May 2019

Vibrations on Networks

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Applications of Graph Spectra

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Outline

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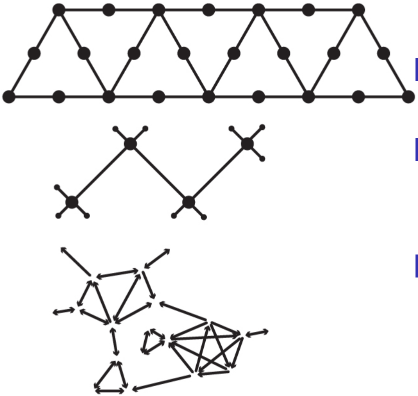
Three Challenges

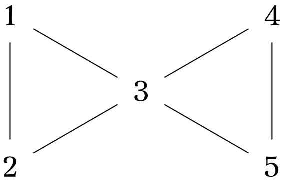


1. Can we ensure structures don't collapse in high winds?
2. Can we predict and characterize absorption spectra molecules?
3. How can we build networks without bottlenecks?

Graph Theory

A Graph G is a structure comprised of Vertices & Edges.

- *Things & Connections between them*

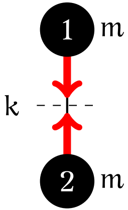


$$\Longleftrightarrow \quad \begin{aligned} V &= \{1, 2, 3, 4, 5\} \\ E &= \{12,\ 13,\ 23,\ 34,\ 35,\ 45\} \end{aligned}$$

Vibrations

A First Example

- Vertices & Edges $\rightarrow$ Masses & Forces
- Spring constant k, $[N/m]$
- Mass m, $[kg]$



$$m\frac{d^2x_1}{dt^2} = -k(x_1 - x_2)$$

$$m\frac{d^2x_2}{dt^2} = -k(x_2 - x_1)$$

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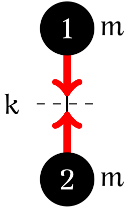

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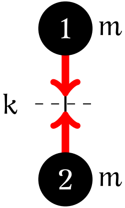

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If $x_1 = x_2 \;\Rightarrow\; \frac{d^2}{dt^2}\vec{x} = 0\vec{x} \Rightarrow f_1 = 0$, *Equilibrium.*

If $x_1 = -x_2 \;\Rightarrow\; \frac{d^2}{dt^2}\vec{x} = -\frac{2k}{m}\vec{x} \Rightarrow f_2 = \sqrt{\frac{2k}{m}}$, *First Resonance*

$$m\frac{d^2x_1}{dt^2} = -k(x_1 - x_2)$$

$$m\frac{d^2x_2}{dt^2} = -k(x_2 - x_1)$$

$$\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{k}{m}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\frac{d^2}{dt^2}\vec{x} = -L\vec{x}$, where L is the discrete Laplacian.

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Such systems are solvable by separation of variables if

$L\vec{x} = \lambda\vec{x}$: λ a real constant.

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Such systems are solvable by separation of variables if

$$L\vec{x} = \lambda\vec{x} \ : \ \lambda \text{ a real constant.}$$

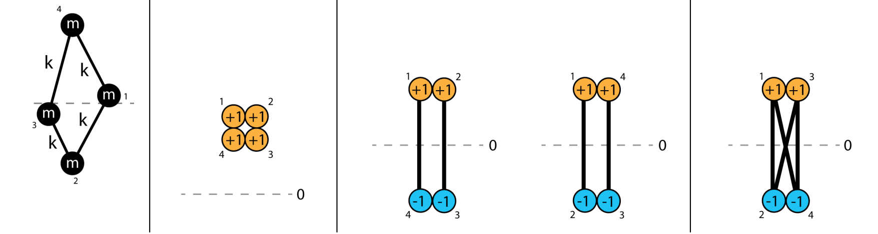
- λ is called an eigenvalue in the spectrum of L.
- System has a resonant frequency of $\sqrt{\lambda}$.
- Units $[\lambda] = [k/m] = N/(m \cdot kg) = 1/s^2 = hz^2$

4-cycle Vibrational Modes

The Discrete Laplacian

$$\text{If } \frac{k}{m} = 1 \text{ then } L = \begin{bmatrix} +2 & -1 & +0 & -1 \\ -1 & +2 & -1 & +0 \\ +0 & -1 & +2 & -1 \\ -1 & +0 & -1 & +2 \end{bmatrix}$$

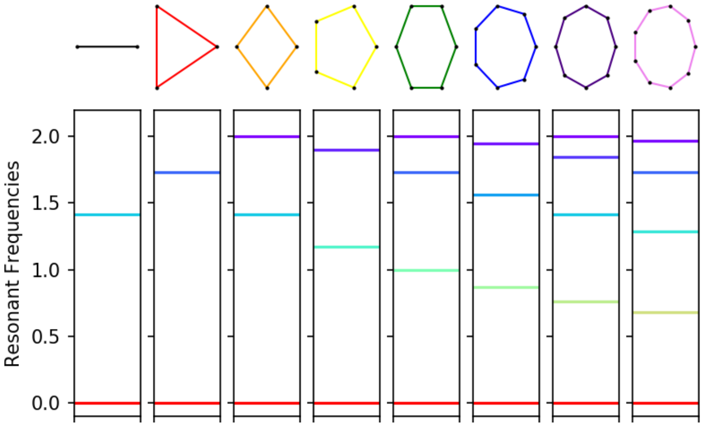
$L\vec{x} = \lambda\vec{x}$	$\lambda_1 = 0$	$\lambda_2 = \lambda_3 = 2$	$\lambda_4 = 4$

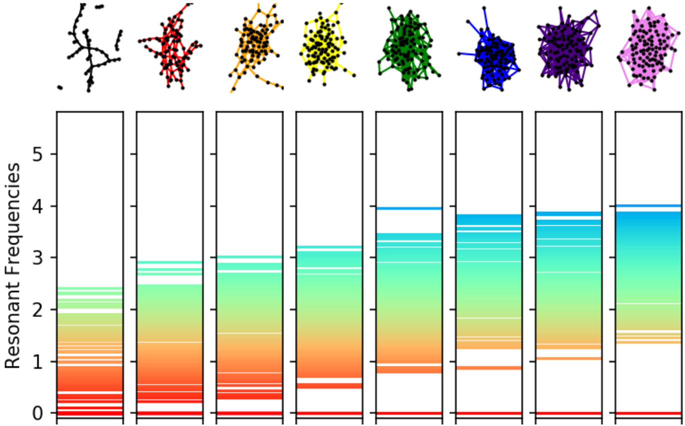


More springs per unit mass $\iff$ higher frequency mode

Cycle Graphs

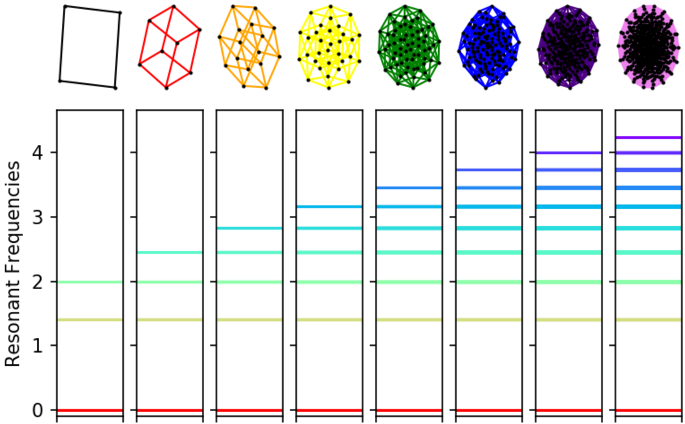
Spectra




Resonant Frequencies
0
1
2
3
4
5

Hypercube Graphs

Spectra



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