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Lagrangian Mechanics and the Motion of a T-Handle

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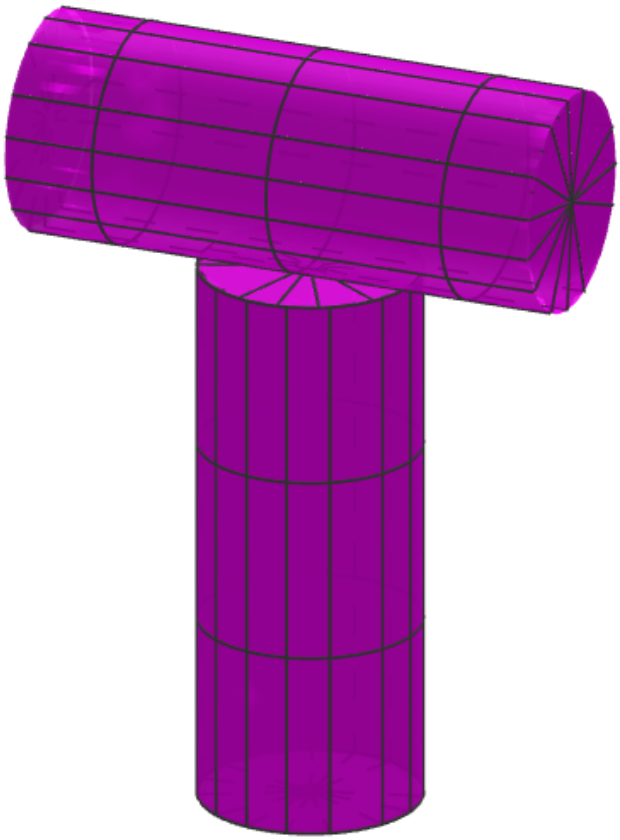

t-handle

Contents

1 Introduction

In this paper, we will cover three topics: the Lagrangian formulation of mechanics, rigid body motion, and finally a specific, interesting case of rigid body motion. Lagrangian mechanics differs from the traditional Newtonian mechanics in the sense that it is modeled around the energy of a system, rather the forces applied on a system. Despite this, it is important to note that the physical contents of Lagrange's equations are the same as those of Newton's equations. In fact, Lagrange's equations can be derived from Newton's equation, which will be done in this paper. There are, however, several advantages that the Lagrangian formulation has over the Newtonian formulation, which will be discussed in that section.

For rigid body motion, we will mostly be looking at the kinetic energy of the body and motion of the angular momentum vector as the body rotates and moves through space. Through that lens, the Intermediate Axis Theorem, also called the Dzhanibekov effect, which describes the motion of a rigid body rotating about its unstable intermediate axis, will be investigated. The goal is to give some explanation for this seeming improbably motion.

Finally, we will apply what was covered in the rigid body section to a t-handle to see its motion as it rotates about its three principal axes. This section includes computing certain values for the t-handle, based on the geometry of the body, and looking at how changing its geometry changes its motion.

2 Lagrangian Mechanics

As mentioned in the introduction, there are several important advantages to using the Lagrangian formulation of mechanics over the Newtonian one. Newtonian mechanics is most easily applied to systems whose area of motion is all of 3-space, where Cartesian coordinates are natural. However, there are other situations when the area of motion of the system is restricted to a specific part of 3-space. This is referred to as a constrained system, and the area in which it is free to move is known as the configuration manifold. It is possible to represent these systems through the Newtonian formulation; however, Lagrangian mechanics is formulated in a way such that the constraint equations are accounted for already. This makes calculation much simpler, as the aforementioned constraint equations can essentially be ignored, as they have already been applied.

Other advantages to the Lagrangian formulation include the fact that it highlights the connection between conservation laws and symmetric properties of dynamical systems and that Lagrange's equations can be derived from a variational principle, which is common in many branches of physics. In this section, we will begin by deriving Lagrange's equations from Newton's equations, then show that those same equations can be derived from a variational principle, before finally looking at Noether's theorem.

2.1 Lagrange's Equations

Beginning with the equations of motion for a particle constrained to a surface[1], we have

$$m\ddot{\mathbf{x}} = \mathbf{F} + \lambda(t)\nabla f, \qquad f(\mathbf{x}, t) = 0. \tag{2.1.1}$$

For the second of Equations (2.1.1), f is the equation for the surface to which the particle is constrained. On the left side of the first of Equations (2.1.1), m is the mass and $\ddot{\mathbf{x}}$ is the second derivative of the position vector with respect to time. On the right side, $\mathbf{F}$ is the external force applied and $\lambda\nabla f$ resembles the constraint equations. The constraint equations hold the particle to the surface, and are thus perpendicular to the surface, so they can be represented as some scalar, $\lambda(t)$ times the gradient vector ∇f.

First we eliminate λ, which is done by taking an arbitrary vector τ, which is tangent to the surface. Since ∇f is orthogonal to the surface, we get

$$m\ddot{\mathbf{x}} = \mathbf{F} + \lambda(t)\nabla f \implies m\ddot{\mathbf{x}} - \mathbf{F} = \lambda(t)\nabla f \implies (m\ddot{\mathbf{x}} - \mathbf{F}) \cdot \tau = 0. \tag{2.1.2}$$

Tangent vectors can be put in the form

$$\tau_l = \varepsilon^\alpha \frac{\partial \mathbf{x_l}}{\partial q^\alpha}, \tag{2.1.3}$$

where α runs from 1 to n, the ε^α are a set of arbitrary constants, the q^α are the generalized coordinates, and $\mathbf{x}_l = \mathbf{x}_l(q^1, q^2, \ldots, q^\alpha)$ is the position vector written in terms of the generalized coordinates. (Note that summation notation is used for the Greek indices, which run from one to n.) Equation (2.1.3) can be inserted into equation (2.1.2), giving us

$$\sum_{l=1}^{N} (m_l \ddot{\mathbf{x}}_\mathbf{l} - \mathbf{F_l}) \cdot \frac{\partial \mathbf{x_l}}{\partial q^\alpha} = 0, \qquad \alpha = 1, \ldots, n \tag{2.1.4}$$

Now we look at $\mathbf{F}$. We assume the forces are conservative, so $\mathbf{F_l} = -\nabla_l V(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ where V is the potential energy of the system. Then

$$\sum_{l=1}^{N} \mathbf{F}_l \cdot \frac{\partial \mathbf{x}_l}{\partial q^\alpha} = -\sum_{l=1}^{N} \nabla_l V \cdot \frac{\partial \mathbf{x}_l}{\partial q^\alpha} = -\frac{\partial V}{\partial q^\alpha}. \tag{2.1.5}$$

Then, when V is written in terms of the q^α, this expression is written in terms of generalized coordinates.

Lastly, we look at the $\ddot{\mathbf{x}}$.

$$\ddot{\mathbf{x}}_l \cdot \frac{\partial \mathbf{x}_l}{\partial q^\alpha} = \frac{d}{dt}\left[\dot{\mathbf{x}}_l \cdot \frac{\partial \mathbf{x}_l}{\partial q^\alpha}\right] - \dot{\mathbf{x}}_\mathbf{l} \cdot \frac{d}{dt}\frac{\partial \mathbf{x}_l}{\partial q^\alpha} \tag{2.1.6}$$

But $\mathbf{v}$, the velocity, is

$$\mathbf{v}_l = \dot{\mathbf{x}}_l = \frac{d\mathbf{x}_l}{dt} = \frac{\partial \mathbf{x}_l}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial \mathbf{x}_l}{\partial t}, \tag{2.1.7}$$

[1]This derivation is adapted from [1] and [2].

so

$$\frac{\partial \mathbf{v}_l}{\partial \dot{q}^\alpha} = \frac{\partial \dot{\mathbf{x}}_l}{\partial \dot{q}^\alpha} = \frac{\partial \mathbf{x}_l}{\partial q^\alpha}. \tag{2.1.8}$$

In the last term of Equation (2.1.6),

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathbf{x}_l}{\partial q^\alpha} &= \frac{\partial^2 \mathbf{x}_l}{\partial q^\alpha \partial q^\beta}\dot{q}^\beta + \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_l}{\partial q^\alpha}\right) \\
&= \frac{\partial}{\partial q^\alpha}\left(\frac{\partial \mathbf{x}_l}{\partial q^\beta}\dot{q}^\beta + \frac{\partial \mathbf{x}_l}{\partial t}\right) \\
&= \frac{\partial \mathbf{v}_l}{\partial q^\alpha}
\end{aligned}$$

Inserting the last two equations into (2.1.6), multiplying by m_l, and summing over l, gives us

$$\begin{aligned}
\sum_{l=1}^{N} m_l \ddot{\mathbf{x}}_l \cdot \frac{\partial \mathbf{x}_l}{\partial q^\alpha} &= \sum_{l=1}^{N}\left[\frac{d}{dt}\left(m_l \mathbf{v}_l \cdot \frac{\partial \mathbf{v}_l}{\partial \dot{q}^\alpha}\right) - m_l \mathbf{v}_l \cdot \frac{\partial \mathbf{v}_l}{\partial q^\alpha}\right] \\
&= \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\alpha} - \frac{\partial T}{\partial q^\alpha},
\end{aligned}$$

where $T = \frac{1}{2}\sum_{l=1}^{N} m_l \mathbf{v}_l$ is the total kinetic energy of the system of particles. Then Equations (2.1.2) becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\alpha} - \frac{\partial T}{\partial q^\alpha} + \frac{\partial V}{\partial q^\alpha} = 0 \tag{2.1.9}$$

If T is written out in terms of the a^α, then these equations are the equations of motions in terms of the q^α. Since $\frac{\partial V}{\partial \dot{q}^\alpha} = 0$, as V depends only on the q^α, not the $\dot{q}^\alpha$, we can define a new function

$$L = T - V, \tag{2.1.10}$$

called the Lagrangian. Written in terms of the Lagrangian, these equations of motion become

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0, \tag{2.1.11}$$

which are known as Lagrange's equations.

2.2 Principle of Least Action

Lagrange's equations resemble equations found from the solution a variational problem, which means the system moves in such a way that maximizes or minimizes something. It turns out that the system minimizes what is known as the action, defined as

$$S \equiv \int L(q, \dot{q}, t)\, dt,$$

where L is the Lagrangian. Of all the possible motions that fix the endpoints, the actual physical motion is the one that minimizes this action.

Now consider an initial time t_0 and final time t_1. Then the action associated with this time interval is

$$S(q : t_0, t_1) \equiv \int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt.$$

When the equation for $q(t)$ is inserted into the expression for $L(q, \dot{q}, t)$, the above integrand becomes a function of only t, and can then be integrated. Because each $q(t)$ just has to have the same $q(t_0)$ and $q(t_1)$, there are many different trajectories that will satisfy this. Each of these different trajectories will yield a different value for S, and, as mentioned above, the physical trajectory will be the one that minimizes S.

2.3 Euler-Lagrange Equations

We will now show that minimizing the action will lead to the Euler-Lagrange equations.[2] Consider a family of trajectories $q(t, \varepsilon)$, all starting at $q(t_0)$ and ending at $q(t_1)$, where ε is an index that labels each particular trajectory of the family. Each $q(t, \varepsilon)$ leads to it own action, $S(\varepsilon)$. The physical trajectory is the one that leads to the smallest $S(\varepsilon)$, and, as per *Hamilton's Variational Principle*, that is independent of the way in which the ε-family of trajectories is chosen. Therefore, we can require each ε to be a real number and that they parameterize the family of trajectories continuously and differentiably. Thus, the partial derivative $\frac{\partial q(t;\varepsilon)}{\partial \varepsilon}$ exists for all values of t in the interval $[t_0, t_1]$. All calculations will depend on ε only through derivatives, so ε can be changed without loss of generality by adding an arbitrary constant, which we choose so $\varepsilon = 0$ is the trajectory that leads to the minimum $S(\varepsilon)$ in the family. The physical trajectory is the one that satisfies

$$\left.\frac{dS}{d\varepsilon}\right|_{\varepsilon=0} \equiv \left[\frac{d}{d\varepsilon}\int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt\right]_{\varepsilon=0} = 0 \tag{2.3.1}$$

From now on, we will abbreviate $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}$ as δ, so Equation (2.3.1) becomes

$$\delta S \equiv \delta \int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt = 0 \tag{2.3.2}$$

Next, we take the derivative of the integral with respect to ε. The integral depends on ε because of the fact that q and $\dot{q}$ in the Lagrangian depend on ε. Thus,

$$\delta S = \int_{t_0}^{t_1} \delta L\, dt \tag{2.3.3}$$

and

$$\delta L = \frac{\partial L}{\partial q^\alpha}\delta q^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha}\delta \dot{q}^\alpha. \tag{2.3.4}$$

$\dot{q}^\alpha \equiv \frac{dq^\alpha}{dt}$ is a time derivative taken for a fixed ε and is thus a partial derivative, which should be written as $\frac{\partial q^\alpha}{\partial t}$. However, for the sake of keeping with tradition of the final resolution of the problem, we will continue to write $\frac{dq^\alpha}{dt}$. Because it is a partial derivative, there is

[2] This derivation is adapted from [2].

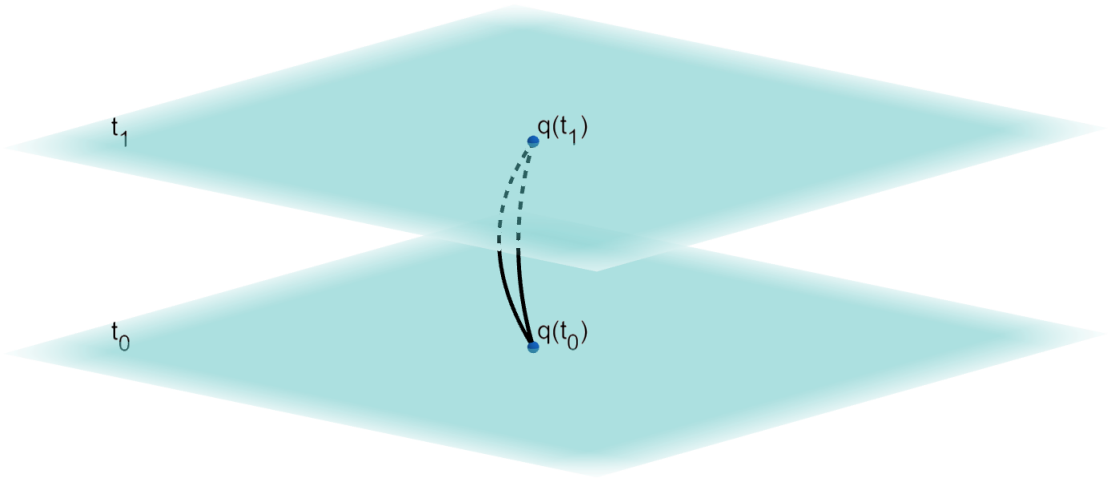

Figure 2.1: Family of trajectories, $q(t, \varepsilon)$, with fixed endpoints

no issue with changing the order of $\frac{d}{dt}$ and $\frac{\partial}{\partial \varepsilon}$ (i.e. changing the order of $\frac{d}{dt}$ and δ). Therefore,

$$\begin{aligned}\frac{\partial L}{\partial \dot{q}^\alpha}\delta\dot{q}^\alpha &= \frac{\partial L}{\partial \dot{q}^\alpha}\frac{d}{dt}q^\alpha \\ &= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^\alpha}\delta q^\alpha\right] - \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha}\right]\delta q^\alpha.\end{aligned}$$

Inserting this into the expression for δL gives us

$$\delta L = \left[\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha}\right]\delta q^\alpha + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^\alpha}\delta q^\alpha\right] \tag{2.3.5}$$

Now, we insert Equation (2.3.5) into Equation (2.3.3). We get

$$0 = \delta S = \int_{t_0}^{t_1}\left[\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha}\right]\delta q^\alpha\, dt + \int_{t_0}^{t_1}\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^\alpha}\delta q^\alpha\right]\, dt. \tag{2.3.6}$$

We can easily solve the second integral; it is

$$\frac{\partial L}{\partial \dot{q}^\alpha}\delta q^\alpha\Big|_{t_0}^{t_1} = 0,$$

which equals 0 because all trajectories are equal at the endpoints, so $\delta q^\alpha = 0$. The first term is then written as

$$\int_{t_0}^{t_1}\Lambda_\alpha \delta q^\alpha\, dt = 0, \tag{2.3.7}$$

where $\Lambda_\alpha = \frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha}$

Now we will use the following theorem. Suppose that f_a, $a = 1, 2, \ldots, n$ is a set of n integrable functions of a real variable t on the interval I. Furthermore suppose that

$$\int_I f_a h_a \, dt = 0$$

for every arbitrary set of integrable functions h_a on the same interval, all of which are equal to 0 at the end points. Then $f_a = 0$ for all a.

For us, every path q^α, no matter the ε, is equal at the end points, so $\delta q^\alpha = 0$ at the endpoints, no matter the α, just like h_a in the theorem. Because our integral is also always equal to 0, we can apply the theorem and see that $\Lambda_\alpha = 0$. Remembering what Λ_α is, we get

$$\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} = 0. \tag{2.3.8}$$

These are, of course, Lagrange's equations, and we see that they can be derived from minimizing the action.

2.4 Noether's Theorem

Noether's theorem states that if a Lagrangian is invariant under a family of transformations, then its dynamical system contains a constant of the motion, and that constant can be found from the Lagrangian and the transformation. Now assume that our Lagrangian, L, is invariant under the ε family of transformations. Then the derivative $\partial L/\partial \varepsilon = 0$. Now we use the fact that we obtained Equation (2.3.5) without any restrictions on ε, so we can use it now. To emphasize that the derivative with respect to ε is being taken, we will go back to writing δ as $\frac{\partial}{\partial \varepsilon}$. Then equation (2.3.5) becomes

$$\frac{\partial L_\varepsilon}{\partial \varepsilon} = \left[\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha}\right]\frac{\partial \psi(q^\alpha)}{\partial \varepsilon} + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^\alpha}\frac{\partial \psi(q^\alpha)}{\partial \varepsilon}\right], \tag{2.4.1}$$

where $\psi(q^\alpha)$ is what was previously called $q^\alpha(\varepsilon)$ (there is no need to have an ε subscript because L is invariant under the ε family of transformations).

Define $\psi_0(q(t)) \equiv q(t)$ and assume that $q(t)$ is a solution of the equations of motion. Then the Euler-Lagrange equations imply that the first term on the right side of Equation (2.4.1) is equal to 0, so that (the derivative at $\varepsilon = 0$ is written δ)

$$\delta L_\varepsilon = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^\alpha}\delta q^\alpha\right]. \tag{2.4.2}$$

We now use the condition that L is invariant under the ε family of transformations (i.e. $\frac{\partial L_\varepsilon}{\partial \varepsilon} = 0$). Then

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^\alpha} q^\alpha\right] = 0$$

or

$$\Gamma \equiv \frac{\partial L}{\partial \dot{q}^\alpha} q^\alpha \tag{2.4.3}$$

is a constant of motion.

3 Rigid Body Motion

Rigid bodies are extremely common in the real world, to the point that one can likely imagine what they are without needing a definition. Despite this, we begin is section by giving a more formal definition of rigid bodies. After doing so, we move onto discussing their kinetic energy and angular momentum. Next, we define two different types of coordinate systems, before finally moving onto the dynamics of a rigid body and the motion of its angular momentum vector as it travels through space.

3.1 Definition

A rigid body is a collection of point particles constrained so that the distance between any two points remains constant. If we consider a triangle, consisting of points, A, B, C, if the distance between each of those points does not change, then neither does the angle. Therefore, the triangle ABC moves rigidly through space. Because this holds true for any triangle, the same idea can be applied to objects consisting of much more than three points.

The configuration manifold, $\mathbb{Q}$, of a rigid body can be given by

1. The coordinates of an arbitrary point A in the rigid body (three coordinates)
2. The direction to another point B in the rigid body (two more coordinates)
3. The orientation of the plane containing A, B, and a third point in the rigid body, C.

Thus $\mathbb{Q}$ has six dimensions, three of which relate to the position of A, and another three which relate to the orientation of the body.

3.2 Kinetic Energy and Angular Momentum

The kinetic energy of a rigid body with mass density $\mu(\mathbf{x})$, where $\mathbf{x}$ is the position vector, is given by

$$T = \frac{1}{2}\int \dot{x}^2(\mathbf{x})\mu(\mathbf{x})d^3x \equiv \frac{1}{2}\int \dot{x}^2 dm, \tag{3.2.1}$$

where $dm = \mu(\mathbf{x})d^3x$. Continuing on, $\dot{\mathbf{x}} = \omega \times \mathbf{x}$, so we have

$$\begin{aligned}\dot{x}^2 &= (\omega \times \mathbf{x}) \cdot (\omega \times \mathbf{x}) \\ &= \det(\omega \times \mathbf{x}, \omega, \mathbf{x}) \\ &= \omega^T \left(|\mathbf{x}|^2 - \mathbf{x}\mathbf{x}^T\right)\omega\end{aligned}$$

Thus,

$$T = \frac{1}{2}\omega \cdot \mathbf{I}\omega, \tag{3.2.2}$$

where $\mathbf{I}$, the inertia tensor, is a 3×3 matrix and its elements, which depend only on the geometry of the body and the mass distribution, are

$$I_{jk} = \int \left(|\mathbf{x}|^2 - \mathbf{x}\mathbf{x}^T\right)\, dm. \tag{3.2.3}$$

The inertia tensor is to rotational motion what mass is to translational motion, so the equation $T = \frac{1}{2}\omega \cdot \mathbf{I}\omega$ is the rotational version of $T = \frac{1}{2}\mathbf{v} \cdot m\mathbf{v}$.

It is important to note that $\mathbf{I}$ has in inverse. Because the body is rotating, the ω_j are not all zero, and the kinetic energy is nonzero and positive. Thus,

$$\omega^T I\omega, \ \forall\omega \neq 0 \tag{3.2.4}$$

where the ω are the column vectors of $\mathbf{I}$ whose components are the ω_j. Therefore, the product of $\mathbf{I}$ and any nonzero column vector is nonzero itself, so it has an inverse.

It can be seen that $\mathbf{I}$ is a symmetric matrix (as $I_{jk} = I_{kj}$), which means that it can be diagonalized. Hence there is an orthogonal coordinate system whose basis vectors are the eigenvectors of $\mathbf{I}$. A coordinate system in which $\mathbf{I}$ is diagonal is called a principal-axis system, and the eigenvalues of $\mathbf{I}$ are the principal values, or moments of inertia of the body, labeled I_1, I_2, and I_3. In the principal-axis system, Equation (3.2.2) becomes

$$T = \frac{1}{2}\omega^T I\omega = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2), \tag{3.2.5}$$

where the ω_k are the components of the ω vector in the principal-axis system.

The angular momentum of a rigid body is given by

$$\mathbf{J} \equiv \int \mathbf{x} \times \dot{\mathbf{x}}\, dm = \int \mathbf{x} \times (\omega \times \mathbf{x})\, dm = \left[\int \left(|\mathbf{x}|^2 - \mathbf{x}\mathbf{x}^T\right)\, dm\right] \omega, \tag{3.2.6}$$

so

$$\mathbf{J} = \mathbf{I}\omega.$$

Just as $\frac{1}{2}\mathbf{I}\omega^2$ was the analog of $\frac{1}{2}m\mathbf{v}^2$, $\mathbf{I}\omega$ is the analog of $m\mathbf{v}$. In the principal axis system, $\mathbf{I}$ is diagonal, so the ith component of $\mathbf{J}$ is just $I_i\omega_i$.

3.3 Space and Body Systems

The next section will require us to use a different kind of coordinate system to avoid complicated calculations. To do so, we will need to define two coordinate systems, the space system and body system.

First we define the body system, $\mathcal{B}$. Its origin, A is at the center of mass of the object or at one of its inertial points (for us, it will be the center of mass we calculated earlier), and its orientation is chosen for convenience. The position vector, $\mathbf{x}_\mathcal{B}$, has fixed coordinates in $\mathcal{B}$, and, since they are fixed, $\dot{\mathbf{x}}_\mathcal{B} = 0$. Thus, it is not possible to describe the motion of the body in terms of the $\mathbf{x}_\mathcal{B}$.

Now we define the space system, $\mathcal{S}$. Similar to the body system, its origin is also at A and its orientation is also chosen for convenience, but the orientation is generally chosen to coincide with $\mathcal{S}$ at a specific time. The position vector in this system, $\mathbf{x}_\mathcal{S}$ does not always have a rate of change equal to 0, so it can be used to describe motion.

However, it is not as convenient to calculate the I_{jk} in $\mathcal{S}$, as they are constantly changing, unlike in $\mathcal{B}$. Because of this, we will show that the angular momentum vector, ω, is equivalent in both $\mathcal{B}$ and $\mathcal{S}$.

Equation (3.3.1), below, shows how the position vector varies in any system that's origin is at A. In particular, if the system is the $\mathcal{S}$ that coincides instantaneously with $\mathcal{B}$, then Equation (3.3.1) reads

$$\dot{\mathbf{x}}_{\mathcal{S}} = \omega_{\mathcal{B}} \times \mathbf{x}_{\mathcal{B}}. \tag{3.3.1}$$

The body components of ω and $\mathbf{x}$ can be used here because $\mathcal{S}$ and $\mathcal{B}$ coincide instantaneously, so they are the same in both $\mathcal{S}$ and $\mathcal{B}$. We can modify Equation (3.3.1) so that it applies to any vector, $\mathbf{s}$, in the body system, not just a fixed one by adding the term $\dot{\mathbf{s}}_{\mathcal{B}}$. This extra term tells how $\mathbf{s}$ moves with respect to the body system, so Equation (3.3.1) becomes

$$\dot{\mathbf{s}}_{\mathcal{S}} = \omega_{\mathcal{B}} \times \mathbf{s}_{\mathcal{B}} + \dot{\mathbf{s}}_{\mathcal{B}}. \tag{3.3.2}$$

Equation (3.3.2) tells how to transform velocity vectors between the space system and body system. So, specifically if $\mathbf{s}$ is the angular velocity, we get

$$\dot{\omega}_{\mathcal{S}} = \omega_{\mathcal{B}} \times \omega_{\mathcal{B}} + \dot{\omega}_{\mathcal{B}} = \dot{\omega}_{\mathcal{B}}, \tag{3.3.3}$$

thus showing that both ω and $\dot{\omega}$ are equal in $\mathcal{S}$ and $\mathcal{B}$, provided that the space system is chosen so that $\omega_{\mathcal{S}} = \omega_{\mathcal{B}}$ at time $t = 0$. Because of this, the subscripts for ω and $\dot{\omega}$ are no longer necessary and will be left off.

3.4 Dynamics

The dynamics of rigid body motion is specified by

$$\mathbf{N} = \dot{\mathbf{J}}, \tag{3.4.1}$$

which is the analog of $\mathbf{F} = \dot{\mathbf{p}} = m\mathbf{a}$. We note that the torque must be calculated about the center of mass or an inertial point. We also note that the Equation (3.4.1) requires finding the time derivative of $\mathbf{J} = \mathbf{I}\omega$ about one such point, which, as indicated by the calculation for the I_{jk} earlier, can end up quite complicated. To deal with this, we will use results proved in the previous section.

If $\mathbf{s}$ in Equation (3.3.2) is the angular momentum, then it becomes

$$\dot{\mathbf{J}}_{\mathcal{S}} = \omega \times \mathbf{J}_{\mathcal{B}} + \dot{\mathbf{J}}_{\mathcal{B}} = \omega \times (\mathbf{I}\omega) + \mathbf{I}\dot{\omega} \tag{3.4.2}$$

This removes the need to calculate the inertia tensor in the space system, which greatly simplifies the calculation. From Equation (3.4.1), we know that $\mathbf{N} = \dot{\mathbf{J}}$, so Equation (3.4.2) becomes

$$\dot{\mathbf{J}} = \mathbf{N} = \omega \times (\mathbf{I}\omega) + \mathbf{I}\dot{\omega}, \tag{3.4.3}$$

which means that $\omega \cdot \mathbf{N} = \omega \cdot \mathbf{I}\dot{\omega}$. (Note that $\omega \cdot (\omega \times \mathbf{I}\omega) = 0$.) The change in kinetic energy (in the principal axis system) is then given by $\dot{T} = \frac{1}{2}(\dot{\omega} \cdot \mathbf{I}\omega + \omega \cdot \mathbf{I}\dot{\omega}) = \dot{\omega} \cdot \mathbf{I}\omega$, using the symmetry of $\mathbf{I}$. Thus,

$$\dot{T} = \omega \cdot \mathbf{N}, \tag{3.4.4}$$

which is the analog of $\dot{T} = \mathbf{v} \cdot \mathbf{F}$.

We can now write out Equation (3.4.2) in the principal axis system:

$$\left.\begin{aligned} N_1 &= (I_3 - I_2)\omega_3\omega_2 + I_1\dot{\omega}_1 \\ N_2 &= (I_1 - I_3)\omega_1\omega_3 + I_2\dot{\omega}_2 \\ N_3 &= (I_2 - I_1)\omega_2\omega_1 + I_3\dot{\omega}_3 \end{aligned}\right\}. \tag{3.4.5}$$

These equations are known as Euler's equations for the motion of a rigid body. As can be seen, everything in these equations is calculated from the body system.

Now we look at a special case, where the rigid body is experiencing torque-free rotation (i.e. the body is free-falling in a gravitational field or it is in a gravity-free environment). In this case, the left hand side of each equation is equal to 0. Additionally, if the body is rotating about one of the principal axes, two of the angular velocities are equal to 0, so the first term on the right side of each equation disappears. This leaves us with just $I_k\dot{\omega}_k = 0$, so $\dot{\omega}_k = 0$ for $k = 1, 2, 3$, meaning the angular velocity is unchanging. The physical description of this that if the body starts rotating about one of its principal axes while in a torque-free environment, its rotation will remain unchanged. However, if the rotation is not about one of the principal axes, the first first term on the right side of the equations will not disappear, and the body will wobble, flip, or move in some other more complicated manner as it spins. This happens despite the fact that angular momentum is constant, as there is not torque on the body.

If we consider ω-space, the vector in space in which ω moves as the motion occurs, we now have three fixed points. We are eventually leading up to the stability of each of the fixed rays, which will be discussed in a later section.

3.5 Motion of the Angular Momentum Vector

Consider a freely rotating rigid body ($\mathbf{N} = \dot{\mathbf{J}} = 0$). The angular momentum vector, $\mathbf{J}$, then remains constant in the space system, but this is not necessarily true in the body system. In fact, by looking at Equation (3.4.2), we see that

$$\dot{\mathbf{J}}_{\mathcal{S}} = \omega \times \mathbf{J}_{\mathcal{B}} + \dot{\mathbf{J}}_{\mathcal{B}} \implies 0 = \omega \times \mathbf{J}_{\mathcal{B}} + \dot{\mathbf{J}}_{\mathcal{B}} \implies \dot{\mathbf{J}}_{\mathcal{B}} = -(\omega \times \mathbf{J}_{\mathcal{B}}),$$

so the angular momentum vector is only constant in the body system when it is parallel to the the angular velocity.

We know that $\mathbf{J} = \mathbf{I}\omega$, so if ω is an eigenvector of $\mathbf{I}$, the angular velocity is parallel to $\mathbf{J}$. Since that means $\dot{\mathbf{J}}_{\mathcal{B}}$ is equal to 0, if the body is spinning around one of its principal axes, $\mathbf{J}$ remains fixed in the body.

We can see this mathematically by looking two scalar quantities, $J^2 = \mathbf{J} \cdot \mathbf{J}$ and the kinetic energy $T = \frac{1}{2}\omega \cdot \mathbf{I}\omega$. Because they are scalars, they are equal in all systems. First, we rewrite T in terms of J by using $\omega = \mathbf{I}^{-1}J$:

$$T = \frac{1}{2}\mathbf{I}^{-1}\mathbf{J}_{\mathcal{B}} \cdot \mathbf{I}\mathbf{I}^{-1}\mathbf{J}_{\mathcal{B}} = \frac{1}{2}\mathbf{I}^{-1}\mathbf{J}_{\mathcal{B}} \cdot \mathbf{J}_{\mathcal{B}} = \text{const.}, \qquad J^2 = \text{const.} \tag{3.5.1}$$

The angular momentum in the body system satisfies both of these equations simultaneously.

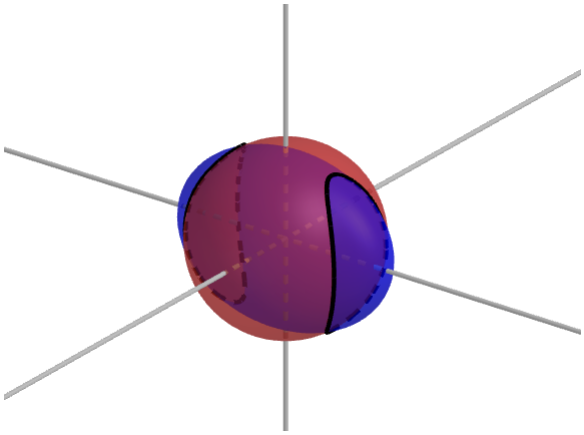

Figure 3.1: Intersection line on $\mathbb{S}$ and $\mathbb{E}$

The second equation in (3.5.1) implies that $\mathbf{J}_{\mathcal{B}}$ lives on a sphere, $\mathbb{S}$. Working with the first equation, and remembering we are working in the principal axis system, we see

$$\begin{aligned} T = \frac{1}{2}\mathbf{I}^{-1}\mathbf{J}_{\mathcal{B}} \cdot \mathbf{J}_{\mathcal{B}} &\iff 2T = \mathbf{I}^{-1}\mathbf{J}_{\mathcal{B}} \cdot \mathbf{J}_{\mathcal{B}} \\ &\iff 2T = \frac{J_1^2}{I_1} + \frac{J_2^2}{I_2} + \frac{J_3^2}{I_3} \\ &\iff \frac{J_1^2}{2TI_1} + \frac{J_2^2}{2TI_2} + \frac{J_3^2}{2TI_3} = 1. \end{aligned}$$

Thus, the first equation is the equation of ellipsoid, $\mathbb{E}$, which is the energy surface. In the next section we consider the motion of $\mathbf{J}_{\mathcal{B}}$ (although we will not use the subscript $\mathcal{B}$ because we will only be discussing angular momentum in the body system) on $\mathbb{E}$, which will tell us about the stability of the fixed points discussed above.

3.6 Fixed Points and Stability

Because $\mathbf{J}$ satisfies both equations in (3.5.1), it must lie on on the intersection of the sphere and ellipsoid mentioned above. These surfaces only intersect if $|J|$, the radius of $\mathbb{S}$, is larger than the smallest semiaxis of $\mathbb{E}$ and smaller than its largest semiaxis. We assume that the I_k are unequal and $I_1 > I_2 > I_3$. Then the three semiaxes are $\sqrt{2TI_1} > \sqrt{2TI_2} > \sqrt{2TI_3}$, so we have

$$\sqrt{2TI_1} > |J| > \sqrt{2TI_3}. \tag{3.6.1}$$

Thus, for a fixed T, there is a an upper and lower bound on the magnitude of the angular momentum.

In most cases, the sphere and ellipsoid will intersect at two curves; however, there are two special cases: when $|J|$ is at its minimum and when $|J|$ is at its maximum. In those two scenarios, the two surfaces intersect only at one point, the end of the smallest semiaxis for the minimum case and the end of the largest semiaxis for the maximum case. Because J moves only on the intersection curves, it is stuck at those two points. Thus, they are two of the fixed points discussed above.

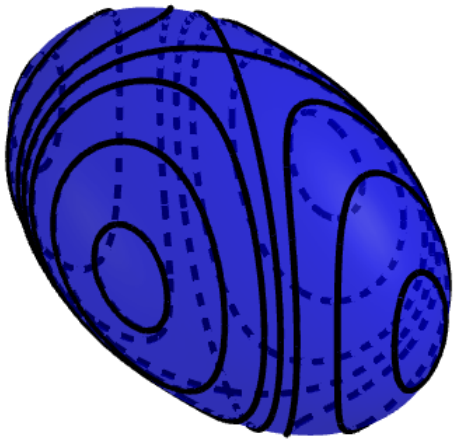

Figure 3.2: Motion of $\mathbf{J}$ on $\mathbb{E}$

As can be seen in Figure 3.2, the three fixed points are $J_{\text{min}} \equiv \sqrt{2TI_1}$, $J_{\text{int}} \equiv \sqrt{2TI_2}$, and $J_{\text{max}} \equiv \sqrt{2TI_3}$. By looking at the intersection curves, it is clear that J_{min} and J_{max} are stable fixed points; when J starts close to them, it stay close to them. On the other hand, J_{int} appears to be unstable, which is what will be shown next. To do so, we look back at Euler's equations (3.4.5).

This a torque-free case, so each N_j is zero. Because we are considering the fixed points, let J be close to one of the principal axes. This means that $\omega = \mathbf{I}^{-1}\mathbf{J}$ is also close to one of them, which means that the body is rotating about an axis close to one of principal axes. We will first look at the two stable axes, to see how they compare to the conclusions reached from the intersection curves on the ellipse.

Assume the rigid body is rotating about an axis close to x_1.[3] Then ω_2 and ω_3 are very small and can be approximated as 0. Plugging those values into the first equation, we get $\dot{\omega}_1 = 0$, or, in other words, $\dot{\omega}_1$ is constant. Then the second and third equations can be rearranged to read $\dot{\omega}_2 = \omega_1\omega_3(I_1 - I_3)/I_2$ and $\dot{\omega}_3 = \omega_1\omega_2(I_2 - I_1)/I_3$, respectively. Since we are considering ω_1 a constant, these are a pair of coupled equations for ω_2 and ω_3.

We can take their time derivatives, yielding $\ddot{\omega}_2 = \omega_1\dot{\omega}_3(I_1 - I_3)/I_2$ and $\ddot{\omega}_3 = \omega_1\dot{\omega}_2(I_2 - I_1)/I_3$ respectively. We can then substitute in the equations for $\dot{\omega}_1$ and $\dot{\omega}_2$ to decouple them, giving us

$$\ddot{\omega}_k = \frac{\omega_1^2(I_1 - I_3)(I_2 - I_1)}{I_2 I_3}\omega_k, \text{ for } k = 2, 3. \tag{3.6.2}$$

[3]This derivation is adapted from [2]

The factor multiplying ω_k on the right side of the equation is negative (remember $I_1 > I_2 > I_3$), so this is the equation for a harmonic oscillator. Them ω_2 and ω_3 oscillate harmonically about 0 with frequency given by

$$\omega_1 \sqrt{\frac{(I_1 - I_3)(I_2 - I_1)}{I_2 I_3}},$$

remaining small.

We get a similar result when the body is rotating about an axis very similar to I_3, verifying that the fixed points at $J_{\min}$ and $J_{\max}$ are stable. However, the case when the body is rotating about an axis similar to I_2 is different.

The equation we get corresponding to Equation (3.7.2) is

$$\ddot{\omega}_k = \frac{\omega_2^2 (I_2 - I_3)(I_1 - I_2)}{I_1 I_3}, \text{ for } k = 1, 3. \tag{3.6.3}$$

The difference is that the multiplying factor on the right hand side of the equation is now positive, making ω_1 and ω_3 to grow exponentially, which causes our approximation to break down. Thus, ω_2 does not remain constant, rather all three components of ω change significantly. This means the angular velocity vector moves far away from I_2 so the angular momentum vector does as well, confirming what appears to be true in Figure 2.

4 Example: T-Handle

This section first deals with the calculation of the $\mathbf{I}$ vector for a t-handle as it spins about itself in a torque-free environment. After making these calculations, we turn to analysing how changing the geometry of the body will change which of the three principal axes are stable and which one is unstable.

4.1 Moments of Inertia Calculation

In this section we will be looking at an example of a rigid body in motion: the spinning t-handle. For the sake of simplicity, our t-handle will be two cylinders, one standing vertically and the second laying horizontally along the top of the first cylinder. The two cylinders are simply connected at the point that they touch. Cylinder 1, the vertical one, has radius R_1 and height H_1, while Cylinder 2 has radius R_2 and height (length) H_2. Our t-handle also has constant density, $\mu(\mathbf{x}) = 1$.

We will eventually calculate the I_{jk} for this t-handle, but first we find the center of mass, as that will be the pivot point when it is rotating about itself. The center of mass in the x_i direction of an object is given by

$$CM_{x_i} = \frac{\iiint_V x_i \mu(\mathbf{x})\, dV}{\iiint_V \mu(\mathbf{x})\, dV}.^4 \tag{4.1.1}$$

[4]Integrating over V refers to integrating over the entire t-handle, while integrating over V_1 and V_2 refers to integrating over Cylinder 1 and Cylinder 2, respectively.

However, for us, $\mu(\mathbf{x}) = 1$, so Equation 4.1.1 becomes

$$CM_{x_i} = \frac{\iiint_V x_i \, dV}{\iiint_V \, dV}, \tag{4.1.2}$$

where the denominator is simply the volume, V, of the t-handle. Thus, the center of mass integral for each x_i can be written as

$$CM_{x_i} = \frac{1}{V} \iiint_V x_i \, dV = \frac{1}{V} \left(\iiint_{V_1} x_i \, dV_1 + \iiint_{V_2} x_i \, dV_2 \right). \tag{4.1.3}$$

To simplify integration, we will convert to a separate cylindrical coordinate system for Cylinder 1 and Cylinder 2. For Cylinder 1, we have

$$x_1 = r\cos(\theta), \; x_2 = r\sin(\theta), \; x_3 = z, \; dV = r \; dr \, d\theta \, dz,$$

and for Cylinder 2 we have

$$x_1 = \rho\cos(\phi), \; x_2 = y, \; x_3 = \rho\sin(\phi) + H_1 + R_2, \; dV = \rho \; d\rho \, d\phi \, dy$$

Now we have everything we need to begin calculating. The final equations will be shown in this section; however, the step-by-step calculation, beginning with Equation (4.1.3), will appear in the Appendix, rather than here. Beginning with the volume, we get

$$V = \pi R_1^2 H_1 + \pi R_2^2 H_2. \tag{4.1.4}$$

Now that we have the volume, we can move onto the centers of mass in the x_1, x_2, and x_3 direction. Those equations yield

$$CM_{x_1} = 0 \tag{4.1.5}$$

$$CM_{x_2} = 0 \tag{4.1.6}$$

$$CM_{x_3} = \frac{\frac{1}{2}\pi H_1^2 R_1^2 + \pi H_2 R_2^2 (H_1 + R_2)}{\pi R_1^2 H_1 + \pi R_2^2 H_2} \tag{4.1.7}$$

Thus, the coordinates of the center of mass of our t-handle are

$$\left(0, 0, \frac{\frac{1}{2}\pi H_1^2 R_1^2 + \pi H_2 R_2^2 (H_1 + R_2)}{\pi R_1^2 H_1 + \pi R_2^2 H_2} \right). \tag{4.1.8}$$

Now that we know our pivot point, we can begin the calculation for the I_{jk} for the t-handle. We use almost the same cylindrical coordinate system as we did for the center of mass calculation, but the origin is now located at the coordinates for the center of mass[5]. For Cylinder 1 we still get the same coordinate system:

$$x_1 = r\cos(\theta), \; x_2 = r\sin(\theta), \; x_3 = z, \; dV = r \; dr \, d\theta \, dz.$$

[5]Due to the length of the center of mass coordinate in the x_3 direction, we will denote it as C_3.

However, for Cylinder 2, we get something slightly different:

$$x_1 = \rho\cos(\phi),\ x_2 = y,\ x_3 = \rho\sin(\phi) + H_1 + R_2 - C_3,\ dV = \rho\, d\rho\, d\phi\, dy$$

After evaluating the integrals for I_{11}, I_{22}, and I_{33} and substituting in the true value of C_3, we get:

$$I_{11} = \frac{\pi\left(H_1^4R_1^4 + 4H_1^3H_2R_2^2R_1^2 + H_1H_2R_2^2R_1^2\left(H_2^2 + 3\left(R_1^2 + 5R_2^2\right)\right)\right)}{12\left(H_1R_1^2 + H_2R_2^2\right)} + \frac{\pi\left(H_2^2R_2^4\left(H_2^2 + 3R_2^2\right) + 3H_1^2\left(4H_2R_2^3R_1^2 + R_1^6\right)\right)}{12\left(H_1R_1^2 + H_2R_2^2\right)} \tag{4.1.9}$$

$$I_{22} = \frac{\pi\left(6H_2^2R_2^6 + 4H_1^3H_2R_1^2R_2^2 + 3H_1H_2R_1^2\left(R_1^2 + 6R_2^2\right)R_2^2\right)}{12\left(H_1R_1^2 + H_2R_2^2\right)} + \frac{\pi\left(H_1^4R_1^4 + 3H_1^2\left(4H_2R_2^3R_1^2 + R_1^6\right)\right)}{12\left(H_1R_1^2 + H_2R_2^2\right)} \tag{4.1.10}$$

$$I_{33} = \frac{1}{12}\pi\left(6H_1R_1^4 + 3H_2R_2^4 + H_2^3R_2^2\right) \tag{4.1.11}$$

Next, we do the same for the I_{jk} with $j \neq k$, yielding

$$I_{12} = I_{21} = I_{13} = I_{31} = I_{23} = I_{32} = 0. \tag{4.1.12}$$

We can now put the I_{jk} into the $\mathbf{I}$ matrix. Notice above that each of the nondiagonal entries are 0, so our coordinate system is already the principal-axis system. Thus,

$$\mathbf{I} = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}. \tag{4.1.13}$$

4.2 Changing Fixed Point Stability Calculation

In the previous section, we calculated the I_{jk} for our t-handle. In this section, we are going to look at how we can modify the dimensions of the t-handle to change which fixed points are stable and which one is unstable. Remembering that the three semiaxes on the energy ellipse, $\mathbb{E}$, are $\sqrt{2TI_1}$, $\sqrt{2TI_2}$, and $\sqrt{2TI_3}$, we see that the only variable in those equations are the I_j. The unstable axis is the middle one, so if changing the dimensions (H_1, R_1, H_2, and R_2) of the t-handle will change the I_{jk}, they could also change which semiaxis is the middle one, hence changing which one is unstable.

First, we require that R_1 and R_2 be equal to each other to simplify the calculation. That gives us

$$(0, 0, C_3) \tag{4.2.1}$$

for the center of mass.

Now, substituting R for R_1 and R_2 into the equations for the I_{jk}, we can simplify them to give:

$$I_1 = \frac{\pi R^2 \left(H_1 \left(18H_2R^2 + H_2^3\right) + 3H_2^2R^2 + 3H_1^2R \left(4H_2 + R\right) + H_1^4 + 4H_2H_1^3 + H_2^4\right)}{12 \left(H_1 + H_2\right)} \tag{4.2.2}$$

$$I_2 = \frac{\pi R^2 \left(21H_2H_1R^2 + 6H_2^2R^2 + 3H_1^2R \left(4H_2 + R\right) + H_1^4 + 4H_2H_1^3\right)}{12 \left(H_1 + H_2\right)} \tag{4.2.3}$$

$$I_3 = \frac{1}{12}\pi R^2 \left(3H_2R^2 + 6H_1R^2 + H_2^3\right). \tag{4.2.4}$$

Now we write H_1 and H_2 in terms of R, with $H_1 = aR$ and $H_2 = bR$. This means that when a is large, the height of Cylinder 1 is large compared to the radius, and, when a is small, the height of cylinder 1 is small compared to the radius. Similarly, a large b implies a long Cylinder 2, and a small b implies a short Cylinder 2. Substituting those into Equations 4.2.2, 4.2.3, and 4.2.4, we get

$$I_1 = \frac{\pi R^5 \left(a^4 + 4a^3b + 3a^2(4b + 1) + ab \left(b^2 + 18\right) + b^2 \left(b^2 + 3\right)\right)}{12(a + b)} \tag{4.2.5}$$

$$I_2 = \frac{\pi R^5 \left(a^4 + 4a^3b + 3a^2(4b + 1) + 21ab + 6b^2\right)}{12(a + b)} \tag{4.2.6}$$

$$I_3 = \frac{1}{12}\pi R^5 \left(6a + b^3 + 3b\right). \tag{4.2.7}$$

We are curious when each of the principal axes are the intermediate axis, so we want to look at when $I_2 < I_1 < I_3$ or $I_3 < I_1 < I_2$, when $I_1 < I_2 < I_3$ or $I_3 < I_2 < I_1$, and when $I_1 < I_3 < I_2$ or $I_2 < I_3 < I_1$. Each of Equations 4.2.5, 4.2.6, and 4.2.7 have a $\pi R^5/12$ term, so it can be divided out of them, and we get

$$I_1 = \frac{a^4 + 4a^3b + 3a^2(4b + 1) + ab \left(b^2 + 18\right) + b^2 \left(b^2 + 3\right)}{a + b} \tag{4.2.8}$$

$$I_2 = \frac{a^4 + 4a^3b + 3a^2(4b + 1) + 21ab + 6b^2}{a + b} \tag{4.2.9}$$

$$I_3 = 6a + b^3 + 3b. \tag{4.2.10}$$

It can be seen that these equations only depend on a and b, so we can plot the regions where each axis is the intermediate axis.

Figure 4.1 shows us this in a small window where $0 < a, b < 10$. The regions extend out past $a, b = 10$ in the manner in which it appears they should, with the intersections between each region remaining linear. As can be seen, it is possible for each of the axes to be unstable, depending on the dimensions of the t-handle. Overlayed on each of the regions in figure 4.1 is an image of how the t-handle might look when each axis is unstable, and which axis is the unstable one.

One interesting feature of the plot is that there is a small region between $a = 0$ and $a = 2$ and near $b = 0$ where x_2 is the intermediate axis, despite it appearing as though x_1 should be. While it seems as though this may be an error, Figure 4.2 shows that zooming in on that region reveals that is in fact correct, and there is also a small region in that area where x_3 is the intermediate axis.

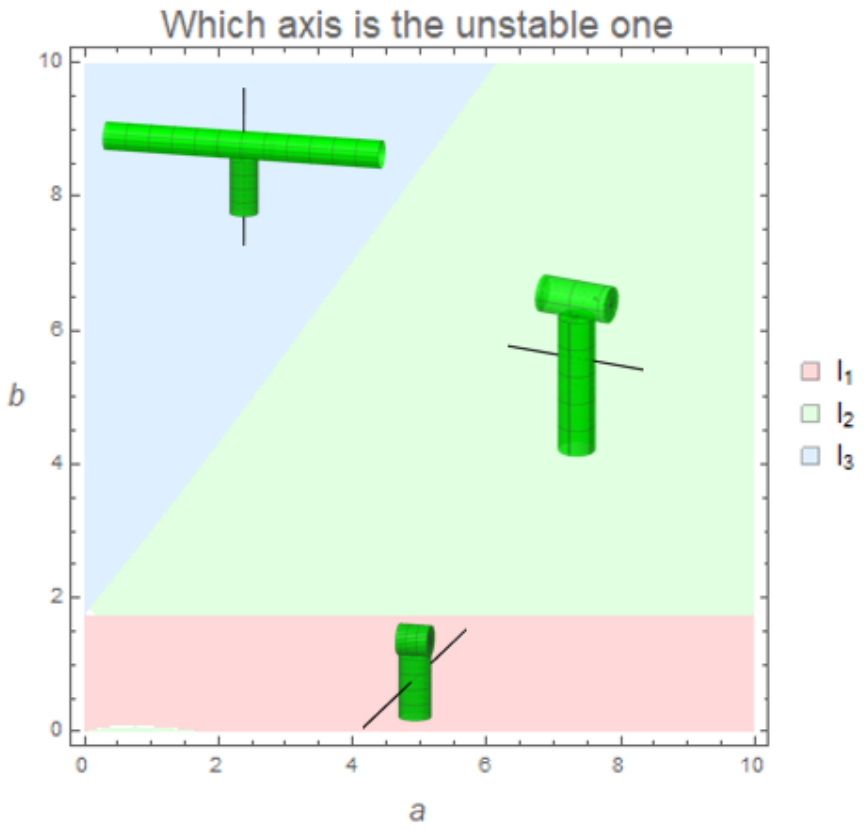

Figure 4.1

5 Appendix

Here we will show the calculation performed to obtain the values and equations in Section 4.

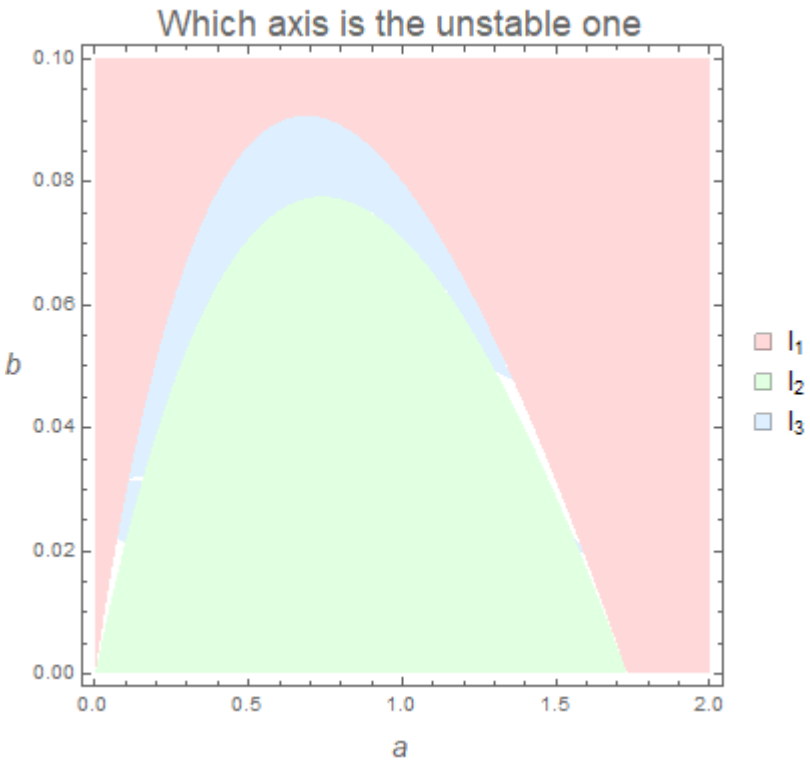

Figure 4.2

5.1 Calculation for Section 4.1

Equation (4.1.4):

$$\begin{aligned} V &= \iiint_V dV \\ &= \iiint_{V_1} dV_1 + \iiint_{V_2} dV_2 \\ &= \int_0^{R_1}\int_0^{2\pi}\int_0^{H_1} dz\, d\theta\, dr + \int_0^{R_2}\int_0^{2\pi}\int_{-\frac{H_2}{2}}^{\frac{H_2}{2}} dy\, d\phi\, d\rho \\ &= \pi R_1^2 H_1 + \pi R_2^2 H_2 \end{aligned}$$

Equation (4.1.5):

$$\begin{aligned}
CM_{x_1} &= \frac{1}{V}\left(\iiint_{V_1} x_1 \, dV_1 + \iiint_{V_2} x_1 \, dV_2\right) \\
&= \frac{1}{V}\left(\int_0^{2\pi}\int_0^{R_1}\int_0^{H_1} r^2\cos(\theta)\, dz\, dr\, d\theta + \int_0^{2\pi}\int_0^{R_2}\int_{-\frac{H_2}{2}}^{\frac{H_2}{2}} \rho^2\cos(\phi)\, dy\, d\rho\, d\phi\right) \\
&= \frac{1}{V}(0+0) \\
&= 0
\end{aligned}$$

Equation (4.1.6):

$$\begin{aligned}
CM_{x_2} &= \frac{1}{V}\left(\iiint_{V_1} x_2 \, dV_1 + \iiint_{V_2} x_2 \, dV_2\right) \\
&= \frac{1}{V}\left(\int_0^{2\pi}\int_0^{R_1}\int_0^{H_1} r^2\sin(\theta)\, dz\, dr\, d\theta + \int_0^{2\pi}\int_0^{R_2}\int_{-\frac{H_2}{2}}^{\frac{H_2}{2}} \rho y\, dy\, d\rho\, d\phi\right) \\
&= \frac{1}{V}(0+0) \\
&= 0
\end{aligned}$$

Equation (4.1.7):

$$\begin{aligned}
CM_{x_3} &= \frac{1}{V}\left(\iiint_{V_1} x_3 \, dV_1 + \iiint_{V_2} x_3 \, dV_2\right) \\
&= \frac{1}{V}\left(\int_0^{2\pi}\int_0^{R_1}\int_0^{H_1} rz\, dz\, dr\, d\theta + \int_0^{2\pi}\int_0^{R_2}\int_{-\frac{H_2}{2}}^{\frac{H_2}{2}} \rho(\rho\sin(\phi) + H_1 + R_2)\, dy\, d\rho\, d\phi\right) \\
&= \frac{1}{V}\left(\frac{1}{2}\pi H_1^2 R_1^2 + \pi H_2 R_2^2(H_1 + R_2)\right) \\
&= \frac{\frac{1}{2}\pi H_1^2 R_1^2 + \pi H_2 R_2^2(H_1 + R_2)}{\pi R_1^2 H_1 + \pi R_2^2 H_2}
\end{aligned}$$

Equation (4.1.9):

$$
\begin{aligned}
I_{11} &= \iiint_{V_1} \left(\delta_{11}x^2 - x_1x_1\right)\, dV_1 + \iiint_{V_2} \left(\delta_{11}x^2 - x_1x_1\right)\, dV_2 \\
&= \iiint_{V_1} x_2^2 + x_3^2\, dV_1 + \iiint_{V_2} x_2^2 + x_3^2\, dV_2 \\
&= \int_0^{2\pi}\int_0^{R_1}\int_{-C_3}^{H_1-C_3} r\left(r^2\sin^2(\theta) + z^2\right)\, dz\, dr\, d\theta \\
&\qquad + \int_0^{2\pi}\int_0^{R_2}\int_{-\frac{H_2}{2}}^{\frac{H_2}{2}} \rho\left(y^2 + (\rho\sin(\phi) + H_1 + H_2 - C_3)^2\right)\, dy\, d\rho\, d\phi \\
&= \frac{\pi\left(H_1^4R_1^4 + 4H_1^3H_2R_2^2R_1^2 + H_1H_2R_2^2R_1^2\left(H_2^2 + 3\left(R_1^2 + 5R_2^2\right)\right)\right)}{12\left(H_1R_1^2 + H_2R_2^2\right)} \\
&\qquad\qquad + \frac{\pi\left(H_2^2R_2^4\left(H_2^2 + 3R_2^2\right) + 3H_1^2\left(4H_2R_2^3R_1^2 + R_1^6\right)\right)}{12\left(H_1R_1^2 + H_2R_2^2\right)}
\end{aligned}
$$

Equation (4.1.10):

$$
\begin{aligned}
I_{22} &= \iiint_{V_1} \left(\delta_{22}x^2 - x_2x_2\right)\, dV_1 + \iiint_{V_2} \left(\delta_{22}x^2 - x_2x_2\right)\, dV_2 \\
&= \iiint_{V_1} x_1^2 + x_3^2\, dV_1 + \iiint_{V_2} x_1^2 + x_3^2\, dV_2 \\
&= \int_0^{2\pi}\int_0^{R_1}\int_{-C_3}^{H_1-C_3} r\left(r^2\cos^2(\theta) + z^2\right)\, dz\, dr\, d\theta \\
&\qquad + \int_0^{2\pi}\int_0^{R_2}\int_{-\frac{H_2}{2}}^{\frac{H_2}{2}} \rho\left(\rho^2\cos^2(\phi) + (\rho\sin(\phi) + H_1 + R_2 - C_3)^2\right)\, dy\, d\rho\, d\phi \\
&= \frac{\pi\left(6H_2^2R_2^6 + 4H_1^3H_2R_1^2R_2^2 + 3H_1H_2R_1^2\left(R_1^2 + 6R_2^2\right)R_2^2\right)}{12\left(H_1R_1^2 + H_2R_2^2\right)} \\
&\qquad\qquad + \frac{\pi\left(H_1^4R_1^4 + 3H_1^2\left(4H_2R_2^3R_1^2 + R_1^6\right)\right)}{12\left(H_1R_1^2 + H_2R_2^2\right)}
\end{aligned}
$$

Equation (4.1.11)

$$
\begin{aligned}
I_{33} &= \iiint_{V_1} \left(\delta_{33}x^2 - x_3x_3\right)\ dV_1 + \iiint_{V_2} \left(\delta_{33}x^2 - x_3x_3\right)\ dV_2 \\
&= \iiint_{V_1} x_1^2 + x_2^2\ dV_1 + \iiint_{V_2} x_1^2 + x_2^2\ dV_2 \\
&= \int_0^{2\pi}\int_0^{R_1}\int_{-C_3}^{H_1-C_3} r\left(r^2\cos^2(\theta) + r^2\cos^2(\theta)\right)\ dz\,dr\,d\theta \\
&\qquad + \int_0^{2\pi}\int_0^{R_2}\int_{-\frac{H_2}{2}}^{\frac{H_2}{2}} \rho\left(\rho^2\cos^2(\phi) + y^2\right)\ dy\,d\rho\,d\phi \\
&= \frac{1}{12}\pi\left(6H_1R_1^4 + 3H_2R_2^4 + H_2^3R_2^2\right)
\end{aligned}
$$

Equation (4.1.12):

$$
\begin{aligned}
I_{12} = I_{21} &= \iiint_V \left(\delta_{12}x^2 - x_1x_2\right)\ dV \\
&= \iiint_{V_1} -x_1x_2\ dV_1 + \iiint_{V_2} -x_1x_2\ dV_2 \\
&= -\int_0^{2\pi}\int_0^{R_1}\int_{-C_3}^{H_1-C_3} r^3\cos(\theta)\sin(\theta)\ dz\,dr\,d\theta - \int_0^{2\pi}\int_0^{R_2}\int_{-\frac{H_2}{2}}^{\frac{H_2}{2}} \rho^2 y\cos(\phi)\ dy\,d\rho\,d\phi \\
&= 0 - 0 = 0 \\
I_{13} = I_{31} &= \iiint_V \left(\delta_{13}x^2 - x_1x_3\right)\ dV \\
&= \iiint_{V_1} -x_1x_3\ dV_1 + \iiint_{V_2} -x_1x_3\ dV_2 \\
&= -\int_0^{2\pi}\int_0^{R_1}\int_{-C_3}^{H_1-C_3} r^2 z\cos(\theta)\ dz\,dr\,d\theta \\
&\qquad -\int_0^{2\pi}\int_0^{R_2}\int_{-\frac{H_2}{2}}^{\frac{H_2}{2}} \rho^2\cos(\phi)\left(H_1 - C_3 + R_2 + \rho\sin(\phi)\right)\ dy\,d\rho\,d\phi \\
&= 0 - 0 = 0 \\
I_{23} = I_{32} &= \iiint_V (\delta_{23}x^2 - x_2x_3)\ dV \\
&= \iiint_{V_1} -x_2x_3\ dV_1 + \iiint_{V_2} -x_2x_3\ dV_2 \\
&= -\int_0^{2\pi}\int_0^{R_1}\int_{-C_3}^{H_1-C_3} r^2 z\sin(\theta)\ dz\,dr\,d\theta \\
&\qquad -\int_0^{2\pi}\int_0^{R_2}\int_{-\frac{H_2}{2}}^{\frac{H_2}{2}} \rho y(H_1 - C_3 + R_2 + \rho\sin(\phi))\ dy\,d\rho\,d\phi \\
&= 0 - 0 = 0
\end{aligned}
$$

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