Quasipositive Braids and Ribbon Surfaces

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1. Introduction

Meant to serve as an accessible exploration of knot theory for undergraduates and those without much experience in topology, this paper will start by exploring the basics of knot theory and will work through investigating the relationships between knots and surfaces, ending with an analysis of the relationship between quasipositive braids and surfaces in 4-space. We will begin by defining a knot and introducing the ways in which we are able to manipulate them. Following that, we will explore the basics of surfaces, building up to a proof that all surfaces are homeomorphic to a series of disks and bands which have a single boundary component and an introduction to how to view surfaces in higher dimension. After that, we will examine the relationship between knots and surfaces, proving that every knot bounds an orientable surface. We will follow that up with an introduction to braids and a proof that every knot is isotopic to a braid through Alexander’s algorithm. Finally, we will dissect quasipositive braids and their special relationship with surfaces in 4-space.

2. A Knot or Not?

Knots are the cornerstone of knot theory, a massive building block of topology, and a key tool for helping us understand many other aspect of mathematics, but what exactly is a knot? A knot can be thought of as a circle in 3-space, but that definition leaves us bumping into a few errors.

The simplest thing to do when starting to imagine a knot is to imagine, or take, a shoelace. You can do anything you want with that shoelace, twisting and tying it in every way imaginable, as long as you don’t cut it. When satisfied with the shape of the shoelace, you would then glue the aglets together. The shoelace would then represent a knot in 3-space.

This shoelace example displays some important qualities that are present in theoretical knots as well. To begin, the length of a knot cannot be infinite, as displayed in Figure 1, just as the length of your shoelace in the real world cannot be infinite.

![Figure 1. Wild Knot](image)

Similarly, when we imagine a knot as a set of points in three space, we can imagine a knotted section of it getting smaller and smaller until it eventually is a single point and that knot becomes an unknotted circle, as seen below in Figure 2.
It’s obvious that if we took our shoelace, no matter how tightly we pulled it to shrink the knotted component, it would never make a knotted shoelace turn into an unknotted shoelace; the only way to unknot it would be to break the circle, untie it, and reconnect the ends. Likewise, the shrinking of a knot into a single point cannot happen in our defined knots.

Finally, as we all have experienced from smacking our funny bone into a desk, no two objects can occupy the same space, thus two different parts of our shoelace cannot occupy the same space. While any knot in 3-space, can stretch and compress like a rubber band, at all crossings a knot cannot occupy the same point in space as itself.

With these restrictions in mind, we can write a formal definition of a knot which satisfies all of these properties.

**Definition 2.1.** A knot is a closed polygonal curve in $\mathbb{R}^3$ in which each straight segment of the curve intersects exactly two other segments and intersects them only at their endpoints.

This is why our example with the shoelace works. When we glue its ends together, our shoelace becomes the thickening of a closed curve in 3-space. We can then imagine placing wires within the shoelace to make various sections within it straight, transforming it into a closed polygonal curve with segments intersecting exactly two other segments at their endpoints. In our example, the shoelace doesn’t have wire in it. It still works to represent a knot, though, because it still follows all of our rules. Thus, when we draw the diagram of a knot in 3-space, called a knot diagram, we can smooth out the curve as long as we are sure to comply to the rules we set for our knot.

If you were to take one of these enclosed shoelace knots and pull on one of the laces, deforming it slightly, it would be the same knot, just with a different presentation of itself. We call two knots which are the equivalent to one another with different presentations isotopic.

**Definition 2.2.** Two knots are isotopic if one can be represented by the sequence of points $(p_1, p_2, ..., p_n)$ and the other is determined by the sequence of points $(p_0, p_1, p_2, ..., p_n)$ where $p_0$ is a point that is not colinear to $p_1$ and $p_n$ and the area of the triangle spanned by $(p_0, p_1, p_n)$ only intersects the knot along $(p_1, p_n)$. 

![Figure 2. Shrinking Knot](image)
For example, the polygonal knot and the smooth knot in Figure 3 are isotopic to one another through an infinite series of isotopic knots. Similar infinite series of isotopic knots allow you to smooth the diagram of any polygonal knot, and this is how knot diagrams are most often drawn for convenience sake.

Within our knot diagram, we can also add a feature called an orientation. The orientation is simply the direction traveled consistently along the knot. As a concrete example, we could think of our knot as being made out of wire. The orientation is the direction that electricity would flow through our wire. Since knots are closed, our choice of orientation doesn’t matter as long as we remain consistent about it, as seen in Figure 4.

The rules we laid out apply both for oriented and non oriented knots. Thankfully, we have a couple of tools to manipulate smooth knots while maintaining the rules we set for our knots. There are three moves that can be performed on a knot diagram at a crossing that leave the knots on either side of the transformation isotopic to one another. Once again, these moves and their inverses can easily be practiced on a shoelace with the ends attached or any other piece of rope or strings.

In the first move, you can take your rope and turn a portion of it upside down, leaving two points of the rope where they originally were. This move, Reidemeister Move 1a, results in a small loop in the rope. The inverse move, Reidemeister Move 1b, takes this loop and flips it the opposite direction that it was originally flipped, bringing the rope back to normal. This move can be seen in Figure 5.

In Reidemeister Move 2a, you take one section of the rope and drape it across another section of the rope. Inversely, if a section of the rope is lying on top of or below another section of the rope, you can perform Reidemeister move 2b, pulling these sections apart from one another. Move 2a is specifically known as bending while move 2b is called tightening.

Finally, there's Reidemeister Move 3, which is slightly more complex to explain but just as simple to preform. In Reidemeister Move 3, two strands cross one another, while a third lays atop both strands and on one side of the crossing. Reidemeister Move 3 allows for lifting this third strand and moving it to the other side of the crossing where it once again sits atop both strands involved in the crossing.

Figure 3. Polygonal and Smooth Isotopic Knots
Note that in addition to the moves shown above, there is a mirror set of moves obtained by changing each crossing in each figure.
Within the study of knots, many unique knots have been identified, and each one has a name. These names are made up of two numbers. The main number is the fewest number of crossings a presentation of the knot can have. Multiple distinct knots can have the same number of crossings, though, as can be seen in Figure 8, which is where the second number comes in. The second number is assigned to knots with the same number of crossings in ascending order of which they were tabulated. For example, the unknot, a circle, is the only knot with zero crossings, so it’s name is $0_1$.

As well as there being connected knots in 3-space, disjoint unions of knots can exist at the same time in 3-space. When this happens, the collection of knots are called links.

**Definition 2.3.** A link is union of disjoint knots.

Thus, all knots are considered links, but we call them knots to specify that there is only one component. If all of the knots in a link are polygonal knots, the link containing them is called a polygonal link. We also have the unlink which is the union of unknots all lying in a plane.

### 3. Skimming the Surface

One of the common things we want to know in topology is whether two topological objects, like surfaces, are the same as one another or homeomorphic. If one topological object, like a surface or a knot, could bend, stretch, or compress to form the shape of another topological object, we can think of the two topological objects as being homeomorphic to one another. A surface can even be cut and reattached in the same place as long as the neighborhood of every point remains the same.
Definition 3.1. Two topological spaces $X$ and $Y$ are homeomorphic if there exists a function $f : X \to Y$ such that $f$ is one-to-one, onto, and continuous and whose inverse is continuous.

Recall that that a function is continuous if for each point $x \in X$ and each neighborhood $N$ of $f(x)$ in $Y$, the set $f^{-1}(N)$ is a neighborhood of $x \in X$. As an example, a sphere is homeomorphic to every polyhedron. Likewise, a band attached to a disk where the band has an even number of twists and a band attached to a disk where the band has no twists are also homeomorphic to each other.

Surfaces are one of the, if not the biggest topic of study in topology. We have been studying surfaces for much longer than we have been studying knots, thus if we can find relationships between surfaces and knots, our prior knowledge about surfaces may help provide us with new information about knots.

Definition 3.2. A surface is a second-countable topological space in which each point has a neighborhood homeomorphic to the plane or half-plane, and for which any two distinct points possess disjoint neighborhoods. The boundary components of a surface are the circles of points where every neighborhood is homeomorphic to the half-plane. If a compact, connected surface has no boundary components, it is called closed.

A few well known examples of surfaces include a plane, various regions of a plane, a hollow sphere, or a hollow doughnut known in topology as a torus. While a section of a plane may have a boundary component and a plane is not compact, a sphere and a torus fulfill these requirements and thus are closed.

In addition to closure, these surfaces have many attributes which we use to describe them. One such attribute is whether a surface is orientable or nonorientable. Orientable surfaces have two distinct sides two them. One example of an orientable surface is the unit disk on the $xy$ plane in $\mathbb{R}^3$. The two sides of this disk are the top (the side of the disk you’d see looking down on it from a point with a positive $z$ value) and the bottom (the side of the disk you’d see looking up at it from a point with a negative $z$ value. Another example would be the hollow sphere, $S^2$, the inside of which is one side and the outside of which is another. The two sides of an orientable surface are distinct from one another in the sense that, if you were painting one side, you would never touch the opposite side with your paintbrush (as would be the case with $S^2$), or if you did, you would have to go over a boundary component (as would be the case with the unit disk).

Most real-world surfaces are orientable. There is at least one nonorientable, nontheoretical surface, though, that is easy to construct called the Möbius Loop. Though it is pictured below, this surface is much easier to understand when you can physically interact with it. To create your own Möbius strip, start with a strip of paper and hold the ends together in a loop. Then, leaving one end where it is, flip the other end upside down and connect them. By running a writing utensil around your Möbius Loop, you can clearly see it has only one side. Other nonorientable surfaces are the same way; they only have one side.

Another way to explore the orientation of a surface is through examining a surface’s triangulation. The triangulation of a surface is a representation of a surface built from 2-simplices.

Definition 3.3. Given $k + 1$ points $v_0, v_1, ..., v_k$ in general position, we call the smallest convex set containing them a $k$-simplex. The points $v_0, v_1, ..., v_k$ are called vertices of the simplex. If $A$ and $B$ are simplices and the vertices of $B$ form a subset of the vertices of $A$ then we say that $B$ is a face of $A$. 
Thus, a dot is a 0-simplex, a line segment is a 1-simplex, a triangle is a 2-simplex, and a solid tetrahedron is a 3-simplex. We call a space triangulable if it is homeomorphic to the union of a finite collection of simplices in which whenever a simplex lies in a collection, so do its faces, and whenever two simplices intersect, they do so on a common face. The triangulation of a space is this presentation of simplices homeomorphic to the original space. It is a deep result that all compact surfaces are triangulable.

Once we triangulate the surface, we can then assign each 2-simplex of the surface an ordering of the vertices. Orderings of the vertices are said to be equivalent if they differ by an even permutation. Thus, \((v_0, v_1, v_2)\) is equivalent to \((v_2, v_0, v_1)\) but is not equivalent to \((v_1, v_0, v_2)\). This equivalence relation results in two distinct orderings of the vertices on a 2-simplex. For a given 2-simplex, we can choose one of these orderings, giving the simplex an orientation. On a given surface we can assign one of these orientations as being positive and the other as being negative.

If on a pair of adjacent 2-simplices, the orientation on the shared 1-simplex is induced in opposite directions from the two 2-simplices, these 2-simplices are said to be compatible. If a surface can be triangulated such that all adjacent 2-simplices are compatible, then the surface is defined to be orientable. Otherwise, the surface is said to be non-orientable. By the definition of homeomorphism, the neighborhood of every point is the same both before and after the homeomorphism, so if two surfaces, \(X\) and \(Y\), are homeomorphic, \(X\) will be orientable if and only if \(Y\) is orientable.

Using this idea of triangulation, we are able to prove a major theorem that gives us a new and very useful way to think about surfaces.

**Theorem 3.4.** Every compact surface with connected boundary is homeomorphic to a disk and bands.

**Proof.** As we mentioned earlier, all closed surfaces are triangulable. Since each of these triangles are closed, you can further triangulate them into what is called a barycentric subdivision by drawing lines from the center of each edge of the triangle to the opposing corner. If this happens once, it is called the first barycentric subdivision, and if this happens again, it is called the second barycentric subdivision. Both can be seen for a single triangle in Figure 10.

We then are able to create various graphs along the edges of these barycentric subdivisions. We will call the set of all edges that make up the original triangulation of our surface the 1-skeleton. Likewise, given two adjacent 2-simplices, \(\sigma_1\) and \(\sigma_2\) there are two edges of 2-simplices within the first barycentric subdivision that goes from the barycenter, the center.
of the triangle in which all barycentric edges meet, to the center of the face connecting $\sigma_1$ and $\sigma_2$. If we consider the union of those two edges as a single edge, then the dual 1-skeleton is the set of all such edges. In Figure 11, a subtree of the 1-skeleton can be seen in dark blue and a subtree of the dual 1-skeleton can be seen in red.

Along these sets of edges, given a tree, the union of all 2-simplices in the second barycentric subdivision adjacent to that tree is homeomorphic to the disk since, at all places, at least one 2-simplex in the second barycentric subdivision keeps the 2-simplices adjacent to a tree in the 1-skeleton or the dual 1-skeleton from forming a loop.

Further, given a surface with a triangulation, let $S$ be a tree in the 1-skeleton of our triangulation. Let $S'$ be the subgraph of the dual 1-skeleton whose edges do not intersect
Since $S$ is a tree, $S'$ is connected. If $S'$ were disconnected, there would be a circuit in $S$ preventing two distinct sections of $S'$ from connecting with one another.

We thus know that $S$ is homeomorphic to a disk, as is the maximal tree $D$ of $S'$. $S' - D$ is then homeomorphic to a finite set of disjoint disks which we will call bands. If there are $n$ of these bands, we can label them $H_i$ where $1 \leq i \leq n$. Our surface, then is equal to $S \cup D \cup \bigcup_{i=1}^{k} H_i$.

Further, $\text{Int} S \cap D \cup \bigcup_{i=1}^{k} H_i = \emptyset$, and for each $i$ such that $1 \leq i \leq n$, $H_i \cap D$ is equal to two disjoint arcs that lie on $\partial H_i \cap \partial D$. As discussed when defining homeomorphism, this is homeomorphic to a surface $M$ such that $M = S \cup D \cup \bigcup_{i=1}^{k} H_i$, and for all $j$ such that $1 \leq j \leq k$, $H_j$ has either one twist or no twist, and all of our other rules about $S$, $D$, and $\bigcup_{i=1}^{k} H_i$ still apply.

Since our surface has zero boundary components, if we remove a disk, which has a single boundary component from it, the resulting surface will have a single boundary component. We know our surface is equal to $S \cup D \cup \bigcup_{i=1}^{k} H_i$, and $S$ is a disk, thus, $D \cup \bigcup_{i=1}^{k} H_i$ has a single boundary component. We can see in Figure 12 that a band connected to a disk

![Figure 12. Boundary Components of Disks With Bands Attached](image)

with a single twist has a single boundary surface, but a band connected to a disk without a twist has two boundary components. Therefore, for every band $H_j$ such that $1 \leq j \leq k$ and $H_j$ has no twists, $H_j$ divides the boundary of the surface into two arcs, $b_1$ and $b_2$. Since $D \cup \bigcup_{i=1}^{k} H_i$ has a single boundary component, there exists an untwisted band $H_n$ such that $1 \leq n \leq k$, and $H_n$ is attached to both $b_1$ and $b_2$.

Therefore every compact surface with connected boundary homeomorphic to a disk and bands attached.

With this in mind, we can introduce an invariant for orientable surfaces, the Euler characteristic.

**Definition 3.5.** For a given surface $S$ the Euler characteristic $\chi(S)$ is equal to the the number of disks minus the the number of bands. For a triangulated surface, this is equal to the number of 0-simplices minus the number of 1-simplices plus the number of 2-simplices.
Thus, a disk has an Euler characteristic of 1, the annulus, which we discussed is homeomorphic to a disk with a band attached, has an Euler characteristic of 0, and the sphere, which we discussed is homeomorphic to two disks attached together has an Euler characteristic of 2. The Euler characteristic is a very important tool for helping us calculate the genus of a surface.

**Definition 3.6.** For an orientable surface \( S \), the *genus*, \( g(S) \) can be computed as

\[
g(S) = \frac{2 - \chi(S) - B}{2}
\]

where \( B \) represents the number of boundary components a surface has.

Thus, a sphere, which has an Euler characteristic of 2, as we discussed earlier, has a genus of 0 since it doesn’t have any boundaries, as does the disk since it has 1 boundary component. In fact, the genus of a surface in \( \mathbb{R}^3 \) without any boundary components is equal to the number of holes it has going through it. This can be seen with the sphere which has genus 0, the torus which has genus 1, and the double torus which has genus 2.

We’ll end this section by discussing techniques of viewing surfaces in higher dimensions. Later in this paper, surfaces in 4-space are going to play an important role, but obviously surfaces in 4-space are incredibly hard to view. Thus, we want to find a way to view them while we remain in simple 3-space.

To solve this problem, let’s approach it with things that are a little more familiar and easy to picture. How would you be able to represent a surface from 3-space in 2-space? One way we can imagine doing this is by passing our surface through 2-space. This results in a “movie” of our three dimensional surface in 2-space, an example of which can be seen in Figure 14.

If we had a video screen, we’d be able to watch every moment of this movie play out, but on paper, one of the things that we can do is grab clear moments that help us to understand the movie as a whole, such as when the two legs of the torus join together again. This allows us to piece together the less important moments happening between these shots of the movie.

This can be applied not only to surfaces but to other items in \( \mathbb{R}^3 \) as well. In Figure 15, you can see what the movie of the knot of \( 8_{20} \) would look like. It begins with a single point which separates into two points. Then a third point appears and splits as well. The middle two points switch position with the right middle point going in front of the left middle point. Then the two leftmost points switch position with the leftmost point going in front. The middle two points then switch positions again with the middle right point going in front of
the middle left point. A fifth point then appears to the left of the rightmost points and splits into two points. The two rightmost points then switch positions with the point further right going in front. The center two points then switch places with the point more to the right going in front. The fourth and fifth point then collide to form a single point and disappear. The middle points switch positions with the left one going in front, followed by the two points on the left switching positions with the point that was originally closer to the center going in front. The center points switch positions one last time with the left point going in front of the right one. The two leftmost points then combine to become a single point and disappear followed by the remaining two points combining to form a single point and disappearing.

In the same way, this process can be applied to four dimensional surfaces going through 3-space, and just as above, we can capture the key moments of it to help us understand what the surface in its entirety would look like. An example of such a movie can be seen in Figure 16.

Sometimes, in the midst of these movies of surfaces in 4-space, a knot will arise. When this happens in a movie of a disk, the knot is called slice since the knot occurs on a slice of the surface in 4-space. Further, we can shift the surface so that the knot occurs when the fourth dimension is equal to zero. When this is true, it bounds a smooth disk in either \( \mathbb{R}^4_+ \) or \( \mathbb{R}^4_- \). Without loss of generality, we will say that the knot bounds a disk in \( \mathbb{R}^4_- \).

Finding these slice knots is one of the really big questions in knot theory which will be explored later in this paper.
In this section, we explore a fundamental connection between knots and surfaces.

**Theorem 4.1.** Every knot is the boundary of an orientable surface in $\mathbb{R}^3$.

*Proof.* To begin constructing such a surface from a given knot, the first step is to choose an orientation for the knot diagram. We then want to construct the diagram’s Seifert circles. To construct said circles, begin tracing the knot diagram starting from a strand and following the orientation. When a crossing is reached, switch the strand you’re tracing from the one you entered the crossing on to the other one, continuing in the direction of the diagram’s orientation. Continue this process until you trace a section that has already been traced. When this happens, switch to a new untraced strand and continue the process. If no such strands exists, all of the Seifert circles have been drawn for this diagram. An example of this process can be seen in Figure 17.

Each of these Seifert circles trivially bounds a disk. For all non-nested or outermost nested Seifert circles, we can think of these disks as lying in a plane in 3-space. We can think of each layer of the inner nested Seifert circles as living on a plane parallel to but higher than the plane of the circle it lies within. Each Seifert circle can then be connected with one another via half twisted bands corresponding both in location and direction to the knot diagram’s crossings. Thus, we have created a surface, called a Seifert surface (for Herbert Seifert the mathematician who created the algorithm), whose boundary is our given knot and can very simply be proven is orientable. An example of such surface and it’s corresponding knot and Seifert circles can be seen in Figure 18.

Now that we have defined Seifert circles, we are able to introduce a number of other definitions that are quite important when studying knots, links, and braids. Let $D$ be a specified knot diagram in $\mathbb{R}^2$. Much like shapes in geometry, when a knot diagram has been drawn in $\mathbb{R}^2$, both edges and faces can be identified.
Definition 4.2. An edge is a strand of a knot that goes between two crossings. A face is a connected surface in $\mathbb{R}^2 - D$. A face, $f$, is adjacent to an edge, $e$, if $e \subseteq \partial f$.

One thing that can be quite important is figuring out if a face can exist in 3-space. When one of these faces can’t, we call it a defect face.

Definition 4.3. A defect face is a face adjacent to two edges, $a_1$ and $a_2$, such that $a_1$ is a subset of Seifert circle $S_1$, $a_2$ is a subset of Seifert circle $S_2$, and $S_1$ and $S_2$ are distinct and incompatible.

Furthermore, defect faces are linked to the compatibility and incompatibility of Seifert circles. One of the most interesting features of various topological spaces is the way that they are connected with one another. One of the most prominent examples of this is $\mathbb{R}^2$’s relationship with $S^2$: We can regard the 2-sphere as $S^2 = \mathbb{R}^2 \cup \{\infty\}$. While we’ve solely been looking at these knot diagrams as living in $\mathbb{R}^2$, the knot that the diagram derives from
lives in $\mathbb{R}^3$. In $\mathbb{R}^3$, you can take any strand of the knot, pull it to an outer section, and move it to the other side of said knot, encircling the entire thing.

For our knot diagrams, this would be equivalent to stretching the corresponding strand out infinitely and bringing it back to the knot from the opposite infinity, thus allowing us the ability to refer to knot diagrams as living on $S^2$. Both of these cases can be seen below in Figures 20 and 21.

When we think about knot diagrams living on $S^2$, not only can the areas thought of as being “within” the knot diagram be seen as face, but the area “outside” the knot diagram can be thought of as one as well. We know that given a 2-sphere, if you cut a hole in it, it is homeomorphic to a disk, and if you cut a hole in a disk, it is homeomorphic to an annulus. Thus if you cut two holes in a 2-sphere, it will be homeomorphic to an annulus. We know that Seifert surfaces cannot overlap, and we can put bands between them, allowing them not to touch as well. Thus, if you cut the 2-sphere which a knot diagram is resting on along two of its Seifert circles, the result will be homeomorphic to an annulus.

If the orientation of the two Seifert circles is the opposite of the orientation induced by the 2-simplex representation of the annulus, the two Seifert circles can be considered compatible with one another, as seen in Figure 22. On the other hand, if the orientation of the two Seifert circles is not equivalent to either orientation induced by the 2-simplex representation of the annulus, the Seifert circles are considered incompatible, an example of which can be seen in Figure 23.

A knot diagram, $D$, can then be given a few traits to help describe it, $n(D)$ and $h(D)$. 

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**Figure 17. Seifert Circles**
Definition 4.4. For a given knot diagram, $D$, $n(D)$ is the number of Seifert circles in the knot diagram.

Definition 4.5. The height, $h(D)$, of a given knot diagram, $D$, is the number of pairs of incompatible Seifert circles in the knot diagram.

We can clearly see that $1 \leq n(D)$ and $0 \leq h(D) \leq \frac{n(D)(n(D)-1)}{2}$. Though $n(D)$ can be important when analyzing knots, height isn’t always. Nonetheless, it is particularly important when analyzing braids.

5. The Braid Group

A braid is a type of knot or link. In the mathematical sense, braids, much like those found in hair, are made of a number of strands woven over and under one another in the same
direction which then can be reattached to the strands entering the braid to form either a knot or a series of links. More rigorously, braids are oriented links where all parts of each
component of the link rotates around an axis in 3-space in the same direction. Braids can therefore be studied using the same techniques as knot theory.

One key reason that braids are so important within the study of knot theory is oriented links and closed braids are equivalent to one another. This is known as Alexander’s Theorem and it specifically states

**Theorem 5.1.** Any oriented link in \( \mathbb{R}^3 \) is isotopic to a closed braid.

**Proof.** We know any knot is isotopic to a geometric polygonal link. Similarly, any link is isotopic to a geometric polygonal link as well since a link is the union of disjoint knots. Thus, let \( l \) represent the \( z \)-axis in \( \mathbb{R}^3 \) and let \( L \) be an oriented polygonal link in \( \mathbb{R}^3 \) such that no point in \( L \) lies on \( l \). We can then trivially shift the vertices around so that none of the edges are parallel to \( l \) when projected onto the \( xz \) or \( yz \) plane. Thus, all edges of \( L \) are are a line rather than a point when projected onto the \( xy \) plane. Let \( A \) and \( C \) be two adjacent vertices on \( L \) such that the edge going between vertex \( A \) and vertex \( C \), \( AC \), is oriented from \( A \) to \( C \). Since \( AC \) is an edge in \( L \), when projected onto the \( xz \) and \( yz \) planes, it is not parallel to \( l \), and when projected onto the \( xy \) plane, \( AC \) is a line rotating around the origin. Assume \( AC \) is positive if \( AC \) is rotating counter clockwise about the origin and negative if it is rotating clockwise about the origin. If all edges of \( L \) are positive or if all edges of \( L \) are negative, \( L \) is a closed braid and our work is done. Thus, for the rest of the proof, assume without loss of generality that \( AC \) is negative and there exists a positive edge in \( L \). \( AC \) is called accessible if there exists a point \( B \in l \) such that the 2-simplex, \( \Delta ABC \) intersects \( L \) only along \( AC \).

If \( AC \) is negative and accessible, define \( B \) to be the point in \( l \) that qualifies \( AC \) as accessible. Let \( P \) be the plane in \( \mathbb{R}^3 \) in which \( \Delta ABC \) sits. Since for all point \( p \in L \), \( p \notin l \), the distance between \( p \) and \( B \) is greater than 0. We will say the point closest to \( B \) has a distance \( \delta \) from \( B \). Since \( \delta > 0 \), there exists at least one point \( r \) such that \( r \in P \) and the distance between \( B \) and \( r \) is less than or equal to \( \frac{\delta}{2} \) which is less than \( \delta \). Therefore, for all point \( r \), the points of 2-simplex \( \Delta ABC \) are a subset of the points of \( \Delta ArC \) and \( \Delta ArC \) only intersects \( L \) along the edge \( AC \). Pick a point \( B' \) such that \( B' \) is in the set of points \( r \). Since the set of points \( r \) contains at least one point and \( B' \) is in the set of points \( r \), \( B' \) exists. Furthermore, all of the points in \( \Delta ABC \) are a subset of all of the points in \( \Delta AB'C \) and \( \Delta AB'C \) only intersects \( L \) along the edge \( AC \). Since all of the points in \( \Delta ABC \) form a subset of all of the points in \( \Delta AB'C \), \( B \in \Delta AB'C \). Thus, if going from \( A \) to \( C \) along \( AC \) travels clockwise along \( l \), traveling from \( A \) to \( C \) along \( AB' \) and \( BC \) goes counterclockwise. Since \( AB'C \) intersects \( L \) only along the edge \( AC \), we can replace \( AC \) with \( AB' \) and \( BC \). The resulting knot would be isotopic to our original knot, and one negative edge has been eliminated.

If \( AC \) is negative and not accessible, for every point \( R \in AC \), there exists a finite number of points \( B \in l \) such that \( RB \) intersects \( L \) at a point on \( L \) other than \( R \) since \( L \) must have a finite length. Thus, since \( l \) has an infinite length, for every point \( R \), there exists some \( B \in l \) such that \( RB \) intersects \( L \) only at the point \( R \). Since \( RB \) intersects \( L \) only at the point \( R \), for all points \( p \in L \) such that \( p \neq R \), the distance between any point on \( RB \) and \( p \) is greater than 0. Thus, if \( \epsilon \) is the distance of the point \( p \) closest to \( RB \), \( \epsilon > 0 \). Thus, the \( \frac{\epsilon}{2} \)-neighborhood around \( RB \) does not intersect \( L \). Therefore if \( D, E \in AC \) and the distance between \( R \) and \( D, E \) is equal to \( \min\{\frac{\epsilon}{2}, \text{ the distance between } R \text{ and } A, \text{ the distance between } R \text{ and } C\} \), then \( \Delta DBE \) intersects \( L \) only on \( DE \). Thus, \( DE \) is accessible. Since \( AC \) is compact, we can split it into a finite number of accessible subsegments that only intersect with one another at the boundary.
points. We can then apply the steps as laid out in the paragraph above for each of these subsegments. Thus, replacing $AC$ with a finite number of positive edges.

This process can be repeated until $L$ has only positive edges. When $L$ has only positive edges, $L$ is a closed braid and our work is done. □

Thus, we know that any oriented knot in $\mathbb{R}^3$ is isotopic to a closed braid. But given an oriented knot in $\mathbb{R}^3$, we remain unaware of how to transform it into a closed braid. Fortunately, an algorithm exists to complete this process. To best understand how the algorithm works, though, a few lemmas must be introduced.

**Lemma 5.2.** If $D$ is an oriented link diagram in $\mathbb{R}^2$ bent to obtain $D'$, then $n(D') = n(D)$ and $h(D') = h(D) - 1$.

**Proof.** Let $D$ be an oriented link diagram in $\mathbb{R}^2$ and let $S_1, S_2$ be distinct incompatible Seifert circles of $D$. Since $S_1$ and $S_2$ are incompatible, they can be represented either in Form A or Form B in Figure 23 below. Since these are diagrams of knots in $\mathbb{R}^3$, their diagrams can be thought of as living in $\mathbb{R}^2 \cup \{\infty\} = S^2$. In $S^2$, Form A and Form B are equivalent, thus we
will only be focusing on Form A. Bending of Form A would then cause it to be represented as it is in Figure 26. Thus, since we have intersecting lines, by the definition of Seifert circles, the Seifert circles would need to be reconfigured. This reconfiguration results in the Seifert circles seen in Figure 27. All of the other Seifert circles in the diagram would remain the same, therefore $n(D) = n(D')$.

Once again, since $D$ is the diagram of a knot in $\mathbb{R}^3$, $D$ can be thought of as living on $\mathbb{R}^2 \cup \{\infty\} = S^2$, and in $S^2$ Forms A and B from Figure 23 can be thought of as being equivalent. Thus without loss of generality, we will solely be focusing on Form A in the diagrams in this section as well. $h(D)$ can be tallied up in special regards to our choice of $S_1$ and $S_2$. To begin, we know that one component of $h(D)$ comes from the fact that $S_1$ and $S_2$ are incompatible with one another. Since any Seifert circle within $S_1 \cup S_2$ is guaranteed to be incompatible with either $S_1$ or $S_2$ but not both, as seen in Figure 28, $h(D)$ includes the number of circles inside $S_1 \cup S_2$. The circles outside $S_1 \cup S_2$ are either compatible with both or neither, as seen in Figure 29, therefore $h(D)$ also includes 2 times the number of Seifert circles outside $S_1 \cup S_2$ that are incompatible with $S_1$. Finally, $h(D)$ contains the number of pairs of incompatible Seifert circles that do not include $S_1$ or $S_2$.

After the bending has occurred, the only Seifert circles that have changed are $S_1$ and $S_2$. Thus, the number of pairs of incompatible Seifert circles that do not include $S_1$ or $S_2$ before the bending is equal to the number of pairs of incompatible Seifert circles that do not include...
$S_1$ or $S_2$ after the bending. Likewise, assume $R$ is a Seifert circle outside $S_1 \cup S_2$. As seen in Figure 30, if $R$ was compatible with $S_1$ and $S_2$ before the bending, $R$ is compatible with $S_0$ and $S_\infty$ after the bending. Likewise, if $R$ was incompatible with $S_1$ and $S_2$ before the bending, $R$ is incompatible with $S_0$ and $S_\infty$ after the bending. Thus, the contribution to $h(D)$ in pairs including $S_1$ or $S_2$ and a Seifert circle $R$ outside of $S_1 \cup S_2$ is equal to the contribution to $h(D')$ in pairs including $S_0$ or $S_\infty$ and a Seifert circle $R$ outside of $S_0 \cup S_\infty$.

Similarly, as seen in Figure 31, if a circle was in $S_1 \cup S_2$ before the bending, it can remain in $S_\infty/S_0$ or $S_0/S_\infty$ depending which circle you label $S_0$ and which you label $S_\infty$ after the bending and remains incompatible with strictly one of $S_0$ or $S_\infty$. Therefore, the contribution to $h(D)$ in pairs including $S_1$ or $S_2$ and a Seifert circle $R$ inside of $S_1 \cup S_2$ is equal to the contribution to $h(D')$ in pairs including $S_0$ or $S_\infty$ and a Seifert circle $R$ inside of $S_0 \cup S_\infty$. Finally, $S_0$ and $S_\infty$ are compatible due to the bending, thus, $h(D') = h(D) - 1$. □

We can explore this lemma by applying it to the knot 8$_{20}$. As can be seen in Figure 32 before we bend 8$_{20}$, it has 5 Seifert circles, and the pair colored green are incompatible with one another while all of the other Seifert circles are compatible. After the bending though, five Seifert circles remain, but all of them are compatible with one another.
Lemma 5.3. An oriented link diagram $D$ in $\mathbb{R}^2$ has a defect face if and only if $h(D) \neq 0$.

Proof. If $D$ has a defect face, two incompatible Seifert circles exist. Therefore $h(D) \geq 1 > 0$.

Now, to prove the converse, assume $h(D) > 0$. This implies there exists distinct Seifert circles, $S_1$ and $S_2$, such that $S_1$ and $S_2$ are incompatible with one another. Let $c$ be an arc going from some point in $S_1$ to some point in $S_2$ and meeting each Seifert circle in $D$ at most once, as seen in Figure 32. Since, by definition, there exists a finite number of Seifert circles, and $c$ hits every Seifert circle in $D$ at most once, there must be a finite number of these crossings.

Assign $c$ to have direction from $S_1$ to $S_2$. Each Seifert circle that crosses $c$, in turn, either goes from the left side of $c$ to the right side of $c$ or from the right side of $c$ to the left side of $c$. Since $S_1$ and $S_2$ are incompatible, one is going from the left side of $c$ to the right side of $c$ and the other is going from the right side of $c$ to the left side of $c$. Thus, since there are a finite number of crossings of $c$, there exists two consecutive Seifert circles, $S_n$ and $S_m$, along $c$ going in opposite directions. Let the region of $\mathbb{R}^2$ containing the segment of $c$ that goes
between $S_n$ and $S_m$ be called $F$, and define an orientation on $F$ such that $S_n$ is induced by the orientation, as seen in Figure 34.
Define a Seifert circle as positive if it is induced by the orientation of $F$ and negative otherwise. Since $S_n$ is induced by the orientation, it is positive, and since $S_n$ and $S_m$ are crossing $c$ in opposite directions, $S_m$ is positive too. We thus have two positive Seifert circles along the boundary of $F$. If $F$ does not contain any boundaries going between Seifert circles, then $F$ is a defect face. If it does, though, we’ll call these segments $\gamma_x$. If we remove all $\gamma_x$ from $F$, we obtain a subsurface of $F$, which we will call $F'$.

Suppose each component $f$ of $F'$ is adjacent to exactly one positive Seifert circle and one negative Seifert circle. This would mean, as we go between adjacent $f$'s across $\gamma_x$s, these two Seifert circles would remain the same, meaning we’d be traveling around an annulus. Therefore, $c$ wouldn’t attach to two distinct positive Seifert circles thus reaching a contradiction.
Thus, there exists some $f$ with two adjacent positive Seifert circle or negative Seifert circles, and $f$ would then be a defect face of $D$. \hfill \square

To better understand this lemma, we can look at our example knot $8_{20}$ before and after we performed the bending on it. Before the bending, as can be seen in Figure 37, there was one defect face, labeled $f_1$, and, as we discussed after lemma 5.2, before the final bending, $h(D) = 1$. After the final bending, though, $h(D) = 0$, and none of the faces are defect.

We have one final lemma to help us reach our algorithm.
Lemma 5.4. An oriented link diagram $D$ in $\mathbb{R}^2$ with $h(D) = 0$ is isotopic in the sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ to a closed braid diagram in $\mathbb{R}^2$.

Proof. Let $D$ be an oriented link diagram in $\mathbb{R}^2$ such that $h(D) = 0$. If any of the faces of $D$ are bounded by three or more Seifert circles, then two of them must be incongruent with one another which goes against our assumption that $h(D) = 0$. A compact connect subsurface in $\mathbb{R}^2$ with one or two boundary components is a disk or an annulus, thus, all of the faces of $D$ are disks and annuli. Through isotopy on the sphere, all of the Seifert circles of $D$ are disjoint concentric circles. Since $h(D) = 0$, all of these concentric circles are oriented in the same direction and thus $D$ represents a closed braid diagram in $\mathbb{R}^2$. \qed

As we saw when investigating Lemma 5.3, after performing the bending on $8_{20}$, $h(D) = 0$. We can take the Seifert circle highlighted with green below and move it around $S^2$ to the other side of the knot. This in turn results in a closed braid.
the strands of $D$ in $S^2$ until it’s Seifert circles are concentric, transforming $D$ to a closed braid diagram in $\mathbb{R}^2$.

Once in a braided form, we can describe the braid algebraically, without having to draw it. To begin, we want to choose a straight line exiting the center of the braid on which no crossings occur. We can cut our braid open upon this line.

We then want to organize the crossings within the braid so that there is a clear order in which the crossings occur. This results in most of the strands being straight at most points along the braid, but leads to a much easier ordering schema within the braid.

Once this happens, we can assign a labeling to each column of strands within the knot, starting with the leftmost column being 1 counting up to the rightmost column, $n$. Then when a strand in the $(i + 1)^{th}$ column crosses over a strand in the $i^{th}$ column, we call this crossing $\sigma_i$. Similarly when a strand in the $i^{th}$ column crosses over a strand of the $(i + 1)^{th}$ column, we call that $\sigma_i^{-1}$ since $\sigma_i$ and $\sigma_i^{-1}$ cancel each other out when next to one another. For convention, when listing these crossings, we list them from top to bottom. As an example, the diagramed knot we have been working with, $8_{20}$, which can be seen in Figure 40, is represented as $\sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_1$. Any series of these crossings can be called a word, $\omega$. For example, we can say $\omega_1 = \sigma_1 \sigma_2 \sigma_1^{-1}$ and $\omega_2 = \sigma_2$ and then represent the braid as $\omega_1 \omega_2 \omega_1^{-1} \omega_2 \sigma_1 \omega_2^{-1}$.

This representation is called the braid group and is an algebraic group. In fact, any braid with $n$ strands can be represented as an element of the group $B_n$, which can be presented as

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_n \left| \begin{array}{c} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_j \sigma_k = \sigma_k \sigma_j \end{array} \right. \right\rangle,$$

where the above relations hold for $1 \leq i, j, k \leq n - 1$ and $|j - k| \geq 2$.

6. Quasipositive Braids

Within the group of closed braids, there are certain types of braids that are particularly special, one of which is quasipositive braids.

Figure 39. Cutting the Braid Open
**Definition 6.1.** A braid is *quasipositive* if it’s braid diagram can take the form

\[ \omega_1 \sigma_{n_1} \omega_1^{-1} \omega_2 \sigma_{n_2} \omega_2^{-1} \cdots \omega_m \sigma_{n_m} \omega_m^{-1} \]

where \( \omega_i \) is a word and \( \sigma_i \) is a crossing for all \( i \) such that \( 0 \leq i \leq m \) and \( n_i \) is inclusively between 1 and one less than the number of strands the braid has.

Thus, if crossing \( \sigma_{n_i} \) was removed, \( \omega_i \) and \( \omega_i^{-1} \) would be next to each other, cancelling one another out. Since we demonstrated above that \( 8_{20} \) could be represented as

\[ \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_1^{-1} \sigma_2 \sigma_1^{-1} = \omega_1 \sigma_2 \omega_1^{-1} \omega_2 \sigma_1 \omega_2^{-1} \]

when \( \omega_1 = \sigma_1 \sigma_2 \sigma_1^{-1} \) and \( \omega_2 = \sigma_2 \), \( 8_{20} \) is an example of a quasipositive braid.

One very distinctive fact regarding quasipositive braids is their relationship with bounding surfaces in 4-space. As discussed, all quasipositive braids are of the form

\[ \omega_1 \sigma_{n_1} \omega_1^{-1} \omega_2 \sigma_{n_2} \omega_2^{-1} \cdots \omega_m \sigma_{n_m} \omega_m^{-1} \]

for some integer \( m \), and for all \( i \) such that \( 0 \leq i \leq m \), and \( n_i \) is inclusively between 1 and one less than the number of strands the quasipositive braid has. As we project a quasipositive braid into \( \mathbb{R}^4 \), for all \( i \) such that \( 1 \leq i \leq m \), we can attach a band at \( \sigma_i \), resulting in \( \sigma_i \)
turning into a loop which we can use Reidemeister move 1 to get rid of. Thus,

$$\omega_i \sigma_n \omega_i^{-1} \rightarrow \omega_1(1)\omega_1^{-1}$$

$$\omega_1(1)\omega_1^{-1} = \omega_1\omega_1^{-1} = 1$$

![Figure 41. Example of Adding a Band](image)

We can apply induction to see that when we attach bands at each $\sigma_i$ in our quasipositive braid, the quasipositive braid is simply equal to 1, thus having capped off our surface. Furthermore, if a knot has a band factorization with $b$ bands and the number of strands is $b+1$, the resulting surface would be a disk with Euler characteristic of 1, and the knot would be slice.

Continuing with our example of quasipositive braid $8_{20}$, we can see in the figure below the movie of the disk $8_{20}$ bounds in 4-space.

Furthermore, as discussed earlier, the Seifert circles for braids are nested. We can then think of each crossing in a braid as being a band with a single twist between these disks. Thus, since quasipositive bands are of the form

$$\omega_i \sigma_n \omega_i^{-1} \omega_2 \sigma_n \omega_2^{-1} \cdots \omega_m \sigma_{m-1} \omega_m^{-1}$$

a structure exists that we can further explore.

There exists a special type of crossing called a ribbon crossing, pictured in Figure 43. As can be observed, if the two perpendicular segments are both considered disks, a ribbon crossing could only truly occur in 4-space.
If we observe a simple $\sigma_1\sigma_2\sigma_1^{-1}$ braid, we can see that it contains a ribbon crossing. Likewise, for any $\omega_i\sigma_i\omega_i^{-1}$, $\omega_i$ and $\omega_i^{-1}$ contain a series of ribbon crossings, each of which can be seen even more clearly when a band is attached to $\sigma_i$ and step by step, $\omega_i\omega_i^{-1} = 1$. This in turn results in ribbon surfaces being a series of disks in $\mathbb{R}^4$.

Furthermore, thanks to this presentation, we are able to calculate the genus of any quasipositive braid. In order to calculate the genus, two things are needed: the Euler characteristic and the number of boundary components. Likewise, two things are needed to calculate the Euler characteristic: the number of disks, and the number of bands. Thus, if we can find the number of disks, bands, and boundary components that make up a given quasipositive braid $B$, we can calculate it’s genus in 4-space.

To begin, suppose there is a disk for each column of strands $B$ has. Thus, if $B$ has $n$ strands in one of it’s given braid presentation, it has $n$ disks. For example, our quasipositive braid $8_{20}$ seen in Figure 40 has three disks since it has three strands of columns.
Next, since $B$ is a quasipositive braid it can be represented as

$$
\omega_1\sigma_i\omega_1^{-1}\omega_2\sigma_i\omega_2^{-1}\cdots\omega_m\sigma_i\omega_m^{-1}
$$
where for all \( j \) such that \( 1 \leq j \leq m \), \( \omega_j \) is a word over \( n \) strands and \( 1 \leq i_j \leq n - 1 \). As discussed above, for all \( j \), \( \omega_j \sigma_i \omega_j^{-1} \) is a series of ribbon crossings with \( \sigma_i \) in the center. In 4-space, all of the ribbon crossings aren’t actually crossings. They could be viewed in 3-space as intersections of the disks, but these intersection points can have one of the applicable portions of the surface pushed into \( \mathbb{R}^4_+ \), resulting in a lack of intersection. Thus, within \( \omega_j \sigma_i \omega_j \) the only truly twisted band between two disks is \( \sigma_i \). Therefore, there are \( m \) bands.

Continuing with our example of \( 8_{20} \), we know \( 8_{20} \) can be represented as

\[
\sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_1 \sigma_2 \sigma_1^{-1} = \omega_1 \sigma_2 \omega_1^{-1} \omega_2 \sigma_1 \omega_2^{-1}
\]

and therefore has two bands.

To find the number of boundary components, I find it easiest to draw a diagram. To start, draw \( n \) circles labeled 1 to \( n \). Then examine each \( \omega_j \). We know for all \( \omega_j \), \( \omega_j = \sigma_{k_1} \sigma_{k_2} ... \sigma_{k_j} \). We then know that \( \omega_j \sigma_i \omega_j^{-1} \) connects disk \( k_1 + 1 \) with disk \( i_j \) via a half twisted band. Thus we can draw a half twisted band between disk \( k_1 + 1 \) and disk \( i_j \) in our diagram. Once we have done this process for all \( j \) between 1 and \( m \), we can examine our diagram to discover how many boundary components it has. The resulting number of boundary components, \( p \), is the number of boundary components \( B \) has in \( \mathbb{R}^4 \). An example of these diagrams can be seen in Figure 45. Through tracing, you can see that that \( 8_{20} \) has one boundary component.

![Figure 45. 8_{20} Disks and Bands Diagram](image)

Thus we can plug \( n \), \( m \), and \( p \) into our equation for genus to get

\[
g(B) = \frac{2 - n + m - p}{2}
\]

As an example, \( 8_{20} \) has genus 0 and thus the surface it bounds is a disk. This is striking: The only knot that bounds a disk in 3-space is the unknot, but \( 8_{20} \) is a nontrivial knot that bounds a disk in 4-space.

7. Bibliography

Chapters 1, 2, and 3 of [Liv93] helped guide Section 1 of this paper.
Chapter 1 of [Arm83] was used. It was used in combination though with Chapter 11 of [SS19], with the information about barycentric subdivisions and every surface being homeomorphic to a series of bands and disks with a single boundary component coming from [SS19], and some of the more generalized information coming from [Arm83]. Chapter 9 of [Liv93] also helped inform Section 2 with information about objects in 4-space.

The use of [Liv93] as a resource was then continued in Section 4 through its discussion on the Seifert algorithm in Chapter 4. Chapter 2 of [KT08], though contributed most of the information in the second part of Section 4.

The aforementioned Chapter 2 of [KT08] also helped inform the vast majority of information in Section 5, but some of the more generalized information about braids came from Chapter 1 of [KT08] as well.

Finally, [Gri19], [Rud93], and Chapter 2 of [KT08] helped guide Section 6.

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