \min\{k \in \mathbb{N} : f^{(k)}(z_0) \ne 0\}$. By the factor theorem, [1], there exist $\rho > 0$ and $h : D(z_0,\rho) \to \mathbb{C}$ analytic with $h(z_0) \ne 0$ such that $f(z) = (z - z_0)^N h(z)$ for all $z \in D(z_0,\rho)$. Let $r > 0$ be small enough so that $h(z) \ne 0$ if $|z - z_0| < r$. Thus, if $s < \min\{\rho, r\}$, we have $|f(z)| = |(z - z_0)^N h(z)| = |z - z_0|^N |h(z)| > 0$ for all $z \ne z_0$ in $D(z_0, s)$. □

Theorem 6.3. *Let D be a domain that is symmetric with respect to the real line in the sense that $D^- = \{Im(z) < 0\} \cap D = \{\overline{z} : z \in D, Im(z) > 0\}$. Let u be a harmonic function on $D^+ = \{Im(z) > 0\} \cap D$ that extends to be continuous and zero on $\{Im(z) = 0\} \cap D$. Then the function*

$$v(z) = \begin{cases} u(z) & z \in D^+ \\ -u(\overline{z}) & z \in D^- \\ 0 & z \in \{Im(z) = 0\} \cap D \end{cases} \tag{10}$$

is harmonic on D. Furthermore, v is the unique function which is harmonic on D and coincides with u on D^+.

Proof. By construction v is continuous on $D^+ \cup (\{Im(z) = 0\} \cap D)$ and on D^-, and since $\lim_{\mathrm{Im}z \nearrow 0} v(z) = \lim_{\mathrm{Im}z \nearrow 0} -u(\overline{z}) = -\lim_{\mathrm{Im}z \searrow 0} u(z) = 0$, it is also continuous on $D \cap \mathbb{R}$. Thus v is continuous on all of D.

We will show that v has the mean value property on D. Given $z_0 \in D^+$ let $r > 0$ be small enough so that $D(z_0, r)$ lies completely in D^+. Since v is harmonic on D^+, we have that v has the mean value property on D^+.

Given $z_0 \in D^-$ we again choose a small enough disk $D(z_0, r)$ that lies completely in D^-. Then

$$\int_0^{2\pi} v(z_0 + re^{i\theta})\frac{d\theta}{2\pi} = \int_0^{2\pi} -u(\overline{z_0} + re^{-i\theta})\frac{d\theta}{2\pi} = \int_0^{-2\pi} u(\overline{z_0} + re^{i\theta})\frac{d\theta}{2\pi} = -u(\overline{z_0}) = v(z_0)$$

since by symmetry, $\overline{z_0} + re^{i\theta}$ parameterizes a circle of radius r contained completely in D^+, on which v has the average value property by hypothesis.

Now finally let $z_0 \in \mathbb{R} \cap D$ and assume $D(z_0, r) \subset D$ for some $r > 0$. Then,

$$\begin{aligned}\int_0^{2\pi} v(z_0 + re^{i\theta})\frac{d\theta}{2\pi} &= \int_0^{\pi} v(z_0 + re^{i\theta})\frac{d\theta}{2\pi} + \int_{\pi}^{2\pi} v(z_0 + re^{i\theta})\frac{d\theta}{2\pi} = \int_0^{\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi} + \int_{\pi}^{2\pi} -u(\overline{z_0} + re^{-i\theta})\frac{d\theta}{2\pi} \\ &= \int_0^{\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi} + \int_{-\pi}^{-2\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi} = \int_0^{\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi} + \int_{\pi}^{0} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi} \\ &= 0 = v(z_0).\end{aligned}$$

Therefore v has the mean value property everywhere on D and so by Theorem 6.1 it is harmonic on D.

Regarding uniqueness, suppose $\phi, \psi : D \to \mathbb{R}$ are two real valued harmonic functions such that $\phi|_{D^+} = \psi|_{D^+} = u$. Then $g = \psi - \phi$ is harmonic on D and vanishes on D^+.

Notice that $f = \frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}$ is analytic, since g is harmonic (hence smooth), and f satisfies the Cauchy Riemann equations. Since $g = 0$ on D^+, f is an analytic function that is zero on an open set. By Lemma 6.2, f is identically the zero function and so $g = 0$. □

This result is called the **reflection principle for harmonic functions** and is originally due to Schwarz.

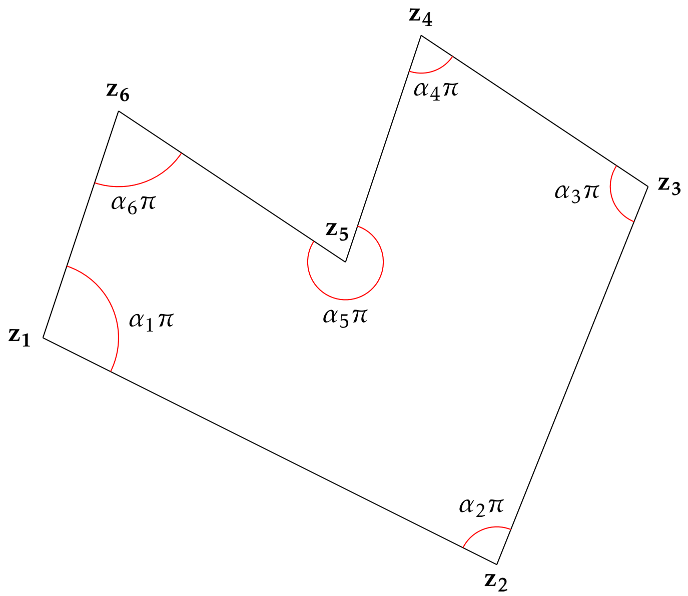

Figure 1: A non-convex 6 sided polygon with vertices labelled in counterclockwise increasing order.

7 Conformal Mapping and the Schwarz-Christoffel Formula

Let $g : U \to \mathbb{C}$ be an analytic function on the domain U. If g is injective and we write $V = g(U)$, then the map $g : U \to V$ is bijective and is said to be *conformal*. Let $\gamma : [0,1] \to U$ be a C^1 and write $z_0 = \gamma(t_0)$ for some $t_0 \in (0,1)$ such that $\gamma'(t_0) \neq 0$. Suppose $\sigma : [0,1] \to U$ is another C^1 curve such that there is $t_1 \in (0,1)$ with $\sigma(t_1) = z_0$ and $\sigma'(t_1) \neq 0$. Consider the quantity $\theta = \arg(\gamma'(t_0) - \arg(\sigma'(t_1))$, where the choice of branch is not important. This is the angle between the vectors $\gamma'(t_0)$ and $\sigma'(t_1)$. Suppose that we want to compute the angle between the image vectors $\frac{d}{dt} g(\gamma(t))|_{t=t_0}$ and $\frac{d}{dt} g(\sigma(t))|_{t=t_1}$. Recall that since g is injective, $g' \neq 0$, so $\frac{d}{dt} g(\gamma(t))|_{t=t_0} = g'(z_0)\gamma'(t_0) \neq 0$ and similarly for σ. Thus, computing $\arg(\frac{d}{dt} g(\gamma(t))|_{t=t_0}) - \arg(\frac{d}{dt} g(\sigma(t))|_{t=t_1}) = \arg(g'(z_0)\gamma'(t_0)) - \arg(g'(z_0)\sigma'(t_0)) = \arg(\gamma'(t_0)) - \arg(\sigma'(t_0)) = \theta$. This is called the angle-preserving property of conformal maps. Said informally, vectors pointing from $z_0 \in U$ will have the same internal angle as their image vectors do under g. We introduce one of the most remarkable theorems in all of complex analysis, originally due to Riemann.

Theorem 7.1. *Let $U \subset \mathbb{C}$ be a simply connected domain that is not all of $\mathbb{C}$. There exists a conformal map $g : U \to \mathbb{D}$, where $\mathbb{D}$ is the unit disk.*

This is called the **Riemann mapping theorem** and we will suppose its truth without proof. Proofs of this theorem are beyond the scope of this paper. A proof may be found in [2], [1], or any good complex analysis text.

We consider the problem of conformally mapping the interior of a polygon to the open unit disk. The argument that follows is due to Tao in [2] that is in turn based on work by Ahlfors. We first clarify some terminology. Let $f : U \to \mathbb{C}$ be an analytic function on the domain U. Let V be an open set in $\mathbb{C}$ such that $U \cup V$ is a domain. We say that f can be analytically continued or extended to V if there is an analytic function $g : U \cup V \to \mathbb{C}$ such that $g|_U = f$. We call g the analytic extension or continuaion of f. Note that if such a g exists, then it is unique. It suffices to find an analytic $g : V \to \mathbb{C}$ such that $g|_{V\cap U} = f|_{V\cap U}$. We will say that f extends *across* some $W \subset \partial U$ if there is an analytic function $g : V \to \mathbb{C}$ where $W \subset V$ and $g|_{V\cap U} = f$.

A note on terminology: If X is a topological space and $Y \subset X$ is given the subspace topology, then a set of the form $A \cap Y$, where A is open in X, will be called a Y-neighbourhood of x_0 for some $x_0 \in A \cap Y$.

Let $z_1, z_2, ..., z_n \in \mathbb{C}$ be distinct. Let $\gamma_{i,i+1}$ denote the straight line path from z_i to z_{i+1} for each $i = 1, 2..., n$, where $n+1$ is taken to be 1. Suppose that the z_i are such that the path concatenation $\gamma := \gamma_{1,2} * \gamma_{2,3} * ... * \gamma_{n-1,n} * \gamma_{n,1}$ is a simple closed curve traversed in the counterclockwise direction. By the Jordan Curve Theorem[1], the image of γ forms the boundary of a bounded simply connected domain U called a polygon. We call the z_j vertices of the polygon U. A polygon constructed in this way is called a simple polygon. By Theorem 6.1, there is a conformal map $f : U \to \mathbb{D}$, where as before $\mathbb{D}$ is the open unit disk. We call a map of this form a

Schwarz-Christoffel map. In some places, any domain that is bounded by some configuration of lines and line segments in the plane is considered a polygon, which may not be simple or even bounded. We utilize some of the nicer topological properties of bounded simple polygons in the initial argument. Later, we will mention how the ideas can be extended to unbounded polygonal domains.

Outline of the proof and the goal of this section

We will analyze the behaviour of f near a boundary point $z^* \in \partial U$. First we pass to a conformal map ϕ_{z^*} from a half disk $D^+(0,\varepsilon) := \{|\zeta| < \varepsilon, \mathrm{Im}(\zeta) > 0\}$ in the ζ-plane to the polygon U in the z-plane. This map should take the portion of the real axis in $D^+(0,\varepsilon)$ to ∂U and take 0 to z^*. We show that $\log|f \circ \phi_{z^*}|$ can be harmonically extended to all of $D(0,\varepsilon)$. This will then allow us to show that $f \circ \phi_{z^*}$ extends to be analytic on all of $D(0,\varepsilon)$. This will show us that f extends to be analytic across the "open" sides $\gamma_{i,i+1}(0,1)$ of U. That is, the boundary of U without the vertices. We then argue that f still extends to be continuous and bijective on all of $\overline{U}$. It then follows that F has a continuous inverse from $\overline{\mathbb{D}}$ to $\overline{U}$ and that this inverse is analytic in $\mathbb{D}$ and across $S^1 \setminus \{w_1, \ldots, w_n\}$ for some $w_i \in S^1$ that are yet to be determined. The goal of this section will be to write down a succinct formula for F.

Constructing the map ϕ

Note that for each $1 \le j \le n$ there is a unique $0 < \alpha_j < 2$ and some $\lambda_j > 0$ such that

$$\lambda_j e^{i\pi\alpha_j}(z_{j+1} - z_j) = z_{j-1} - z_j.$$

That is, the vector pointing from z_j to z_{j-1} is rotated through $\alpha_j \pi$ radians to obtain a scalar multiple of the vector pointing from z_j to z_{j+1}. This is the interior angle at the vertex z_j.

Now for a $z^* \in \partial U$ we either have that z^* is an interior point to some line segment from z_j to z_{j+1} for some j or that z^* is identically z_j for some j. Let $D^+(0,\varepsilon)$ denote the open upper half disk of radius $\varepsilon > 0$ (as in it does not include the real line). In the case that z^* is not a vertex, the map $\phi_{z^*} : D^+(0,\varepsilon) \to \mathbb{C}$ defined by

$$\phi_{z^*}(\zeta) = \zeta(z_{j+1} - z_j) + z^*$$

is injective and analytic, with range contained in U for $\varepsilon > 0$ small enough. When $z^* = z_j$ for some $1 \le j \le n$, we consider the map $\phi_{z^*} : D^+(0,\varepsilon) \to \mathbb{C}$ defined by

$$\phi(\zeta) = (z_{j+1} - z_j)\zeta^{\alpha_j} + z_j,$$

where we choose the branch of $\zeta \mapsto \zeta_j^{\alpha}$ that is positive and real on the positive real axis and which has branch cut on the negative imaginary axis. Thus, ϕ_{z^*} is analytic and injective in both cases. Noting that the interior angle at z_j is $\alpha_j \pi$, for $\varepsilon > 0$ small enough, the range of ϕ_{z^*} is contained in U.

Arguing that $|f|$ extends continuously to $\overline{U}$

We see that $f \circ \phi_{z^*}$ analytically and injectively maps $D^+(0,\varepsilon)$ to $\mathbb{D}$. Let $1 > \epsilon > 0$ be given. Note that the set $A_\epsilon = \{z : |f(z)| \le 1 - \epsilon\}$ is closed in U for every $0 < \epsilon < 1$. A_ϵ is a closed set in the compact Hausdorff space $\overline{U}$ and hence A_ϵ is compact. Let $z^* \in \partial U$ be given. Note that z^* is not a limit point of the compact set A_ϵ and so $d(z^*, A_\epsilon) > 0$. Let $z_n \in U$ be a sequence of complex numbers where $z_n \to z^*$ in $\overline{U}$. There exists an N_ϵ such that $n \ge N_\epsilon$ implies $|z^* - z_n| < d(z^*, A_\varepsilon)$. This implies that $z_n \notin A_\epsilon$ by the definition of infimum. Hence $|f(z_n)| > 1 - \epsilon$ so $|f(z_n)| \to 1$. Thus $|f|$ can be made continuous on all of $\overline{U}$ by setting $|f| = 1$ on ∂U.

Arguing that $f \circ \phi_{z^*}$ extends analytically to $D(0,\varepsilon)$

Note that for both z^* a vertex and z^* a non-vertex, ϕ_{z^*} continuously maps the real portion of the boundary of $D^+(0,\varepsilon)$ to ∂U. Thus, as $\mathrm{Im}(\zeta) \to 0$, we have $|(f \circ \phi_{z^*})(\zeta)| \to 1$, and so $\log|(f \circ \phi_{z^*})(\zeta)| \to 0$. For $\varepsilon > 0$ small enough, we can make $|f \circ \phi_{z^*}| > 0$ on $\overline{D^+(0,\varepsilon)}$, where we use the definition of $|f|$ extended to ∂U. With this in mind together with the analyticity of $f \circ \phi_{z^*}$, we have that $\ln|f \circ \phi_{z^*}|$ is harmonic on $D^+(0,\varepsilon)$ and tends to 0 as $\mathrm{Im}(\zeta) \to 0$. To see that $\ln|f|$ is harmonic whenever $f = u + iv$ is analytic and non-zero, consider $2\ln|f| = \ln|f|^2 = \ln(u^2 + v^2)$, then compute the laplacian of the right hand side and use the Cauchy-Riemann equations to conclude that $\nabla 2 \ln|f| = 0$ and thus $\nabla \ln|f| = 0$.

Applying the Schwarz reflection principle

We apply the Schwarz reflection principle to conclude that there is a harmonic function $G : D(0,\varepsilon) \to \mathbb{C}$ with $G|_{D^+(0,\varepsilon)} = \ln|f \circ \phi_{z^*}|$. Since $D(0,\varepsilon)$ is simply connected, G is the real part of some analytic function $g_{z^*} : D(0,\varepsilon) \to \mathbb{C}$. However, note that on $D^+(0,\varepsilon)$, $\ln|f \circ \phi_{z^*}|$ is the real part of the analytic function $\mathrm{Log}_\beta f \circ \phi_{z^*}$ for some branch of the complex logarithm Log_β with branch cut $\{re^{i\beta} : r \ge 0\}$ disjoint from the range of

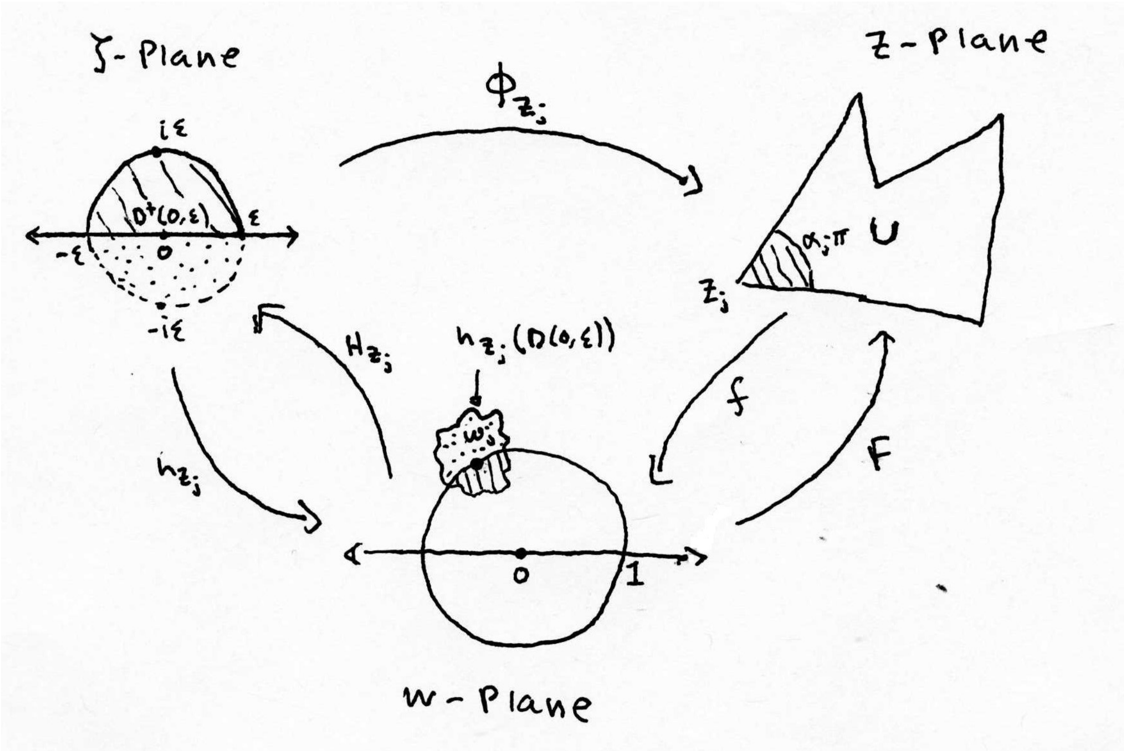

Figure 2: Diagram of the three domains that we are working with and the maps that relate them

$f \circ \phi_{z^*}$ in $\mathbb{D}$. We conclude that g_{z^*} is the unique analytic continuation of $\log_\beta f \circ \phi_{z^*}$ to the disk $D(0,\varepsilon)$. But then $\exp(g_{z^*})$ is also analytic on $D(0,\varepsilon)$, and since this coincides with $f \circ \phi_{z^*}$ on $D^+(0,\varepsilon)$, we see that $f \circ \phi_{z^*}$ can be analytically continued to $D(0,\varepsilon)$. In particular, $f \circ \phi_{z^*}$ extends to be analytic on the real axis in $D(0,\varepsilon)$. Let h_{z^*} be the analytic continuation of $f \circ \phi_{z^*}$ to $D(0,\varepsilon)$. Note that since h_{z^*} is continuous, and

$$\lim_{\mathrm{Im}(\zeta)\searrow 0} (f \circ \phi_{z^*})(\zeta) \in S^1,$$

we have

$$\lim_{\mathrm{Im}(\zeta)\to 0} h_{z^*}(\zeta) \in S^1,$$

so h_{z^*} maps the real axis to S^1. In particular, h_{z^*} maps $D(0,\varepsilon)$ to a $\mathbb{C}$-neighbourhood of some point in S^1. In general, for $z^* \in \partial U$, we set $f(z^*) := h_{z^*}(0)$. There is now a notion of f defined on all of $\overline{U}$. In particular we write $w_j := f(z_j) = h_{z_j}(0)$.

Arguing that the extended f is a continuous bijective map from $\overline{U}$ to $\overline{D(0,1)}$

As a brief aside, we will need later that for any $z^* \in \partial U$, $h'_{z^*}(0) \neq 0$. Suppose otherwise, and we consider the power series expansion of h_{z^*} about 0 as $h_{z^*}(\zeta) = \sum a_k \zeta^k$. Note that $a_0 = h_{z^*}(0)$ and $a_1 = h'_{z^*}(0) = 0$. Therefore, the first term in the expansion of $h_{z^*}(\zeta) - h_{z^*}(0)$ with non-zero coefficient has degree $m \geq 2$. Factoring, we have $h_{z^*}(\zeta) - h_{z^*}(0) = \zeta^m p(\zeta)$, where p is some analytic function and $p(0) \neq 0$. We would like to measure how much the image under $\zeta \mapsto \zeta^m p(\zeta)$ of a curve in $D^+(0,\varepsilon)$ winds around the origin. This is called the increase in argument and is given by an integral. More discussion can be found in [1].

Note that for $\varepsilon > 0$ small enough, $p \neq 0$ on $D^+(0,\varepsilon)$ and $\zeta^m \neq 0$ on $D^+(0,\varepsilon)$. The range of p can not enclose the origin, since it is contained in some disk away from the origin. Thus, there is some ray $\{re^{i\beta} : r \geq 0\}$ disjoint from the range of p, and we let $\arg_\beta$ be the branch of arg with branch cut on this ray. Thus $\arg_\beta(p(\zeta)\zeta^m) = \mathrm{Im}\left(\int_{i\varepsilon/2}^{\zeta} \frac{(p(s)s^m)'}{p(s)s^m} ds + i\arg_\beta(p(i\varepsilon/2)(i\varepsilon/2)^m)\right)$ is harmonic on $D^+(0,\varepsilon)$, since the right hand side is the imaginary part of an analytic function. Let $\gamma : [0,1] \to D^+(0,\varepsilon)$ be a continuous curve. The differential $d\arg_\beta(p(\zeta)\zeta^m)$ exists on $D^+(0,\varepsilon)$ since $\arg_\beta$ is harmonic here, and we have the equality

$$\int_\gamma d\arg_\beta(p(\zeta)\zeta^m) d\zeta = \arg_\beta(p(\zeta)\zeta^m)\Big|_{\gamma(0)}^{\gamma(1)}.$$

If this integral can be made larger than π for some curve γ with range in $D^+(0,\varepsilon)$, we have that the range of $p(\zeta)\zeta^m|_{D^+(0,\varepsilon)} = (h_{z^*}(\zeta) - h_{z^*}(0))|_{D^+(0,\varepsilon)}$ would not be contained in any half plane bordering the origin. Note that we can always contain $\mathbb{D} - h_{z^*}(0)$ in a half plane bordering the origin by simply taking the tangent line at $h_{z^*}(0) \in S^1$ and then translating by $h_{z^*}(0)$. This would be a contradiction because by hypothesis, $h_{z^*}|_{D^+(0,\varepsilon)} - h_{z^*}(0) = f \circ \phi_{z^*} - h_{z^*}(0)$ has range contained in $\mathbb{D} - h_{z^*}(0)$.

Shrinking $\varepsilon > 0$ as necessary, we can get uniform continuity of $\arg p(\zeta)$ on a closed disk $\overline{D(0,\varepsilon)}$. Let $\varepsilon > \delta > 0$ be such that for all $\zeta_1, \zeta_2 \in \overline{D(0,\varepsilon)}$, we have $|\zeta_1 - \zeta_2| < \delta$ implies $|\arg p(\zeta_1) - \arg p(\zeta_2)| < \frac{\pi}{3}$. Now pick $0 < a < \frac{\pi}{3m}$ and choose $\zeta_1 = \frac{\delta}{2}e^{ia}$, and $\zeta_2 = \frac{\delta}{2}e^{(\pi-a)i}$. Note that both $\zeta_1, \zeta_2 \in D^+(0,\varepsilon)$. Now consider a continuous curve λ with initial point ζ_1 and terminal point ζ_2. Now we compute

$$\begin{aligned}\int_\lambda d\arg(h_{z^*} - h_{z^*}(0)) =& \arg(\zeta^m p(\zeta))|_{\zeta_1}^{\zeta_2} \\ =& m(\arg\zeta_2 - \arg\zeta_1) + \arg p(\zeta_2) - \arg p(\zeta_1) \\ >& m(\pi - 2a) - \frac{\pi}{3} \\ >& (m-1)\pi \\ \geq& \pi.\end{aligned}$$

Here is where the assumption $m \geq 2$ is crucial. We conclude that we could not have had $h'_{z^*}(0) = 0$. By the complex version of the inverse function theorem [1], we may form the local inverse of h_{z^*}, which we write as $H_{z^*} : h_{z^*}(D(0,\varepsilon)) \to D(0,\varepsilon)$. This analytically maps a $\mathbb{C}$-neighbourhood of the point $h_{z^*}(0) \in S^1$ to $D(0,\varepsilon)$. Furthermore, for $w \in \mathbb{D} \cap h_{z^*}(D(0,\varepsilon))$, we have $H_{z^*}(w) = (f \circ \phi_{z^*})^{-1}(w)$. We will use this later.

Continuity on $\overline{U}$

Note that f is continuous on U by hypothesis. Let $z^* \in \partial U$ be given, which may or may not be a vertex. Let V be a neighbourhood of $f(z^*) = h_{z^*}(0)$ in $\mathbb{C}$. Let $\varepsilon > 0$ be such that $\phi_{z^*}(\overline{D^+(0,\varepsilon)}) \subset \overline{U}$. Note that ϕ_{z^*} is continuous and bijectve on $\overline{D^+(0,\varepsilon)}$ and has a continuous inverse Φ_{z^*} from $\phi_{z^*}(\overline{D^+(0,\varepsilon)})$ as a subset of $\overline{U}$ to $\overline{D^+(0,\varepsilon)}$. By continuity of h_{z^*}, there is a neighbourhood $\mathcal{O} \subset D(0,\varepsilon)$ of 0 such that $h_{z^*}(\mathcal{O}) \subset V$. Now pick a neighbourhood $\tilde{\mathcal{O}}$ of z^* in $\overline{U}$ such that $\Phi_{z^*}(\tilde{\mathcal{O}}) \subset \mathcal{O}$. Then, $f(\tilde{\mathcal{O}}) = (h_{z^*} \circ \Phi_{z^*})(\tilde{\mathcal{O}}) \subset h_{z^*}(\mathcal{O}) \subset V$. We see by definition, that f is continuous at z^*. Thus, since z^* was arbitrary, f is continuous on ∂U and is therefore continuous on all of $\overline{U}$.

Injectivity on $\overline{U}$

To see that we retain injectivity on the boundary of U, suppose for some z_1, z_2 we have that $f(z_1) = f(z_2) = w_0 \in s^1$. Now suppose that $z_1 \neq z_2$ and pick γ_1 to be a continuous, injective curve in $\overline{U}$, such that $\gamma_1(0,1) \subset U$ and γ_1 has initial point z_1 and terminal point z_2. Let γ_2 be chosen similarly, but with initial point z_2 and terminal point z_1. We may choose γ_1, γ_2 such that $\gamma_1(0,1) \cap \gamma_2(0,1) = \emptyset$. This is possible in any domain. Note that $\gamma = \gamma_1 * \gamma_2$ parameterizes the boundary of a connected set, $\text{int}(\gamma)$. Now by continuity, $f(\text{int}(\gamma))$ should be a connected set in $\overline{\mathbb{D}}$. However, by the injectivity and continuity of f, we have that $f(\gamma_1), f(\gamma_2)$ are closed loops in $\overline{\mathbb{D}}$ whose only point in common is w_0. Note that $\text{int}(f(\gamma_1)) \cup \text{int} f(\gamma_2) = \text{int}(f(\gamma))$ is then not a connected set. This is a contradiction and we conclude that $z_1 = z_2$.

Thus, f continuously and injectively maps $\overline{U}$ to $\overline{\mathbb{D}}$ and f takes ∂U to S^1. Furthermore $f(\overline{U})$ is a compact set in $\overline{\mathbb{D}}$ and contains all of $\mathbb{D}$ since we supposed that $f|_U$ was bijective. Thus, since each $w \in S^1$ is a limit point of $D(0,1)$ and therefore of $f(\overline{U})$, we must have $w \in f(\overline{U})$ since this is a closed set. Thus, the extension of f is continuous and bijective.

We give the following well-known argument that f is a homeomorphism of topological spaces. Firstly, we show every continuous, bijective, open map is a homeomorphism. Let X, Y be toplogical spaces and $\psi : X \to Y$ bijective, continuous and open. Let $\Psi : Y \to X$ be the inverse of ψ. Let A be open in X. Then $\Psi^{-1}(A) = \psi(A)$, which is open in Y by hypothesis. Thus Ψ is continuous and so ψ is a homeomorphism. Now we show that if X is compact and Y is Hausdorff and compact, bijectivity and continuity of $f : X \to Y$ implies that f is an open map. Let A be open in X. Then $X \setminus A$ is closed. Since X is compact, closed subsets are compact. Then $f(X \setminus A)$ is compact in Y by continuity. Bijectivity gives us the set equality $f(X \setminus A) = Y \setminus f(A)$. Since Y is Hausdorff and $Y \setminus f(A)$ is compact, it is also closed. Thus $f(A)$ is open. Thus f is bijective, continuous and open and so it is a homeomorphism. We apply this result to $f : \overline{U} \to \overline{D(0,1)}$, noting that both are compact and Hausdorff. There is therefore a continuous inverse $F : \overline{D(0,1)} \to \overline{U}$.

Recall the factorization $f(z) = (h_{z^*} \circ \Phi_{z^*})(z)$ for $z \in \overline{U}$ near $z^* \in \partial U$. Notice that when z^* is a non-vertex point, there is a neighbourhood V_{z^*} of z^* in $\mathbb{C}$ (in particular a neighbourhood that does not intersect any vertex of

U), such that the right hand side is analytic on V_{z^*} (since Φ_{z^*} is an affine map when z^* is not a vertex point). Since the right hand side corresponds with f in U and is analytic on V_{z^*}, f extends to be analytic on this neighbourhood. In particular, f extends to be analytic across the "open sides" of U, that is $\partial U \setminus \{z_1,...,z_n\}$. Using the chain rule and the fact that $h'_{z^*}(0) \neq 0$, note that $f' \neq 0$ across $\partial U \setminus \{z_1,...,z_n\}$. It follows that F extends analytically across $S^1 \setminus \{w_1,...,w_n\}$ and that $F' \neq 0$ on $\overline{\mathbb{D}} \setminus \{w_1,...,w_n\}$.

Deriving the formula for F

Recall that we constructed ∂U to be traversed in a counterclockwise fashion. Since we can now extend to the boundary, consider traversing the open segments $\gamma_{1,2}|_{(0,1)}, \gamma_{2,3}|_{(0,1)}...\gamma_{n,1}|_{(0,1)}$. Note that f extends to be conformal on these segments (and hence orientation preserving). Thus, $(f \circ \gamma_{j,j+1})'(t_0)$ points in the counterclockwise direction for any $t_0 \in (0,1)$ and any $1 \leq j \leq n$. Let θ_j be defined by $w_j = e^{i\theta_j}$ such that $\theta_1 < \theta_2 < ... < \theta_n < \theta_1 + 2\pi$. Here we use $\arg_{-\pi/2}$ to compute these phases.

Write $S_j := f(\gamma_{j,j+1}(0,1))$. From the above analysis, we see that this is the open sector of complex numbers $e^{i\theta}$, such that $\theta_j < \theta < \theta_{j+1}$ for $j = 1, 2, .., n-1$ and when $j = n$, S_n is the sector of $e^{i\theta}$ with $\theta_n - 2\pi < \theta < \theta_1$. When we parameterize S_j as $z(\theta) = e^{i\theta}$ for $\theta_j < \theta < \theta_{j+1}$, then $\frac{d}{d\theta}F(e^{i\theta})$ should be a scalar multiple of $z_{j+1} - z_j$ for any $\theta_j < \theta < \theta_{j+1}$.

We can factor F in $\mathbb{D} \cap h_{z_j}(D(0,\varepsilon))$ as $F = \phi_{z_j} \circ H_{z_j}$ and note that H_{z_j} is analytic in a $\mathbb{C}$ neighbourhood of w_j. We have

$$\frac{d}{d\theta}H_{z_j}(e^{i\theta}) = ie^{i\theta}H'_{z_j}(e^{i\theta}),$$

Evaluating at $e^{i\theta_j}$ we obtain $iw_jH'_{z_j}(w_j)$.

Note that H_{z_j} is analytic at w_j, so the above computation is valid. Furthermore, F maps $S_j \cap h_{z_j}(D(0,\varepsilon))$ to a small portion of the segment from z_j to z_{j+1} in ∂U. Then pulling back through $\phi_{z_j}^{-1}$, which takes this segment to the positive real axis in the ζ-plane, we see that H_{z_j} must map S_j to the positive real axis and in particular $H_{z_j}(w_j) = 0$. Using the given orientations of ∂U and S^1 together with this observation, we see that $\frac{d}{d\theta}H_{z_j}(e^{i\theta})|_{\theta_j}$ is a positive real number, say c_j.

Now, consider $w \in h_{z_j}(D(0,\varepsilon)) \setminus \{w_j\}$. We have the factorization

$$H_{z_j}(w) = c_j \frac{w - w_j}{iw_j} \frac{H_{z_j}(w)}{H'_{z_j}(w_j)(w - w_j)}.$$

Now for $\varepsilon > 0$ small enough, by the definition of the derivative, we can make $\frac{H_{z_j}(w)}{H'_{z_j}(w_j)(w-w_j)}$ sufficiently close to 1 for all $w \in h_{z_j}(D(0,\varepsilon)) \setminus \{w_j\}$. In particular, it is away from the negative imaginary axis, and if we define

$$G_j(w) = \begin{cases} \frac{H_{z_j}(w) - H_{z_j}(w_j)}{H'_{z_j}(w_j)(w-w_j)} & w \neq w_j \\ 1 \quad w = w_j \end{cases},$$

for $w \in h_{z_j}(D(0,\varepsilon))$, then we see that G_j is non-vanishing, analytic, and has range that misses the negative imaginary axis. Furthermore, for $w \in \mathbb{D}$, the factor $\frac{w-w_j}{iw_j}$ has

$$\mathrm{Im}(\frac{w - w_j}{iw_j}) = \mathrm{Im}(\frac{w}{iw_j}) + 1 \geq 1 - |\frac{w}{w_j}| > 0.$$

Note that this factor is on the negative imaginary axis if and only if w is a real scalar multiple of w_j with scale factor greater than 1, but this is a ray outside of $\mathbb{D}$. Recall that we are using the branch of $\zeta \mapsto \zeta^{\alpha_j}$ with branch cut on the negative imaginary axis and that is positive and real on the positive real axis. For any $w \in h_{z_j}(D(0,\varepsilon)) \cap \mathbb{D}$,

$$H_{z_j}(w)^{\alpha_j} = c_j^{\alpha_j}(\frac{w - w_j}{iw_j})^{\alpha_j} G_j^{\alpha_j}(w),$$

but now recall that $\zeta \mapsto (z_{j+1} - z_j)\zeta^{\alpha_j} + z_j$ defines the map ϕ_{z_j} and we have factored F as $\phi_{z_j} \circ H_{z_j}$ in $\mathbb{D} \cap h_{z_j}(D(0,\varepsilon))$ so

$$F(w) = c_j^{\alpha_j}(\frac{w - w_j}{iw_j})^{\alpha_j} G_j^{\alpha_j}(w)(z_{j+1} - z_j) + z_j = (\frac{w - w_j}{iw_j})^{\alpha_j} \tilde{G}_j(w) + z_j.$$

The symbol $\tilde{G}_j$ is just a new name that we have given to $c_j^{\alpha_j} G_j^{\alpha_j}(w)(z_{j+1} - z_j)$, which is analytic and non-zero. The latter claim is true since $G_j^{\alpha_j}(w)$ tends to be positive and real for w close to w_j, and $c_j^{\alpha_j}$ is positive and real. Thus, this term tends to a real scalar multiple of $z_{j+1} - z_j \neq 0$ as $w \to w_j$. Now, computing F' with the product rule, we have

$$F'(w) = \alpha_j(\frac{w - w_j}{iw_j})^{\alpha_j - 1}\tilde{G}_j(w) + (\frac{w - w_j}{iw_j})^{\alpha_j}\tilde{G}_j{}'(w) = (\frac{w - w_j}{iw_j})^{\alpha_j - 1}(\alpha_j\tilde{G}_j(w) + (\frac{w - w_j}{iw_j})\tilde{G}_j{}'(w)) = (\frac{w - w_j}{iw_j})^{\alpha_j - 1}\hat{G}_j(w). \tag{11}$$

Here $\hat{G}_j(w)$ is the quantity in the parentheses. Note that as $w \to w_j$, the term $(\frac{w-w_j}{iw_j})\tilde{G}_j{}'(w) \to 0$, since $\tilde{G}_j$ is analytic and so is $\tilde{G}_j{}'$, so overall $\alpha_j\tilde{G}_j(w) + (\frac{w-w_j}{iw_j})\tilde{G}_j{}'(w) \to \alpha_j\tilde{G}_j(w_j)$ which is non-zero. We see that $\hat{G}_j$ is non-zero and analytic in a small ball around w_j. Thus there is a small ball $O_j \subset \mathbb{C}$ containing w_j, and an analytic function $\hat{q}_j : O_j \to \mathbb{C}$, such that $\hat{G}_j = \exp(\hat{q}_j)$. This result is from exercise 8.8.5 in [1]. Similarly, since F' is non-zero on $\mathbb{D}$ and does not vanish across the arcs from (θ_j, θ_{j+1}) there is an analytic $J : \mathbb{D} \to \mathbb{C}$ with $F' = e^J$, and J extends to be analytic across the arcs S_j for all j by the same analysis that was used to show that F extends across these arcs.

For $w \in S^1$, we write $w = e^{i\theta}$, and we would like to consider the map on $S^1 \setminus \{w_1, w_2, ..., w_n\}$ defined by $e^{i\theta} \mapsto \theta + \mathrm{Im}(J(e^{i\theta}))$. Then, since

$$\begin{aligned}\arg(z_{j+1} - z_j) = \arg(ie^{i\theta}F'(e^{i\theta})) &= \frac{\pi}{2} + \theta + \arg F'(e^{i\theta}) \\ &= \frac{\pi}{2} + \theta + \arg\exp\left(\mathrm{Re}(J(e^{i\theta})) + i\mathrm{Im}(J(e^{i\theta}))\right) \\ &= \frac{\pi}{2} + \theta + \mathrm{Im}(J(e^{i\theta})),\end{aligned} \tag{12}$$

we see that on each S_j, this map takes the constant value $\arg(z_{j+1} - z_j) - \frac{\pi}{2}$ dependent only on j. There is a jump as we pass each θ_j, where the map is undefined. Consider the increase in the value of this map from the sector S_{j-1} to the sector S_j. The jump can be computed geometrically, or from (12) to be $\arg(z_{j+1} - z_j) - \arg(z_j - z_{j-1}) = \pi(1 - \alpha_j)$. Note that this quantity is the exterior angle of the polygon at z_j. Now consider the value of this map for some $e^{i\theta}$ in (θ_1, θ_2), and then consider the increase in that value if we increase θ by 2π. Since $\mathrm{Im}(J(e^{i\theta})$ is 2π periodic, the increase in $\theta + \mathrm{Im}(J(e^{i\theta}))$ is 2π. This increase can also be written as the summation over all the individual increases at the jump discontinuities $\theta = \theta_2, \theta_3, ..., \theta_n, \theta_1$, and we conclude

$$\sum_{j=1}^{n} \pi(1 - \alpha_j) = 2\pi.$$

Returning to (11), we use the branch of log with branch cut on the negative imaginary axis to write

$$\log_{-\pi/2} F'(w) = J(w) = (\alpha_j - 1)\log_{-\pi/2}(\frac{w - w_j}{iw_j}) + \hat{q}_j(w) \tag{13}$$

for w near w_j in $\mathbb{D}$. We would like to show that in particular,

$$J(w) = \sum_{k=1}^{n}(\alpha_k - 1)\log_{-\pi/2}(\frac{w - w_k}{iw_k}) + K_1.$$

Where K_1 is some constant. Note that the expression on the right can be analytically continued across each arc S_k, since $\frac{w-w_k}{iw_k}$ has negative imaginary part only for $w \in \{re^{i\theta_k} : r > 1\}$, but these are disjoint from each S_k by definition. We introduce a new function $\tilde{J} : \mathbb{D} \to \mathbb{C}$ defined by

$$\tilde{J}(w) = J(w) + \sum_{k=1}^{n}(1 - \alpha_k)\log_{-\pi/2}(\frac{w - w_k}{iw_k}).$$

$\tilde{J}$ is the sum of analytic functions on $\mathbb{D}$. Consider the behaviour of $\tilde{J}$ near w_j in $\mathbb{D}$ for some $j = 1, 2, ..., n$. First write J in its local form using equation (13). The j^{th} term in the sum, which would tend to be problematic near w_j, is cancelled out and we are left with $\hat{q}_j(w)+\sum_{k=1,k\neq j}^{n}\log_{-\pi/2}(\frac{w-w_k}{iw_k})$, which is analytic in a $\mathbb{C}$ neighbourhood of w_j, since the w_k are distinct. Thus, $\tilde{J}$ extends to be analytic on $\overline{\mathbb{D}}$ so $\text{Im}\tilde{J}$ is harmonic on $\overline{\mathbb{D}}$. Let $j=1,2,...,n$ be arbitrary and let $w\in S_j$. Write $w=e^{i\theta}$. Then,

$$\text{Im}(\tilde{J}(e^{i\theta})=\text{Im}(J(e^{i\theta}))+\sum_{k=1}^{n}(1-\alpha_k)\arg_{-\pi/2}(\frac{e^{i\theta}-w_k}{iw_k})$$

with the same branch cut of arg from before. We have

$$e^{i\theta}-e^{i\theta_k}=e^{\frac{\theta+\theta_k}{2}}(e^{\frac{\theta-\theta_k}{2}}-e^{\frac{\theta_k-\theta}{2}})=e^{\frac{\theta+\theta_k}{2}}2i\sin(\frac{\theta-\theta_k}{2}).$$

Thus, computing the quantity, $\arg_{-\pi/2}(\frac{e^{i\theta}-w_k}{iw_k})-\frac{\theta}{2}$ we have

$$\arg_{-\pi/2}(\frac{e^{i\theta}-w_k}{iw_k})-\frac{\theta}{2}=\arg_{-\pi/2}(\frac{2ie^{i\frac{\theta+\theta_k}{2}}\sin(\frac{\theta-\theta_k}{2})}{ie^{i\theta_k}})-\frac{\theta}{2}=\frac{\theta_k}{2}$$

But then,

$$\arg_{-\pi/2}(\frac{e^{i\theta}-w_k}{iw_k})=\frac{\theta+\theta_k}{2},$$

and a basic substitution gives,

$$\text{Im}(\tilde{J}(e^{i\theta})=\text{Im}(J(e^{i\theta}))+\sum_{k=1}^{n}(1-\alpha_k)\frac{\theta+\theta_k}{2}=\text{Im}(J(e^{i\theta}))+\frac{\theta}{2}\sum_{i=1}^{n}(1-\alpha_k)+C=\text{Im}(J(e^{i\theta}))+\theta+C,$$

where C is the constant $\sum\frac{\theta_k(1-\alpha_k)}{2}$. Note that the right hand side is a constant for w on S_j. Thus, since $\text{Im}\tilde{J}$ is continuous on $\overline{\mathbb{D}}$ and constant on each open sector S_j, it is constant on S^1. But then by the maximum principle, $\text{Im}\tilde{J}$ is constant on all of $\overline{\mathbb{D}}$. Thus, $\text{Re}\tilde{J}$ is also constant on all of $\overline{\mathbb{D}}$ and so $\tilde{J}$ is constant as well. We conclude that for some constant K_1 we have that for all $w\in\mathbb{D}$,

$$J(w)=K_1+\sum_{k=1}^{n}(\alpha_k-1)\log_{-\pi/2}(\frac{w-w_k}{iw_k}).$$

Then,

$$e^{J(w)}=F'(w)=K_1\prod_{k=1}^{n}(\frac{w-w_k}{iw_k})^{\alpha_k-1}.$$

Here we make e^{K_1} our new K_1, and by the fundamental theorem of calculus, up to a constant, $K_2:=F(0)$, we have

$$F(w)=K_1\int_0^w\prod_{k=1}^{n}(\frac{u-w_k}{iw_k})^{\alpha_k-1}du+K_2.$$

A formula for $F(w)$ such as this, is called a Schwarz-Christoffel formula. It is extremely useful in a variety of applications. Many times it is impossible to solve in closed form. Using computers we can approximate solutions to the differential equation that is posed by asking for the integral involved in computing $F(w)$. There is a collection of MATLAB functions available online called SC-toolbox developed by Tobin Driscoll. SC-toolbox has a variety of useful visualization and computational options. In particular it allows one to display the images of orthogonal gridlines in the target domain, which gives one a sense of what the map is actually doing. [2]

Infinite polygonal regions We give an overview of how the formula for $F(w)$ looks similar even if we consider a polygonal region that is unbounded. Let $\mathbb{C}\cup\{\infty\}$ be the extended complex plane with the topology given by the one point compactification of $\mathbb{C}$. Let $z_i,z_j\in\mathbb{C}$ be distinct and consider the rays $R_i=\{z_i+re^{i\theta_i}:r\geq 0\}$, $R_j=\{z_j+re^{i\theta_j}:r\geq 0\}$. We can parameterize say R_i as the image of a straight line path $\gamma_{\infty,i}$, (meaning it is a continuous map from $[0,1]$ to $\mathbb{C}\cup\{\infty\}$) with initial point ∞ and terminal point z_i and R_j similarly as the image of $\gamma_{j,\infty}$, but with initial point z_j and terminal point ∞. Let $\gamma_{i,j}$ denote the straight line path from z_i to z_j. Suppose that we have chosen z_i,z_j such that $\gamma=\gamma_{\infty,i}*\gamma_{i,j}*\gamma_{j,\infty}$ has the

[2] Just google 'SC-toolbox' and it's the first result

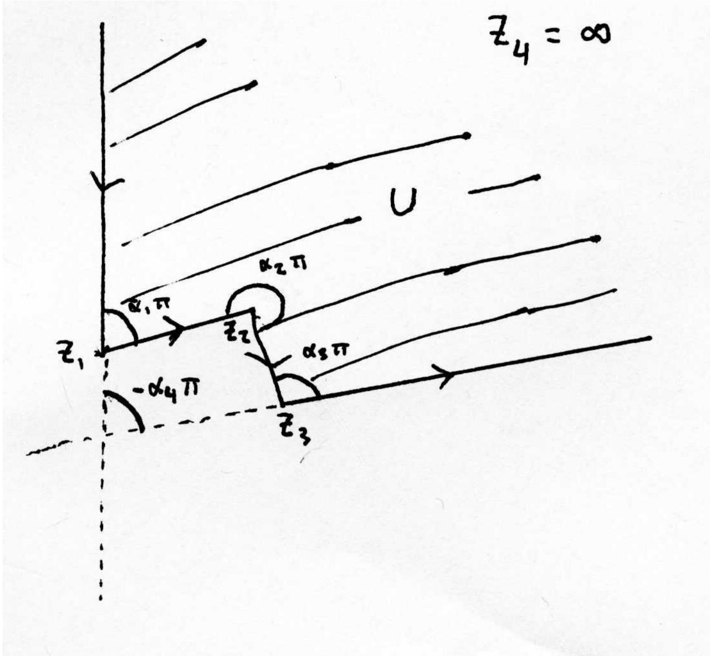

Figure 3: Infinite Polygonal Region

positive orientation. Rather than a straight line path connecting z_i, z_j, we can have any simple concatenation of straight line paths, such that this path also does not intersect R_i, R_j. We let U be the simply connected domain that lies to the left as we traverse γ in the positive direction. U is called an unbounded polygonal domain. Its vertices are $z_i, z_1, ..., z_n, z_j, \infty$. For convenience, we re-index and make z_i our new z_1, and so on, until $z_n = z_{m-2}, z_j = z_{m-1}, \infty = z_m$ totalling $m = n+3$ vertices.

By the Riemann Mapping Theorem, there is again a conformal map $f : U \to \mathbb{D}$. We claim that its conformal inverse $F : \mathbb{D} \to U$ also follows the SC-formula when interpreted in the right sense. We will outline the steps needed to verify this and where it differs from the proof for bounded polygons. Our claim is that when U is given the subspace topology from $\mathbb{C} \cup \{\infty\}$, f extends to be a homeomorphism from $\overline{U}$ to $\overline{\mathbb{D}}$. Furthermore, there are phases $w_1, w_2, ..., w_m \in S^1$ such that $f(z_j) = w_j$ for all j, complex constants K_1, K_2, and for all $j \neq m$, there are real numbers $0 < \alpha_j \leq 2$ and some $-2 \leq \alpha_m < 0$ such that for all $w \in \mathbb{D}$, we have

$$F(w) = K_1 \int_0^w \prod_{k=1}^{n} (\frac{u - w_k}{i w_k})^{\alpha_k - 1} du + K_2.$$

The argument to prove this is nearly identical to the one given for the bounded case, we just need to give the correct notion of angle for the vertex at infinity. We let $-\alpha_m \pi$ be the angle of intersection of the lines obtained by extending R_i, R_j. Without loss of generality let z_∞ be the point of intersection. In the case that the extended lines are parallel, we define the angle of their intersection to be $-\pi$. Let $R > 0$ be large enough that $D(z_\infty, R)$ contains the non-infinite vertices of U. We can parameterize the "corner" at infinity with a conformal map $\phi_\infty : D^+(0,1) \to U$ defined by $\zeta \mapsto Re^{i\theta_j} \zeta^{\alpha_m} + z_\infty$. We again can show that as $\text{Im}(\zeta) \to 0$, we have $\log(f \circ \phi_\infty) \to 0$, so by the Schwarz Reflection Principle, this extends to the whole $D(0,1)$. Taking exponentials, we let h_∞ be the analytic extension of $f \circ \phi$ to $D(0,1)$. Again f extends to be analytic across $\partial U \setminus \{z_1, z_2, ..., z_m\}$. We then must show that f extends to be a homeomorphism from $\overline{U}$ to $\overline{D}$ and form its inverse F. The argument here would be the same.

If necessary, we shrink $D(0,1)$ in the ζ plane to $D(0,\varepsilon)$ and show that $h'_\infty \neq 0$ on $D(0,\varepsilon)$. We again form the analytic inverse H_∞, which maps from a small $\mathbb{C}$-neighbourhood of some point $w_m \in S^1$ to $D(0,\varepsilon)$ in the ζ plane. Thus, we may factor $F = \phi_\infty \circ H_\infty$ and we proceed with the same analysis as before to write $F'(w) = (\frac{w - w_m}{i w_m})^{\alpha_m - 1} \hat{G}_m(w)$ for $w \in h_\infty(D(0,\varepsilon) \cap \mathbb{D}$ and some non-zero analytic $\hat{G}_m : h_\infty(D(0,\varepsilon)) \to \mathbb{C}$. The only difference comes when we tracked $\theta + \arg(F'(e^{i\theta})$ around $S^1 \setminus \{w_1, ..., w_m\}$. The jump turned out to be the exterior angle $\pi(1 - \alpha_j)$, at the corner z_j. Defining the exterior angle at ∞ to be this jump, we see that it is $\pi(1 + \alpha_m)$ which allows us to see that the summation $\pi((1-\alpha_1) + (1-\alpha_2)... + (1+\alpha_m)) = 2\pi$ still holds. This is the reason for taking $\alpha_m < 0$. The formula then follows using the same logic and computations from before.

Frequently, we want to use the upper half plane $\mathbb{H} := \{\zeta \in \mathbb{C} : \operatorname{Im}(\zeta) > 0\}$ as our target domain. Note that the function $C(\zeta) = \frac{\zeta - i}{\zeta + i}$, known as the *Cayley Transformation*, maps conformally the upper half plane onto the unit disk. Furthermore, C bijectively maps the extended real line onto the unit circle, where the point at ∞ goes to 1. Suppose that $F : \mathbb{D} \to U$ is a polygon with vertices $z_1, z_2, ..., z_n$ as we have described, and $w_1, w_2, .., w_n$ the 'pre-vertices' as in the SC-formula. Let $\zeta_i = C^{-1}(w_i)$. If we put $w = C(\zeta)$ in the SC-formula, we obtain a conformal map $g : \mathbb{H} \to U$, that after some simplification looks like

$$g(w) = K_1 \int_i^{\zeta} \prod_{k=1}^{n} (u - \zeta_k)^{\alpha_k - 1} du.$$

This integral is sometimes more simple to solve than the corresponding one for the unit disk.

8 Möbius Transformations and the Uniqueness of SC-maps

Recall that we supposed the existence of *some* map $f : U \to \mathbb{D}$, but we should say something about the uniqueness of this map. To do so, we have to understand a particularly nice family of conformal maps known as fractional linear transformations or Möbius transformations. Let $a, b, c, d \in \mathbb{C}$ and define

$$f(z) = \frac{az + b}{cz + d} \tag{14}$$

for all $z \in \mathbb{C} \setminus \{-\frac{d}{c}\}$. We can regard f as a map from $\mathbb{C} \cup \{\infty\}$, i.e the extended complex plane, to $\mathbb{C} \cup \{\infty\}$ if we put $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c}$. f has a well-defined inverse given by $z = F(w) = \frac{b - wd}{cw - a}$ for $w \neq \frac{a}{c}$, $z = \infty$ for $w = \frac{a}{c}$, and $z = -\frac{d}{c}$ for $w = \infty$. In the case that $ad - bc = 0$, things sort of fall apart since then $f(0) = \frac{b}{d} = \frac{a}{c} = f(\infty)$. We obviously can not then have a well-defined inverse. In fact, the quotient rule shows that $f'(z) = 0$ for all $z \in \mathbb{C}$ if and only if $ad - bc = 0$. Thus, such an f would be the constant map $f(z) = \frac{b}{d}$. When this is not the case, our inverse is well defined and it follows that f is an automorphism of $\mathbb{C} \cup \{\infty\}$. Note that f is clearly analytic on $\mathbb{C} \setminus \{-\frac{d}{c}\}$. We call an f as in (13) with $ad - bc \neq 0$ a Möbius transformation. These conformal maps have an intrinsic relationship with the following.

Definition 8.1. *The cross ratio is the quantity defined by*

$$\{z_1, z_2, z_3, z_4\} = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

for some distinct complex numbers z_1, z_2, z_3, z_4.

It is a relatively non-trivial computation to show that for z_1, z_2, z_3, z_4 distinct complex numbers and f a Möbius transformation, we have

$$\{z_1, z_2, z_3, z_4\} = \{f(z_1), f(z_2), f(z_3).f(z_4)\}. \tag{15}$$

Now supposing that we have shown (14) and also that for some $z_1, z_2, z_3 \in \mathbb{C}$ distinct, we have $f(z_1) = w_1, f(z_2) = w_2$ and $f(z_3) = w_3$ for some $w_1, w_2, w_3 \in \mathbb{C}$ distinct. Using (14) written in the form $\{w, w_1, w_2, w_3\} = \{z, z_1, z_2, z_3\}$, we can solve for w to see that $w = f(z)$ defines a Möbius transformation $f(z) = \frac{az+b}{cz+d}$ with

$$\begin{aligned} a &= w_2(w_1 - w_3)(z_1 - z_2) - w_3(z_1 - z_3)(w_1 - w_2), \\ b &= w_3 z_2(z_1 - z_3)(w_1 - w_2) - z_3 w_2(w_1 - w_3)(z_1 - z_2), \\ c &= (w_1 - w_3)(z_1 - z_2) - (w_1 - w_2)(z_1 - z_3), \\ d &= z_2(z_1 - z_3)(w_1 - w_2) - z_3(w_1 - w_3)(z_1 - z_2). \end{aligned} \tag{16}$$

Thus, if f is a Möbius transformation mapping three distinct points z_1, z_2, z_3 to distinct images w_1, w_2, w_3, we have that f is uniquely defined. Note that since f is a conformal mapping, if we consider the triangle defined by the vertices z_1, z_2, z_3 in the z-plane, we must be able to traverse its boundary in the same order as we can traverse the triangle defined by the vertices w_1, w_2, w_3 in the w-plane. In other words, the z_i have the same orientation as the w_i.

A result of [1] shows that every analytic automorphism $g : \mathbb{D} \to \mathbb{D}$ can be written in the form

$$g(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

for some $a \in \mathbb{D}$ and some $\theta \in \mathbb{R}$. Since $a \in \mathbb{D}$, a g of this form extends to be analytic on $D(0, |\frac{1}{a}|)$ which contains $\partial\mathbb{D}$. We conclude that g extends to be analytic across $\partial\mathbb{D}$ and a simple computation shows that $g|_{\partial\mathbb{D}}$ has range contained in $\partial\mathbb{D}$. Writing $g(z) = \frac{e^{i\theta}z - e^{i\theta}a}{-\bar{a}z+1}$, we see that g has the form of (14), and computing $ad - bc = e^{i\theta} - |a|e^{i\theta} = e^{i\theta}(1-|a|)$, we have $|ad - bc| = |1 - |a|| \geq 1 - |a| > 0$, since $a \in \mathbb{D}$. Thus, every analytic automorphism of the unit disk is a Möbius transformation. As a result of the previous discussion, an analytic automorphism of the open unit disk (which must extend to be analytic across the unit circle) is unique up to a choice of distinct $z_1, z_2, z_3 \in \partial\mathbb{D}$ and their images $w_1, w_2, w_3 \in \partial\mathbb{D}$ with a compatible orientation.

With some results about Möbius transformations in our toolbox, we return to the discussion of SC-maps. Suppose that there are $h, f : U \to D$, which are SC-maps as discussed in the previous section. Let z_i again denote the i^{th} vertex in U. Now suppose that h, f are such that for some distinct $i, j, k \in \{1, 2, .., n\}$, we have that $f(z_i) = h(z_i) = w_i$, $f(z_j) = h(z_j) = w_j$, and $f(z_k) = h(z_k) = w_k$, for some w_i, w_j, w_k with the same orientation as z_i, z_j, z_k. Let $F : \mathbb{D} \to U$ be the analytic inverse of f. Thus, $h \circ F$ is an analytic automorphism of the unit disk that fixes the points $w_i, w_j, w_k \in \partial\mathbb{D}$. Any Möbius transformation that fixes 3 points is the identity map. Thus, $f = H^{-1} = h$.

In a case where $n > 3$, once we specify three vertices z_i, z_j, z_k and their orientation-compatible images w_i, w_j, w_k, the remaining images $f(z_l) = w_l$, $l \neq i, j, k$ are then fixed and must be determined. This is done using the data of the chosen pairs $(z_i, w_i), (z_j, w_j), (z_k, w_k)$ and the polygon itself. This process is known as the 'parameter problem'.

Let U be a simple polygon and $f : U \to \mathbb{D}$ an SC-map. Recall that we can write the analytic inverse $F : \mathbb{D} \to U$ in the form

$$F(w) = K_1 \int_0^w \prod_{k=1}^{n} \left(\frac{u - w_k}{i w_k}\right)^{\alpha_k - 1} du + K_2 \tag{17}$$

Note that $K_2 = F(0)$. Applying the fundamental theorem of calculus and setting $w = 0$, we see that $F'(0) = K_1$. We claim that specifying K_1, K_2 fixes the w_i. Suppose that we have two SC-maps $f, h : U \to \mathbb{D}$ with $f(z_0) = h(z_0) = 0$ for some $z_0 \in U$ and that we also have $f'(z_0) = h'(z_0)$. Let F be the analytic inverse of f. Then $h \circ F$ is an analytic automorphism of the unit disk that fixes the origin. $h \circ F$ is therefore a rotation by a result from [1]. $(h \circ F)'(0) = h'(F(0))F'(0) = h'(z_0)\frac{1}{f'(z_0)} = 1$ is a unimodular constant whose argument gives the amount of rotation. Thus $h \circ F$ is the identity and we conclude that $f = h$. Thus, we can specify K_1, K_2 to uniquely determine a conformal map $f : U \to \mathbb{D}$.

9 Conformal Mapping and Fluid Flow: Some simple examples

Let $D \subset \mathbb{R}^2$ be a domain whose boundary is a piecewise smooth curve. Let $F : D \to \mathbb{R}^2$ be C^1 with $F = (F_1, F_2)$ and such that

$$\text{curl}(F) := \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 \tag{18}$$

and

$$\text{Div}(F) := \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0. \tag{19}$$

Furthermore, suppose $z_0 \in \partial D$ and that ∂D is parameterized near z by the C^1 function $\gamma : [0,1] \to \partial D$ with $\gamma(t_0) = z_0$ for some $t_0 \in [0,1]$. Let $\gamma^{\perp}(t_0)$ be the normal vector to ∂D at z_0. We suppose that $\lim_{z \to z_0} F(z)$ exists and that

$$\lim_{z \to z_0} F(z) \cdot \gamma^{\perp}(t_0) = 0 \tag{20}$$

Such an F is called an *ideal fluid flow*. For us F represents the velocity of a particle of fluid at a given point in D. (18) is taken to mean that the fluid is *irrotational*, in that the particles do not tend to form vortices (little whirlpools) around any points in D. (19) is taken to mean that the fluid is *incompressible*, in that the effects of pressure on the fluid are negligible and so the density does not change. Water is an example of an incompressible fluid, and in some flow conditions it is also approximately irrotational. (20) is taken to mean that F tends to be tangent to the boundary of its domain. We may equivalently consider F as the complex valued function $F_1 + iF_2$. When we do this, we compute the 'dot product' of two complex numbers $z = x + iy$ and $w = u + iv$ as $z \cdot w = xu + yv$. Note that this is not the usual inner product on $\mathbb{C}$, which is $\langle z, w \rangle = z\overline{w}$.

Recall from multivariable calculus that every conservative (satisfying $\text{curl}(F) = 0$) vector field is a gradient field. Thus, there exists a C^2 function $\phi : D \to \mathbb{R}$ with $\nabla\phi = F$. In the context of fluid flow, such a ϕ is called a *velocity potential*. Note that from (19) we have that $\mathrm{Div}(F) = \mathrm{Div}(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ and so ϕ is harmonic. In the case that D is simply connected, ϕ admits a harmonic conjugate ψ such that the function $\mathcal{F} : D \to \mathbb{C}$ with $\mathcal{F} = \phi + i\psi$ is analytic. Using the Cauchy-Riemann equations as well as the condition that $\nabla\phi = F$, we have $\overline{\mathcal{F}'} = F_1 + iF_2$.

Alternatively, we may start with an analytic $\mathcal{F}$ such that $\overline{\mathcal{F}'}$ satisfies (20) on some domain D with piecewise smooth boundary and recover an ideal fluid flow $F := \overline{\mathcal{F}'}$. The function $\mathcal{F}$ is called the *complex potential* for the ideal fluid flow in D.

Conformal mapping ends up being an indispensable tool in finding ideal fluid flows for various domains. suppose that U is a domain with a piecewise smooth boundary and $g : U \to D$ is a conformal mapping such that g extends to be conformal across ∂U and that furthermore $g(\partial U) = \partial D$. We claim that $G := \overline{(\mathcal{F} \circ g)'} = F(g(z))\overline{g'(z)}$ is an ideal fluid flow in U. Note that $(\mathcal{F} \circ g)'$ is analytic on U. Thus, this function satisfies the Cauchy-Riemman equations and we have that $\overline{(\mathcal{F} \circ g)'}$ satisfies (18) and (19). The conformality of g across ∂U tells us that G satisfies (20). We know that if $n_0^\perp$ is a complex number that is normal (in the sense of dot product as discussed above) to ∂D at w_0, then since F is an ideal fluid flow in D,

$$\lim_{w \to w_0} F(w) \cdot n_0^\perp = 0.$$

However, let $w = g(z)$ and note that $n_0^\perp = \lambda m_0^\perp g'(z_0)$, by the chain rule, where $m_0^\perp$ is normal to ∂U at z_0 and $\lambda > 0$ is some scalar. Thus,

$$\begin{aligned} 0 &= \lim_{w \to w_0} F(w) \cdot n_0^\perp \\ &= \lim_{z \to z_0} F(g(z)) \cdot n_0^\perp \\ &= \lim_{z \to z_0} F(g(z)) \cdot (\lambda m_0^\perp g'(z_0)) \\ &= \lambda \lim_{z \to z_0} F(g(z))\overline{g'(z_0)} \cdot m_0^\perp \\ \implies 0 &= \lim_{z \to z_0} G(z) \cdot m_0^\perp. \end{aligned} \tag{21}$$

This is equivalent to (20) for G. We have thus verified that G is an ideal fluid flow in U.

Example: Note that the function $g(z) = z + \frac{1}{z}$ is conformal everywhere except $z = \pm 1, 0$. If we take U to be the exterior of the unit disk note that g conformally maps U to $\mathbb{C}$ with the solid segment from -2 to 2 removed. The ideal fluid flow in this target domain is $F = A$, for some $A > 0$. The complex potential for F is $\mathcal{F}(w) = Aw$. Thus, the complex potential for the ideal fluid flow in the z-plane around the unit disk is $\mathcal{G}(z) = (\mathcal{F} \circ g)(z) = A(z + \frac{1}{z})$. We see that

$$G(x+iy) = A\frac{(x^2+y^2)^2 - (x^2-y^2)}{(x^2+y^2)^2} - iA\frac{2xy}{(x^2+y^2)^2}.$$

In general, finding the flow around an object whose boundary is a simple closed curve is an example of what is known as the 'exterior mapping problem'. Suppose that U is a simply connected domain whose boundary is a smooth jordan curve, parameterized by γ and without loss of generality $0 \in U$. Note that since $z \mapsto \frac{1}{z}$ is injective on ∂U, the map $\hat{\gamma(t)} = \frac{1}{\gamma(t)}$ is a simple closed curve, parameterizing the boundary of a simply connected domain $\hat{U}$. Note also that under $z \mapsto \frac{1}{z}$, U is mapped conformally to $\mathrm{ext}(\hat{U})$ in the extended complex plane and similarly $\mathrm{ext}(U)$ is mapped to $\hat{U}$. Now suppose that we know $\hat{f} : \hat{U} \to \mathbb{D}$ is a conformal map. Then, $f(z) = \frac{1}{\hat{f}(\frac{1}{z})}$ is a conformal map from $\mathrm{ext}(U)$ to $\mathrm{ext}(\mathbb{D})$, and the complex potential for the ideal fluid flow in $\mathrm{ext}(U)$ is $\mathcal{G} \circ f$, where $\mathcal{G}$ is defined as in the previous paragraph.

We present an example where SC-maps help us find ideal fluid flows in relatively easy domains.

The following example comes from exercise 9 in 11.3 of [1]. Let $D = \{z \in \mathbb{C} : \mathrm{Im}(z) > a > 0\} \cup \{z \in \mathbb{C} : \mathrm{Im}(z) > 0, \mathrm{Re}(z) > 0\}$ be the 'step' domain. We find a conformal map G of the upper half plane $\mathbb{H}$ onto D. We think of D as an infinite polygonal region, with vertices $z_1 = ia, z_2 = 0$ and $z_3 = \infty$. We may fix the map G by picking the 'pre-vertices' $w_i \in \mathbb{R}$ to be $w_1 = G(z_1) = -1$, $w_2 = G(z_2) = 0$ and $w_3 = G(z_3) = \infty$. The angles at the vertices are $\alpha_1\pi = \frac{3\pi}{2}$, $\alpha_2\pi = \frac{\pi}{2}$. Note that the factor corresponding to the vertex at ∞ disappears since ∞ is mapped to ∞. First note that for $w \in \mathbb{H}$, we have

$$\frac{d}{dw}\left(\frac{1}{2}\mathrm{Log}_{-\pi/2}\left(\frac{(w+1)^{1/2} + w^{1/2}}{(w+1)^{1/2} - w^{1/2}}\right) + w^{1/2}(w+1)^{1/2}\right) = \frac{(w+1)^{1/2}}{w^{1/2}},$$

where the roots are computed using the branch of the square root function with branch cut on the negative imaginary axis and that takes positive, real values on the positive real axis.

Now, the SC-formula then tells us that for all $w \in \mathbb{H}$, we have for some $A, B \in \mathbb{C}$.

$$\begin{aligned} z = G(w) &= A\int_i^w \frac{(u+1)^{1/2}}{u^{1/2}}du + B \\ &= A\left(\frac{1}{2}\left(\text{Log}_{-\pi/2}\left(\frac{(w+1)^{1/2}+w^{1/2}}{(w+1)^{1/2}-w^{1/2}}\right) - \text{Log}_{-\pi/2}\left(\frac{(i+1)^{1/2}+i^{1/2}}{(i+1)^{1/2}-i^{1/2}}\right)\right) + w^{1/2}(w+1)^{1/2} - i^{1/2}(i+1)^{1/2}\right) + B. \end{aligned} \tag{22}$$

From the condition that $G(0) = 0$, we find that

$$B = A\left(\frac{1}{2}\text{Log}_{-\pi/2}\left(\frac{(i+1)^{1/2}+i^{1/2}}{(i+1)^{1/2}-i^{1/2}}\right) + i^{1/2}(i+1)^{1/2}\right).$$

Now we use the condition that $G(-1) = ia$ to see that

$$A = \frac{2ia}{\text{Log}_{-\pi/2}(-1)},$$

and hence

$$B = \frac{2ia}{\text{Log}_{-\pi/2}(-1)}\left(\frac{1}{2}\text{Log}_{-\pi/2}\left(\frac{(i+1)^{1/2}+i^{1/2}}{(i+1)^{1/2}-i^{1/2}}\right) + i^{1/2}(i+1)^{1/2}\right).$$

To find the ideal fluid flow in D, we would need to find the inverse of $G(w)$, which from our calculation we see can not be found in closed form. If g is the inverse of G, then taking our ideal fluid flow in $\mathbb{H}$ to have uniform velocity $\nu > 0$, the complex potential in D is just $\mathcal{G}(z) = \nu g(z)$.

An example where we can find a closed form solution to the inverse is the following well-known problem. Let $D = \{z \in \mathbb{C} : \text{Im}(z) > 0, -1 < \text{Re}(z) < 1\}$ be the 'half-strip' in the upper-half plane. Suppose that we want to find a conformal map $G : \mathbb{H} \to D$ such that $G(-1) = -1$, $G(1) = 1$ and $G(\infty) = \infty$. We can think of D as an infinite polygonal domain with vertices $z_1 = -1$, $z_2 = 1$, $z_3 = \infty$ and interior angles $\alpha_1\pi = \pi/2, \alpha_2\pi = \pi/2$ and $\alpha_3 = 0$. Thus, our SC-formula tells us that for some constants $A, B \in \mathbb{C}$ and all $w \in \mathbb{H}$, we have

$$G(w) = A\int_i^w \frac{1}{((u-1)(u+1))^{1/2}}du + B = A(\arcsin(w) - \arcsin(i)) + B. \tag{23}$$

Now using the conditions that $G(-1) = -1$ and $G(1) = 1$, we have that

$$A = \frac{2}{\pi}$$

and

$$B = \frac{2}{\pi}\arcsin(i).$$

Thus $z = G(w) = \frac{2}{\pi}\arcsin(w)$ and so the inverse function is $w = g(z) = \sin(\frac{\pi}{2}z)$. Therefore, the ideal fluid flow in the 'half strip' is $\mathcal{G}(z) = \nu\sin(\frac{\pi}{2}z)$. We display the stream-lines of the two flows in Figure 2 and Figure 3.

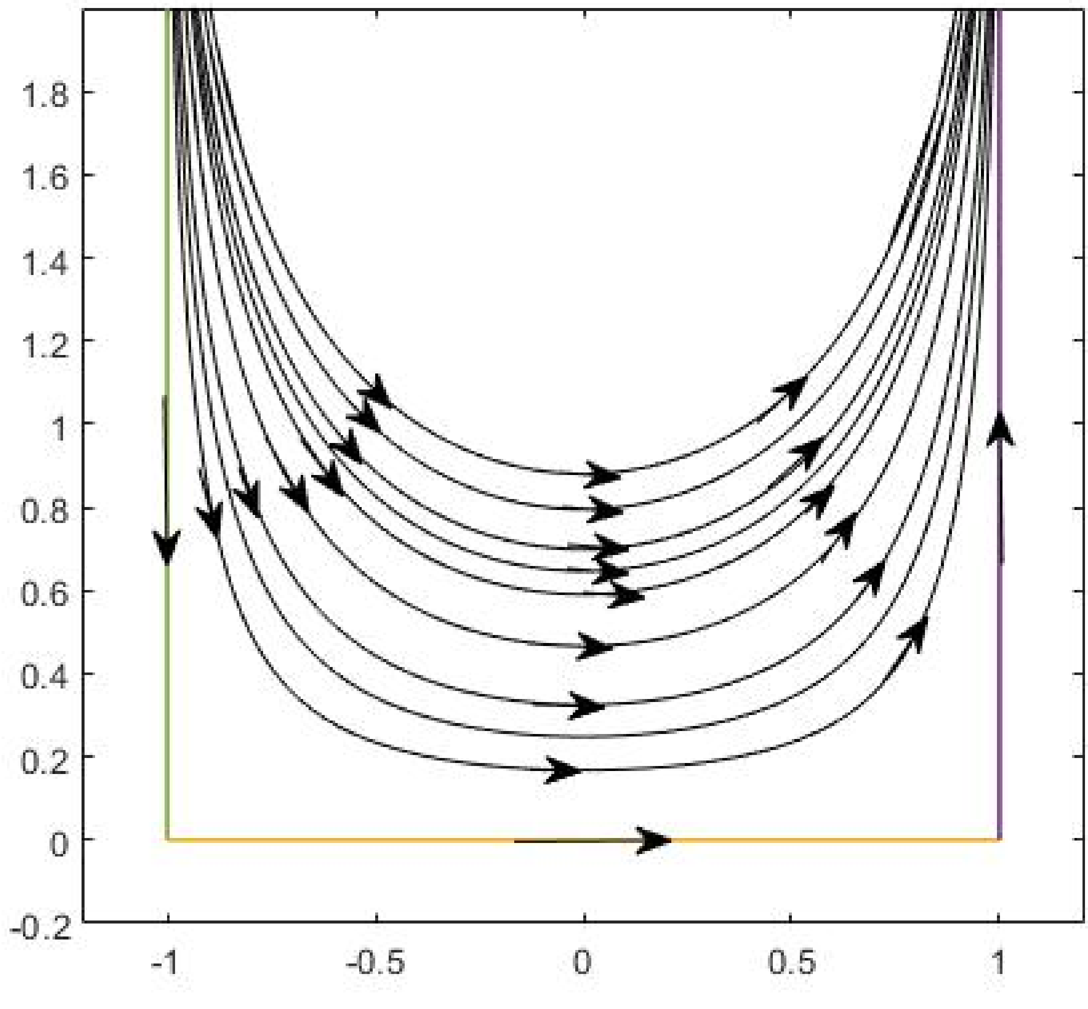

Figure 4: Ideal fluid Flow in the half strip domain

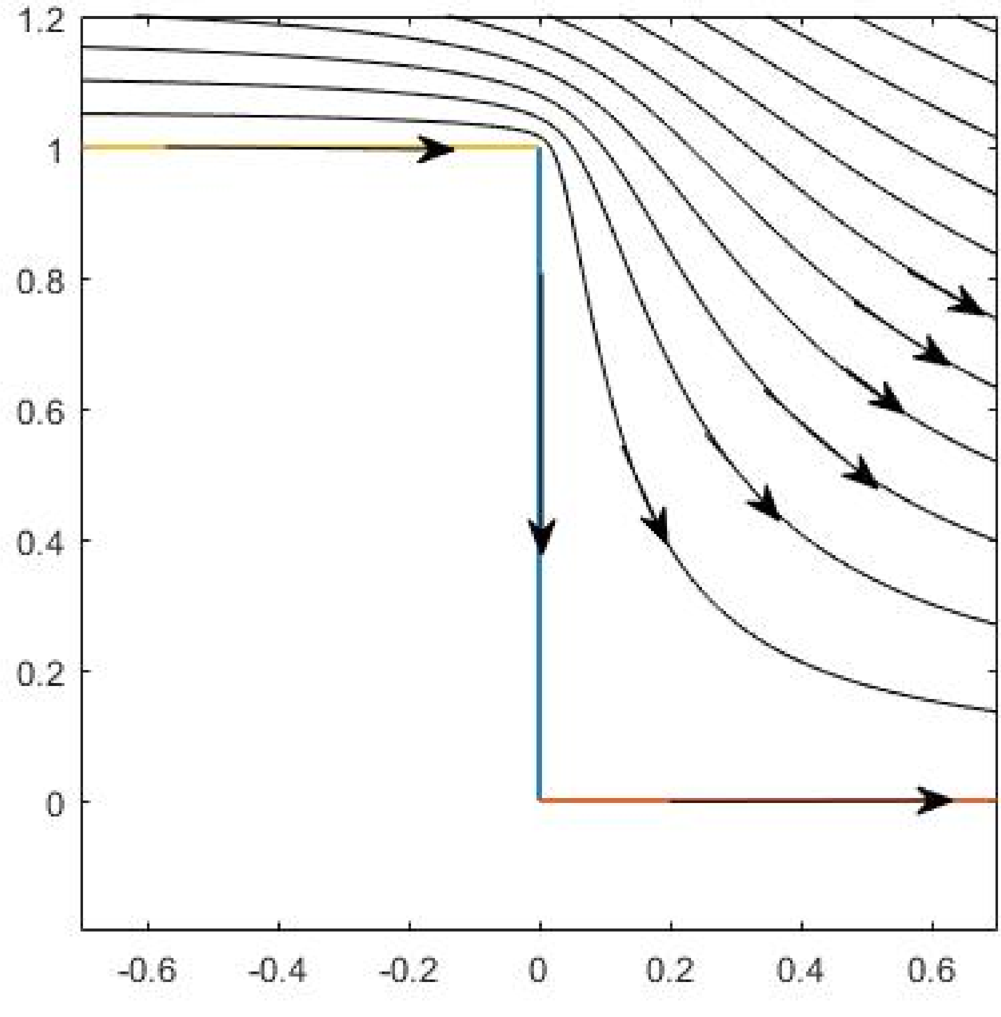

Figure 5: Ideal fluid flow in the step domain

References

[1] T. W. Gamelin, *Complex Analysis*. Springer, 2001.

[2] T. Tao, "246a notes 5: Confomal mapping," Oct 2016. [Online]. Available: https://terrytao.wordpress.com/2016/10/18/246a-notes-5-conformal-mapping/