

9-2000

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Recommended Citation

Ćurgus, Branko; Gheondea, Aurelian; and Langer, Heinz, "On Singular Critical Points of Positive Operators in Krein Spaces" (2000). *Mathematics*. 8.

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ON SINGULAR CRITICAL POINTS OF POSITIVE OPERATORS IN KREIN SPACES

BRANKO ČURĀUS, AURELIAN GHEONDEA, AND HEINZ LANGER

(Communicated by David R. Larson)

ABSTRACT. We give an example of a positive operator B in a Krein space with the following properties: the nonzero spectrum of B consists of isolated simple eigenvalues, the norms of the orthogonal spectral projections in the Krein space onto the eigenspaces of B are uniformly bounded and the point ∞ is a singular critical point of B .

An operator A in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ is said to be *positive* if $[Ax, x] > 0$ for all nonzero x in the domain of A . A bounded positive operator A in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ has a projection valued spectral function E with 0 being its only possible critical point (see [1, Theorem IV.1.5] or [5, Section II.3.]). Recall that, by [5, Proposition 5.6], the condition

$$(1) \quad \|E((-\infty, \alpha])\| \leq C_- < \infty \text{ for all } \alpha < 0$$

is equivalent to the existence of the limit $\lim_{\alpha \uparrow 0} E((-\infty, \alpha])$ in the strong operator topology. Similarly,

$$(2) \quad \|E([\beta, \infty))\| \leq C_+ < \infty \text{ for all } \beta > 0$$

is equivalent to the existence of the limit $\lim_{\beta \downarrow 0} E([\beta, \infty))$ in the strong operator topology. Since 0 is not an eigenvalue of a positive operator A , [5, Proposition 3.2] implies that (1) and (2) are equivalent. Also, if 0 is a critical point, it is said to be *regular* if one of the conditions (1) or (2) is fulfilled. If the critical point 0 is not regular, it is called *singular*.

In the sequel the operator A considered will have a discrete spectrum outside 0. Examples of bounded positive operators in \mathcal{K} having 0 as a singular critical point can be constructed as follows (see also the examples in [2, Section 1], [3], [4]). Consider a sequence of two-dimensional Krein spaces $\mathcal{K}_n = \mathbb{C}^2$ with fundamental symmetry $J_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and positive operators A_n in \mathcal{K}_n ; denote by λ_n^+ (λ_n^- , respectively) its positive (negative, respectively) eigenvalues and by P_n^+ (P_n^- , respectively) the orthogonal (in \mathcal{K}_n) projection onto the corresponding eigenspace.

Received by the editors October 15, 1998.

2000 *Mathematics Subject Classification.* Primary 47B50, 46C50.

Key words and phrases. Krein space, definitizable operator, critical point.

The third author was supported by Fonds zur Förderung der wissenschaftlichen Forschung of Austria, Project P 12176 MAT.

If A_n is chosen such that $\|A_n\| \leq C$ for all n , $\lambda_n^+ \downarrow 0$, $\lambda_n^- \uparrow 0$, $\|P_n^\pm\| \rightarrow \infty$ if $n \rightarrow \infty$, then $A = \bigoplus_{n=1}^\infty A_n$ is a bounded positive operator in $\mathcal{K} = \bigoplus_{n=1}^\infty \mathcal{K}_n$ having 0 as a singular critical point. Evidently,

$$\sigma(A) = \{\lambda_n^+, \lambda_n^- | n \in \mathbb{N}\} \cup \{0\},$$

and $\|E(\{\lambda_n^\pm\})\| \rightarrow \infty$ if $n \rightarrow \infty$, that is, the eigenvectors f_n^+, f_n^- of A corresponding to λ_n^+ and λ_n^- , respectively, become arbitrarily close if n is large.

The question arises whether or not 0 can be a singular critical point of a positive operator A in \mathcal{K} with discrete spectrum $\{\lambda_n^+, \lambda_n^- | n \in \mathbb{N}\}$ in $\mathbb{C} \setminus \{0\}$ if the projections $E(\{\lambda_n^\pm\})$ are uniformly bounded. It is the aim of this note to show that the answer is yes: We will construct a bounded positive operator A in a Krein space \mathcal{K} , such that the projections $E(\{\lambda_n^\pm\})$ corresponding to the single eigenvalues are uniformly bounded but, nevertheless,

$$\|E(\{\lambda_1^\pm, \dots, \lambda_n^\pm\})\| \rightarrow \infty, \quad n \rightarrow \infty.$$

Our construction is based on the following two lemmas.

Lemma 1. *Let \mathcal{H}_n be an n -dimensional vector space with a positive definite scalar product (\cdot, \cdot) . Then there exist a basis f_{n1}, \dots, f_{nn} of \mathcal{H}_n and a positive contraction S_n in \mathcal{H}_n such that*

$$0 < 1 \leq \|f_{nk}\| \leq 2, \quad \|S_n^{-1}\| = n, \quad (S_n f_{nj}, f_{nk}) = \delta_{jk}, \quad j, k = 1, \dots, n.$$

Proof. Let e_{n1}, \dots, e_{nn} be an orthonormal basis of \mathcal{H}_n , let T_n be the selfadjoint transformation in \mathcal{H}_n given by $T_n e_{n1} = \sqrt{n} e_{n1}$, $T_n e_{nj} = e_{nj}$, $j = 2, \dots, n$, and put $S_n = T_n^{-2}$. Evidently, S_n is a positive selfadjoint contraction in \mathcal{H}_n , and $\min \sigma(S_n) = 1/n$. Therefore $\|S_n^{-1}\| = n$. Let $(u_{k1} \dots u_{kn})$, $k = 1, \dots, n$, be an orthonormal basis of the n -dimensional space of row vectors with components in \mathbb{C} , such that $u_{1j} = 1/\sqrt{n}$, $j = 1, \dots, n$. Then $U = (u_{kj})_{k,j=1}^n$ is a unitary matrix with $u_{1j} = 1/\sqrt{n}$, $j = 1, \dots, n$. Put

$$\phi_{nj} = \sum_{k=1}^n u_{kj} e_{nk}, \quad j = 1, \dots, n.$$

Then ϕ_{nj} , $j = 1, \dots, n$, is an orthonormal basis of \mathcal{H}_n and

$$\|T_n \phi_{nj}\|^2 = n \frac{1}{n} + \sum_{k=2}^n |u_{kj}|^2 = 1 + 1 - \frac{1}{n}, \quad j = 1, \dots, n.$$

Hence $1 \leq \|T_n \phi_{nj}\| \leq 2$. Let $f_{nj} = T_n \phi_{nj}$, $j = 1, \dots, n$. Then $1 \leq \|f_{nj}\| \leq 2$ and $(S_n f_{nj}, f_{nk}) = (\phi_{nj}, \phi_{nk}) = \delta_{jk}$, $j, k = 1, \dots, n$. The lemma is proved. \square

Lemma 2. *Let $(\mathcal{H}, (\cdot, \cdot))$ be a separable Hilbert space and let P be a positive, bounded and boundedly invertible operator in \mathcal{H} . Let ϕ_j , $j \in \mathbb{N}$, be a Riesz basis of \mathcal{H} such that $(P\phi_j, \phi_k) = \delta_{jk}$, $j, k \in \mathbb{N}$, and let $\lambda_j \in \mathbb{C}$, $j \in \mathbb{N}$, be a bounded sequence. Define the operator A in \mathcal{H} by $A\phi_j = \lambda_j \phi_j$, $j \in \mathbb{N}$. Then, A can be extended by continuity to a bounded linear operator in \mathcal{H} such that $\|A\| \leq \sqrt{\|P\| \|P^{-1}\|} \sup\{|\lambda_j|, j \in \mathbb{N}\}$.*

Proof. For a bounded and boundedly invertible positive operator P we have

$$(3) \quad \|P^{-1}\|^{-1}(x, x) \leq (Px, x) \leq \|P\|(x, x), \quad x \in \mathcal{H}.$$

Since the vectors $\phi_j, j \in \mathbb{N}$, are orthonormal with respect to the inner product $(P \cdot, \cdot)$, it follows that

$$(4) \quad (PAx, Ax) \leq (\sup\{|\lambda_j|, j \in \mathbb{N}\})^2(Px, x), \quad x \in \mathcal{H}.$$

Combining (3) and (4) we get

$$\begin{aligned} \|Ax\|^2 &= (Ax, Ax) \leq \|P^{-1}\|(PAx, Ax) \leq \|P^{-1}\|(\sup\{|\lambda_j|, j \in \mathbb{N}\})^2(Px, x) \\ &\leq \|P^{-1}\| \|P\|(\sup\{|\lambda_j|, j \in \mathbb{N}\})^2\|x\|^2 \end{aligned}$$

and the lemma follows. □

Theorem. *There exist a Krein space $(\mathcal{K}, [\cdot, \cdot])$ and a bounded positive operator A in \mathcal{K} with the following properties:*

- (a) *The nonzero spectrum of A consists of isolated simple eigenvalues.*
- (b) *The point 0 is a singular critical point of A .*
- (c) *The norms of the orthogonal projections in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ onto the eigenspaces of A are uniformly bounded.*

Proof. With the notation as in Lemma 1, choose $\mathcal{H}_n^+ = \mathcal{H}_n^- = \mathcal{H}_n$. Let $\mathcal{K}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$ be the direct sum of the Hilbert spaces $(\mathcal{H}_n^\pm, (\cdot, \cdot))$. The positive definite inner product on \mathcal{K}_n is also denoted by (\cdot, \cdot) . All norms in \mathcal{K}_n correspond to this inner product. Endow $\mathcal{K}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$ with the indefinite inner product $[\cdot, \cdot]$

given by the fundamental symmetry $J_n = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. Consider the operator

$K_n^+ = (I_n - S_n)^{1/2}$ acting from \mathcal{H}_n^+ into \mathcal{H}_n^- as an angular operator in \mathcal{K}_n . Here S_n is the operator constructed in Lemma 1. Let \mathcal{L}_n^+ be the graph of K_n^+ in $\mathcal{K}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$. Then \mathcal{L}_n^+ is an n -dimensional maximal positive subspace in \mathcal{K}_n . It is spanned by

the vectors $f_{nk}^+ = \begin{pmatrix} f_{nk} \\ K_n^+ f_{nk} \end{pmatrix}, k = 1, \dots, n$, and

$$(5) \quad [f_{nk}^+, f_{nj}^+] = (f_{nk}, f_{nj}) - (K_n^+ f_{nk}, K_n^+ f_{nj}) = (S_n f_{nk}, f_{nj}) = \delta_{kj},$$

$$(6) \quad \|f_{nk}^+\|^2 = \|f_{nk}\|^2 + \|K_n^+ f_{nk}\|^2 \leq 2\|f_{nk}\|^2 \leq 8.$$

Denote by \mathcal{L}_n^- the orthogonal complement of \mathcal{L}_n^+ in the Krein space \mathcal{K}_n . Then \mathcal{L}_n^- is a maximal negative subspace of \mathcal{K}_n . The operator $K_n^- = (I_n - S_n)^{1/2}$, acting from \mathcal{H}_n^- into \mathcal{H}_n^+ , is the angular operator of \mathcal{L}_n^- . The subspace \mathcal{L}_n^- is spanned by the

vectors $f_{nk}^- = \begin{pmatrix} K_n^- f_{nk} \\ f_{nk} \end{pmatrix}, k = 1, \dots, n$. This follows from the linear independence of f_{n1}, \dots, f_{nn} and the relation

$$(7) \quad \begin{aligned} [f_{nj}^+, f_{nk}^-] &= (f_{nj}, K_n^- f_{nk}) - (K_n^+ f_{nj}, f_{nk}) \\ &= (f_{nj}, (I - S_n)^{1/2} f_{nk}) - ((I - S_n)^{1/2} f_{nj}, f_{nk}) = 0. \end{aligned}$$

The decomposition $\mathcal{K}_n = \mathcal{L}_n^+ \dot{+} \mathcal{L}_n^-$ is a fundamental decomposition of $(\mathcal{K}_n, [\cdot, \cdot])$. Solving a corresponding system of vector equations we find that the orthogonal (fundamental) projections Q_n^\pm of the Krein space \mathcal{K}_n onto \mathcal{L}_n^\pm are given by

$$Q_n^+ = \begin{pmatrix} I_n \\ K_n^+ \end{pmatrix} S_n^{-1} \begin{pmatrix} I_n & -K_n^- \end{pmatrix}, \quad Q_n^- = \begin{pmatrix} K_n^- \\ I_n \end{pmatrix} S_n^{-1} \begin{pmatrix} -K_n^+ & I_n \end{pmatrix}.$$

From Lemma 1 it follows that $\|S_n^{-1}\| = n$. This and the above matrix representations of Q_n^\pm imply that

$$(8) \quad n \leq \|Q_n^\pm\| \leq 2n.$$

Consequently, for any $f \in \mathcal{K}_n$ we have

$$\|Q_n^\pm f\| \leq 2n\|f\| .$$

It follows from (5) that the vectors $f_{n1}^+, \dots, f_{nn}^+$ form an orthonormal basis in the Hilbert space $(\mathcal{L}_n^+, [\cdot, \cdot])$. Denote by

$$P_{nk}^+ = \frac{[\cdot, f_{nk}^+]}{[f_{nk}^+, f_{nk}^+]} f_{nk}^+, \quad k = 1, \dots, n ,$$

the orthogonal projection in the Krein space \mathcal{K}_n onto the subspace spanned by the vector $f_{nk}^+, k = 1, \dots, n$. Then, by (5) and (6),

$$(9) \quad 1 \leq \|P_{nk}^+\| = \frac{\|f_{nk}^+\|^2}{[f_{nk}^+, f_{nk}^+]} \leq 8, \quad k = 1, \dots, n .$$

Further, the operator

$$J_{n1} := Q_n^+ - Q_n^-$$

is a fundamental symmetry in $(\mathcal{K}_n, [\cdot, \cdot])$. In particular, the inner product

$$(x, y)_1 := [J_{n1}x, y], \quad x, y \in \mathcal{K}_n,$$

is positive definite. Therefore, the operator $J_n J_{n1}$ is positive and invertible in the Hilbert space $(\mathcal{K}_n, (\cdot, \cdot))$. Note also that $J_{n1} = J_{n1}^{-1}$. It follows from (8) that $\|J_{n1}\| = \|J_{n1}^{-1}\| \leq \|Q_n^+\| + \|Q_n^-\| \leq 4n$. Consequently,

$$(10) \quad \|J_n J_{n1}\| = \|(J_n J_{n1})^{-1}\| \leq 4n .$$

The vectors $f_{nj}^+, f_{nk}^-, j, k = 1, \dots, n$, are orthonormal in $(\mathcal{K}_n, (\cdot, \cdot)_1)$. This follows from (5), (7) and the relation

$$(f_{nj}^+, f_{nk}^-)_1 = [(Q_n^+ - Q_n^-)f_{nj}^+, f_{nk}^-] = [Q_n^+ f_{nj}^+, f_{nk}^-] = [f_{nj}^+, f_{nk}^-] = 0 .$$

Now we can apply Lemma 2 to the vectors $f_{nj}^+, f_{nk}^-, j, k = 1, \dots, n$, and the positive operator $J_n J_{n1}$: For given $\lambda_1^\pm, \dots, \lambda_n^\pm \in \mathbb{C}$ define an operator A_n by

$$A_n f_{nj}^\pm = \lambda_{nj}^\pm f_{nj}^\pm, \quad j = 1, \dots, n,$$

and then extend it by linearity to \mathcal{K}_n . It follows from Lemma 2 and (10) that

$$(11) \quad \|A_n\| \leq 4n \max\{|\lambda_j^\pm|, j = 1, \dots, n\} \leq 4C .$$

Let \mathcal{K} be the Krein space which is the direct orthogonal sum of the Krein spaces $\mathcal{K}_n, n \in \mathbb{N}$,

$$\mathcal{K} := \bigoplus_{n=1}^\infty \mathcal{K}_n .$$

The vectors $f_{nj}^\pm, j = 1, \dots, n, n \in \mathbb{N}$, constructed above are considered as elements of \mathcal{K} and the Krein spaces $\mathcal{K}_n, n \in \mathbb{N}$, are considered as mutually orthogonal subspaces of \mathcal{K} . The vectors $f_{nj}^\pm, j = 1, \dots, n$, form a basis for \mathcal{K}_n . Let $\lambda_{nj}^\pm, j = 1, \dots, n$, be distinct real numbers such that $\pm\lambda_{nj}^\pm > 0, j = 1, \dots, n$, and such that there exists a constant C with

$$(12) \quad n \max\{|\lambda_{nj}^\pm|, j = 1, \dots, n\} \leq C$$

for all $n \in \mathbb{N}$.

Put

$$A := \bigoplus_{n=1}^{\infty} A_n .$$

Then A is a positive operator in the Krein space $(\mathcal{K}, [\cdot, \cdot])$, and from (11) and (12) we get $\|A\| \leq 4C$. Since the linear span of the vectors f_{nj}^{\pm} , $j = 1, \dots, n$, $n \in \mathbb{N}$, is dense in \mathcal{K} , it follows from the spectral theorem (see [1, Theorem IV.1.5] or [5, Theorem 3.1]) that the nonzero spectrum of A consists of the simple eigenvalues λ_{nj}^{\pm} , $j = 1, \dots, n$, $n \in \mathbb{N}$. Consequently, the left-hand side of the inequality (8) implies that 0 is a singular critical point of A and the right-hand side of the inequality (9) implies that the norms of the orthogonal projections in $(\mathcal{K}, [\cdot, \cdot])$ onto the eigenspaces of A are uniformly bounded by 8. The theorem is proved. \square

Remark. We can arrange the numbers λ_{nj}^{\pm} , $j = 1, \dots, n$, $n \in \mathbb{N}$, in a lower triangular table. Also, we can put the sequence $\{\frac{1}{m}, m \in \mathbb{N}\}$ in a lower triangular table by ending each row with a triangular number $\frac{n(n+1)}{2}$ in the denominator. A comparison of these two tables leads to

$$(13) \quad \lambda_{nj}^{\pm} := \pm \left(\frac{n(n-1)}{2} + j \right)^{-1}, \quad j = 1, \dots, n, \quad n \in \mathbb{N} .$$

In this way we get

$$\{ \lambda_{nj}^{\pm}, j = 1, \dots, n, n \in \mathbb{N} \} = \left\{ \pm \frac{1}{m}, m \in \mathbb{N} \right\} .$$

The numbers λ_{nj}^{\pm} in (13) satisfy (12) with $C = 2$. The proof of the Theorem implies that the nonzero spectrum of the operator A , which was constructed by means of the numbers λ_{nj}^{\pm} from (13), consists of the simple eigenvalues $\pm \frac{1}{m}$, $m \in \mathbb{N}$.

If we consider the inverse $B = A^{-1}$ of the operator A from the previous theorem and with the specific choice of numbers λ_{nj}^{\pm} as in the Remark, we get:

Corollary. *There exist a Krein space $(\mathcal{K}, [\cdot, \cdot])$ and an unbounded positive operator B in \mathcal{K} with the following properties:*

- (a) *The nonzero spectrum of B consists of isolated simple eigenvalues.*
- (b) *The point ∞ is a singular critical point of B .*
- (c) *For each positive number μ we have*

$$\|E([a, b])\| \leq 8[\mu] \quad \text{whenever} \quad b - a < \mu ,$$

where E is the spectral function of B and $[\mu]$ denotes the largest integer smaller than μ .

Proof. Let A be the operator defined in the proof of the Theorem with the specific choice of the numbers $\pm \lambda_{nj}$ as in the Remark. Then $B = A^{-1}$ is a positive operator with a nonempty resolvent set (see e.g. [5, Proposition 3.1]), and $\sigma(B) = \mathbb{Z} \setminus \{0\}$. Let $\mu > 0$ be arbitrary and let $0 < b - a < \mu$. Then the interval $[a, b)$ contains at most $[\mu]$ eigenvalues of B . Therefore, $\|E([a, b])\| \leq 8[\mu]$. \square

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