



May 2019

# Vibrations on Networks

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# Vibrations on Networks

## Applications of Graph Spectra

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15 May 2019

# Outline

## 1 Graph Theory

# Outline

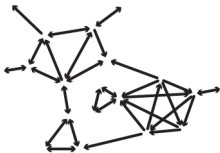
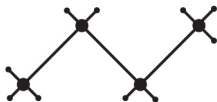
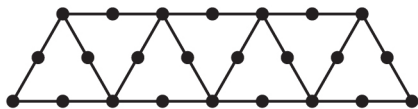
**1** Graph Theory

**2** The Discrete Laplacian

# Outline

- 1 Graph Theory
- 2 The Discrete Laplacian
- 3 Graph Spectra

# Three Challenges

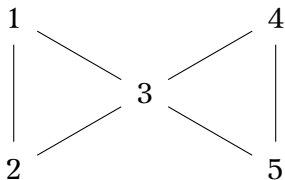


- 1 Can we ensure structures don't collapse in high winds?
- 2 Can we predict and characterize absorption spectra molecules?
- 3 How can we build networks without bottlenecks?

# Graph Theory

A Graph  $G$  is a structure comprised of Vertices & Edges.

■ *Things & Connections between them*



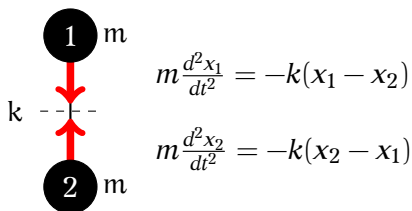
$$V = \{1, 2, 3, 4, 5\}$$

$$E = \{12, 13, 23, 34, 35, 45\}$$

# Vibrations

## A First Example

- Vertices & Edges  $\rightarrow$  Masses & Forces
- Spring constant  $k$ ,  $[N/m]$
- Mass  $m$ ,  $[kg]$

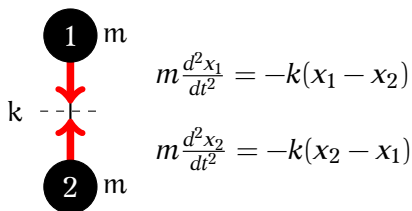




# Vibrations

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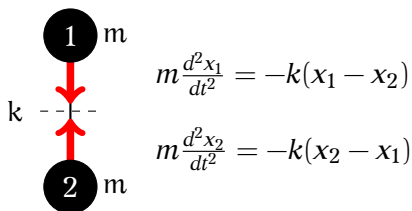


If  $x_1 = x_2 \Rightarrow \frac{d^2}{dt^2} \vec{x} = 0 \vec{x} \Rightarrow f_1 = 0$  , *Equilibrium*.

# Vibrations

## A First Example

- Vertices & Edges  $\rightarrow$  Masses & Forces
- Spring constant  $k$ ,  $[N/m]$
- Mass  $m$ ,  $[kg]$



If  $x_1 = x_2 \Rightarrow \frac{d^2}{dt^2}\vec{x} = 0\vec{x} \Rightarrow f_1 = 0$  , *Equilibrium*.

If  $x_1 = -x_2 \Rightarrow \frac{d^2}{dt^2}\vec{x} = -\frac{2k}{m}\vec{x} \Rightarrow f_2 = \sqrt{\frac{2k}{m}}$  , *First Resonance*

# Solutions

## The Discrete Laplacian

$$\begin{aligned} m \frac{d^2 x_1}{dt^2} &= -k(x_1 - x_2) \\ m \frac{d^2 x_2}{dt^2} &= -k(x_2 - x_1) \end{aligned} \quad \frac{d^2}{dt^2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{k}{m} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\frac{d^2}{dt^2} \vec{x} = -L\vec{x}$ , where  $L$  is the discrete Laplacian.

# Solutions

## The Discrete Laplacian

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Such systems are solvable by separation of variables if

$$L\vec{x} = \lambda\vec{x} : \lambda \text{ a real constant.}$$

# Solutions

## The Discrete Laplacian

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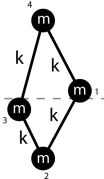
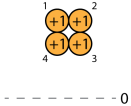
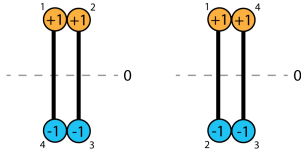
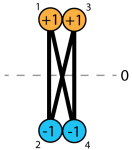
$$L\vec{x} = \lambda\vec{x} : \lambda \text{ a real constant.}$$

- $\lambda$  is called an eigenvalue in the spectrum of  $L$ .
- System has a resonant frequency of  $\sqrt{\lambda}$ .
- Units  $[\lambda] = [k/m] = N/(m \cdot kg) = 1/s^2 = Hz^2$

# 4-cycle Vibrational Modes

## The Discrete Laplacian

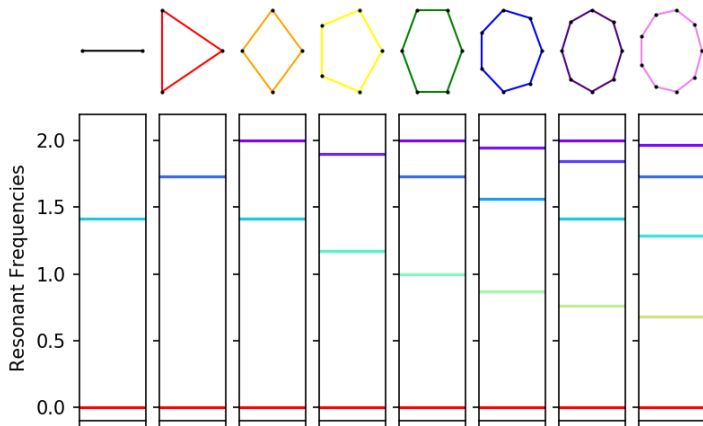
If  $\frac{k}{m} = 1$  then  $L = \begin{bmatrix} +2 & -1 & +0 & -1 \\ -1 & +2 & -1 & +0 \\ +0 & -1 & +2 & -1 \\ -1 & +0 & -1 & +2 \end{bmatrix}$

| $L\vec{x} = \lambda\vec{x}$   | $\lambda_1 = 0$   | $\lambda_2 = \lambda_3 = 2$  | $\lambda_4 = 4$   |
|---|---|--|---|
|  |  |  |  |

More springs per unit mass  $\iff$  higher frequency mode

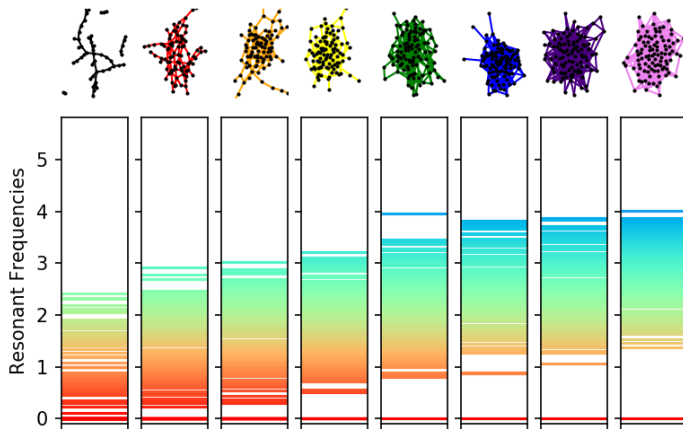
# Cycle Graphs

## Spectra



# Random Graphs

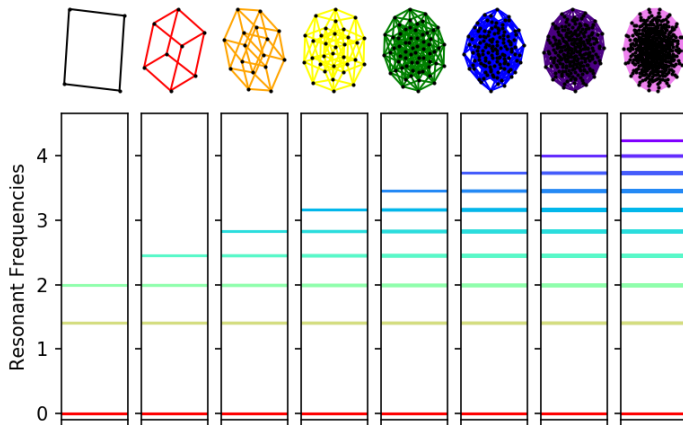
## Spectra





# Hypercube Graphs

## Spectra



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