9-1994

A Positive Answer to the Busemann-Petty Problem in 3 Dimensions

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Annals of Mathematics, **140** (1994), 435–447

A positive answer to the Busemann-Petty problem in three dimensions

By R.J. GARDNER*

Dedicated to Professor C.A. Rogers

ABSTRACT. We prove that in $\mathbb{E}^3$ the Busemann-Petty problem, concerning central sections of centrally symmetric convex bodies, has a positive answer. Together with other results, this settles the problem in each dimension.

1. Introduction

In [8], Busemann and Petty asked the following question, which resulted from reformulating a problem in Minkowskian geometry. Suppose K_1 and K_2 are convex bodies in n-dimensional Euclidean space $\mathbb{E}^n$ and are centrally symmetric with center at the origin, and such that

$$\lambda_{n-1}(K_1 \cap u^\perp) \leq \lambda_{n-1}(K_2 \cap u^\perp),$$

for all u in the unit sphere $\mathbb{S}^{n-1}$. Then is it true that

$$\lambda_n(K_1) \leq \lambda_n(K_2)?$$

(Here λ_k denotes k-dimensional Lebesgue measure; see Section 2 for other notation.)

The question, now generally known as the Busemann-Petty problem, has often appeared in the literature. More than 30 years ago, Busemann [7] gave the problem wide exposure, and Klee raised it again in [22]. The problem attracted the attention of those working in the local theory of Banach spaces; see, for example, the paper [25, p. 99] of Milman and Pajor. It surfaced again in Berger's article [1, p. 663], and it is also stated in the books [6, p. 154], [9, Problem A9, p. 22] and [30, p. 423]. Many papers have contributed partial

1991 *Mathematics Subject Classification.* 52A40, 28A75, 44A12, 52A15, 52A20, 52A30, 52A38, 92C55.
Supported in part by NSF Grant DMS 9201508.

solutions to the problem. We refer the reader to [11], [26], [33] and [35] for detailed historical comments, and confine ourselves here to a few remarks.

In [35], Zhang shows that the answer is negative for all $n \geq 4$. Since the question is trivial in $\mathbb{E}^2$ (the hypotheses imply that $K_1 \subset K_2$), our answer for $n = 3$ settles the problem in all dimensions. The unexpected positive nature of the solution stands in complete contrast to the situation in higher dimensions; this is especially interesting from the point of view of geometric tomography, in which one attempts to obtain information about a geometric object from data concerning its sections or projections. Geometric tomography has connections with functional analysis, and possible applications to robot vision and stereology (see [12] and the references given there).

Several previous papers concern the case $n = 3$. In [32], evidence is provided to indicate that the method found by Larman and Rogers [23] to construct examples in $\mathbb{E}^n$ for $n \geq 12$ will not work when $n = 3$. Bourgain [4] proves that his method, which provides examples for $n \geq 7$, fails for $n = 3$. Hadwiger [19] and Giertz [13] independently showed that the Busemann-Petty problem has an affirmative answer when K_1 and K_2 are coaxial centered convex bodies of revolution in $\mathbb{E}^3$. Of course, the present paper explains why techniques previously used to construct counterexamples in higher dimensions cannot succeed, and demonstrates that the extra assumptions in the theorems of Hadwiger and Giertz are quite redundant.

The most important concept in the solution for $n = 3$ (and, indeed, in Zhang's solution for $n \geq 4$) is that of an intersection body. This was introduced by Lutwak in [24], and it is now apparent that Lutwak's theorem [24, Theorem 10.1] represents the first step towards the full solution of the Busemann-Petty problem. The class of intersection bodies is, in a sense, dual to the better known class of projection bodies. The latter, which are just the centered zonoids, have been intensively studied and have many applications; see, for example, [5], [16], [31] and [30, Section 3.5]. In fact, the Busemann-Petty problem has a dual form, due to G.C. Shephard, in which sections are replaced by projections. Shephard's problem was solved, by Petty and Schneider independently, shortly after it was posed, using tools from the Brunn-Minkowski theory (see, for example, [30, p. 422]). Lutwak's theorem, dual to a corresponding theorem of Petty and Schneider, similarly utilizes the machinery of a dual Brunn-Minkowski theory. It is interesting to note that the answer to Shephard's problem is negative for all $n \geq 3$; so this paper provides a rare example of a result in the dual Brunn-Minkowski theory with a counterpart in the Brunn-Minkowski theory which is false.

Lutwak's theorem implies that the answer to the Busemann-Petty problem is affirmative if K_1 is an intersection body. Conversely, results in [11] and [34] imply that if there is a centered convex body in $\mathbb{E}^n$ which is not an intersection body, then the problem has a negative answer. In [11], the author proved that a centered cylinder in $\mathbb{E}^n$, $n \geq 5$, is not an intersection body, and Zhang [35] shows that a centered cube in $\mathbb{E}^n$, $n \geq 4$, is also not an intersection body.

In Theorem 4.1 of this paper we obtain a new inversion formula for a spherical Radon transform which is the radial function of a centered convex body. This enables us to conclude in Theorem 5.2 that a dense set of centered convex bodies in $\mathbb{E}^3$ consists of those which are the intersection body of a star body. (It is essential to work with star bodies in this context, since a centered cylinder in $\mathbb{E}^3$, for example, is the intersection body of a nonconvex star body; see [11, Remark 5.2(ii)].) We deduce in Corollary 5.3 that every centered convex body in $\mathbb{E}^3$ is an intersection body. The solution to the Busemann-Petty problem in $\mathbb{E}^3$ is an immediate corollary of either Theorem 5.2 or Corollary 5.3.

I am grateful to Professor R. Gorenflo and Dr. Vu Kim Tuan for the information presented in Section 3 concerning continuous solutions of the Abel integral equation, and to Professor Eric Grinberg for providing the proof of Proposition 2.3.

2. Preliminaries

We denote the unit sphere and closed unit ball in n-dimensional Euclidean space $\mathbb{E}^n$ by $\mathbb{S}^{n-1}$ and $\mathbb{B}$, respectively. If $u \in \mathbb{S}^{n-1}$, then $u^\perp$ is the subspace orthogonal to u. We write λ_k for k-dimensional Lebesgue measure, which we identify throughout with k-dimensional Hausdorff measure.

As is usual, we denote by C (or C^∞) the class of continuous (or infinitely differentiable, respectively) functions. By C_{e} or C_{e}^∞ we mean the even functions in these classes.

Suppose φ is the vertical angle for spherical polar coordinates in $\mathbb{E}^n$, that is, the angle between a vector and the positive x_n-axis. We say that a function f on $\mathbb{S}^{n-1}$ is *rotationally symmetric* (with respect to the x_n-axis) if its values depend only on φ.

A *convex body* is a compact convex set with nonempty interior. A set L is *star-shaped* at the origin if it contains the origin, and every line through the origin meets L in a (possibly degenerate) line segment. If L is star-shaped at the origin, its *radial function* ρ_L is defined by

$$\rho_L(u) = \sup\{c \geq 0 \colon\ cu \in L\},$$

for $u \in \mathbb{S}^{n-1}$. The radial function ρ_L can be extended to a function defined on $\mathbb{E}^n$ by requiring positive homogeneity of degree -1; we shall refer to this as the extended radial function of L. By a *star body* we mean a compact set which is star-shaped at the origin and whose radial function is continuous. We say a set is *centered* if it is centrally symmetric with center at the origin.

Suppose L is a star body of revolution. Then L is said to be *axis-convex* if each line parallel to its axis which meets it does so in a (possibly degenerate) line segment.

Let K be a convex body in $\mathbb{E}^n$, and $u \in \mathbb{S}^{n-1}$. Let l_u be the line parallel to u through the origin. For each $x \in l_u$, let D_x denote the (possibly degenerate) $(n-1)$-dimensional ball contained in the hyperplane $u^\perp + x$, with center at x and with λ_{n-1}-measure equal to that of $K \cap (u^\perp + x)$. (We take $D_x = \emptyset$ if $K \cap (u^\perp + x) = \emptyset$.) The union of all the sets D_x for $x \in l_u$ is called the *Schwarz symmetral* of K in the direction u. It follows directly from the Brunn-Minkowski theorem that a Schwarz symmetral is a convex body of revolution with axis l_u; see, for example, [3, Section 41].

Suppose $g \in C(\mathbb{S}^{n-1})$, and f is defined by

$$f(u) = \int_{\mathbb{S}^{n-1} \cap u^\perp} g(v)\, d\lambda_{n-2}(v),$$

for all $u \in \mathbb{S}^{n-1}$; that is, $f(u)$ is the integral of g over the great sphere in $\mathbb{S}^{n-1}$ orthogonal to u. Then we write

$$f = Rg,$$

and say that f is the *spherical Radon transform* of g. The following useful fact is known about R (see [21, p. 161]).

PROPOSITION 2.1. *Suppose $f \in C_e^\infty(\mathbb{S}^{n-1})$. Then there is a $g \in C_e^\infty(\mathbb{S}^{n-1})$ such that $f = Rg$.*

It is also known (see [21, p. 144]) that R is self-adjoint, in the sense that for f and g in $C_e(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} f(u)(Rg)(u)\, d\lambda_{n-1}(u) = \int_{\mathbb{S}^{n-1}} (Rf)(u) g(u)\, d\lambda_{n-1}(u).$$

From this and the fact that by Proposition 2.1, the range of R is dense in $C_e(\mathbb{S}^{n-1})$, it follows that R is injective on $C_e(\mathbb{S}^{n-1})$. (Direct proofs of the injectivity are given in [20], [27] and [28].)

The next proposition is proved in [18, p. 193] for the Radon transform on complex projective space. For the convenience of the reader, we provide the details for the case which interests us here.

PROPOSITION 2.2. *The spherical Radon transform commutes with rotations.*

Proof. Let $\Phi \in \mathrm{SO}_n$, $f \in C(\mathbb{S}^{n-1})$ and $u \in \mathbb{S}^{n-1}$. Then

$$\Phi(Rf)(u) = Rf(\Phi^{-1}u) = \int_{\mathbb{S}^{n-1}\cap(\Phi^{-1}u)^\perp} f(v)\, d\lambda_{n-2}(v).$$

Substitute $w = \Phi v$. Note that $\Phi^{-1}u \cdot v = 0$ if and only if $u \cdot \Phi v = u \cdot w = 0$. Using the SO_n-invariance of λ_{n-2}, the last integral becomes

$$\int_{\mathbb{S}^{n-1}\cap u^\perp} f(\Phi^{-1}w)\, d\lambda_{n-2}(w) = R(\Phi f)(u),$$

as required. □

Suppose that $f \in C(\mathbb{S}^2)$. For $0 \le \varphi \le \pi$, denote by $\tilde{f}(\varphi)$ the average of f over the circle of latitude with angle φ from the north pole in $\mathbb{S}^2$; thus

$$\tilde{f}(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta,\varphi)\, d\theta.$$

We identify $\tilde{f}$ with its natural extension to a rotationally symmetric function on $\mathbb{S}^2$. The following observation was made by P. Funk in [10, VI, p. 285].

PROPOSITION 2.3. *If f, $g \in C(\mathbb{S}^2)$ and $f = Rg$, then $\tilde{f} = R\tilde{g}$.*

Proof. We have

$$\tilde{f}(\varphi) = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m f(\theta_i,\varphi),$$

for suitable choices of θ_i, $1 \le i \le m$, and a similar expression for $\tilde{g}(\varphi)$. Denote also by θ_i the rotation by angle θ_i about the z-axis. Then, using Proposition 2.2, we have

$$\begin{aligned} f(\theta_i,\varphi) &= (\theta_i^{-1}f)(0,\varphi) \\ &= \big(\theta_i^{-1}(Rg)\big)(0,\varphi) = \big(R(\theta_i^{-1}g)\big)(0,\varphi). \end{aligned}$$

Consequently,

$$\begin{aligned} \tilde{f}(\varphi) &= \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m \big(R(\theta_i^{-1}g)\big)(0,\varphi) \\ &= R\left(\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m (\theta_i^{-1}g)(0,\varphi)\right) \\ &= R\left(\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m g(\theta_i,\varphi)\right) \\ &= R\tilde{g}(\varphi). \end{aligned}$$

□

Using Proposition 2.1, one can extend the spherical Radon transform R in $\mathbb{E}^n$ to a continuous linear map from the space of finite signed Borel measures in $\mathbb{S}^{n-1}$ into itself; see the discussion in [15]. A star body L in $\mathbb{E}^n$ is called an *intersection body* if $\rho_L = R\mu$, where μ is an even (nonnegative) Borel measure in $\mathbb{S}^{n-1}$. Any such L is centered.

The latter definition is introduced in [14]. We shall almost always work with a special case of this, which corresponds to the original definition of an intersection body in [24]. We say that a star body L is the *intersection body of a star body* M if

$$\rho_L(u) = \lambda_{n-1}(M \cap u^{\perp}),$$

for all $u \in \mathbb{S}^{n-1}$. We write $L = IM$. It is easy to see, using the polar coordinate formula for volume, that $L = IM$ for some M if and only if $\rho_L = Rg$ for some nonnegative continuous function g; just take $g = \rho_M^{n-1}/(n-1)$. It is known (see [24, (8.3)]) that if $L = IM$, then there is a unique such M which is also centered.

The following theorem is from [11, Theorem 3.1] (see also [35]). Most of the work is done in Lutwak's theorems [24, Theorems 10.1 and 12.2].

THEOREM 2.4. *Let $n \geq 3$. The Busemann-Petty problem has a positive answer in $\mathbb{E}^n$ if and only if each centered convex body K in $\mathbb{E}^n$, with everywhere positive Gaussian curvature and $\rho_K \in C_{\mathrm{e}}^{\infty}(\mathbb{S}^{n-1})$, is the intersection body of a star body.*

Zhang [34] has noted that it is possible to prove the following refinement of the previous theorem.

THEOREM 2.5. *Let $n \geq 3$. The Busemann-Petty problem has a positive answer in $\mathbb{E}^n$ if and only if each centered convex body K in $\mathbb{E}^n$ is an intersection body.*

3. Some inversion formulae

In this section we study some known inversion formulae for the spherical Radon transform in three dimensions.

Suppose that $f \in C_{\mathrm{e}}(\mathbb{S}^2)$ is rotationally symmetric with respect to the z-axis, and $f = Rg$ for $g \in C_{\mathrm{e}}(\mathbb{S}^2)$. Then, using the facts (see Section 2) that R is injective on $C_{\mathrm{e}}(\mathbb{S}^2)$ and commutes with rotations, we see that g is also rotationally symmetric with respect to the z-axis. Let $u \in \mathbb{S}^2$, and suppose the angle between u and the z-axis is $\frac{\pi}{2} - \psi$. Let φ be the vertical angle in spherical polar coordinates, and suppose that $x = \cos\psi$ and $t = \cos\varphi$. Then the equation $f = Rg$ becomes

$$f(\sin^{-1} x) = 4\int_0^x \frac{g(\cos^{-1} t)}{\sqrt{x^2 - t^2}}\, dt, \tag{1}$$

for $0 < x \leq 1$, and $f(\sin^{-1} 0) = 2\pi g(\cos^{-1} 0)$. (This is explained in detail in [11, Section 4].) Conversely, if $g \in C_{\mathrm{e}}(\mathbb{S}^2)$ is rotationally symmetric and satisfies (1), then $f = Rg$.

Equation (1) is an Abel integral equation, which can be solved by standard techniques. Suppose we define h by $h(x) = f(\sin^{-1} x)$, $0 \leq x \leq 1$. The existence and integrability of a solution g, unique in $L^1(0,1)$, was proved by L. Tonelli, under the assumption that h is absolutely continuous; see Chapter 1 of [17], especially Theorem 1.2.1. One form of the solution is (cf. [17, (1.B.5i), p. 24])

$$g(\cos^{-1} t) = \frac{1}{2\pi}\frac{d}{dt}\int_0^t \frac{x f(\sin^{-1} x)}{\sqrt{t^2 - x^2}}\, dx, \tag{2}$$

for $0 < t \leq 1$, and $g(\cos^{-1} 0) = f(\sin^{-1} 0)/2\pi$. An alternative form of the solution of (1) is (cf. [17, (1.B.5ii), p. 24])

$$g(\cos^{-1} t) = \frac{1}{2\pi}\left(f(0) + t\int_0^t \frac{f'(\sin^{-1} x)}{\sqrt{(t^2 - x^2)(1 - x^2)}}\, dx\right), \tag{3}$$

for $0 \leq t \leq 1$. The existence of a unique *continuous* g of either form follows from the assumption that the function h defined above satisfies $h \in C^1([0,1])$. Indeed, by substituting (3) into (1), we see that

$$\begin{aligned} 4\int_0^x \frac{g(\cos^{-1} t)}{\sqrt{x^2 - t^2}}\, dt &= \frac{2}{\pi}\int_0^x \frac{f(0)\, dt}{\sqrt{x^2 - t^2}} \\ &\quad + \frac{2}{\pi}\int_0^x \frac{t}{\sqrt{x^2 - t^2}}\int_0^t \frac{f'(\sin^{-1} y)}{\sqrt{(t^2 - y^2)(1 - y^2)}}\, dy\, dt \\ &= f(0) + \frac{2}{\pi}\int_0^x \frac{f'(\sin^{-1} y)}{\sqrt{1 - y^2}}\int_y^x \frac{t}{\sqrt{(x^2 - t^2)(t^2 - y^2)}}\, dt\, dy \\ &= f(0) + \int_0^x \frac{f'(\sin^{-1} y)}{\sqrt{1 - y^2}}\, dy = f(\sin^{-1} x), \end{aligned}$$

so that (3) is a solution of (1). (It is also not difficult to obtain (3) from (2), by means of integration by parts and differentiating under the integral sign.) According to [17, Theorem 5.1.5], with $m = 1$, $K(x,t) \equiv 1$, $\beta = 0$, $\mu = 2$ and $\alpha = 1/2$, there is a continuous solution g of (1) if $h \in C^1([0,1])$; since the solution is unique, we conclude that the g given by (2) or by (3) is continuous for $0 \leq t \leq 1$. (A direct proof of the continuity of g is fairly straightforward using (3).)

Setting $t=1$ in (3), and substituting $x=\sin\varphi$, we obtain

$$(4)\qquad 2\pi g(0)=f(0)+\int_0^{\frac{\pi}{2}} f'(\varphi)\sec\varphi\,d\varphi.$$

Let us now suppose that $f=Rg$, for arbitrary $f\in C_{\mathrm{e}}(\mathbb{S}^2)$ and $g\in C_{\mathrm{e}}(\mathbb{S}^2)$. Then $\tilde{f}=R\tilde{g}$, by Proposition 2.3. Denote the north pole in $\mathbb{S}^2$ by u_0. When (4) is valid, we can use it with f and g replaced by $\tilde{f}$ and $\tilde{g}$, respectively, together with the equalities $f(u_0)=\tilde{f}(0)$ and $g(u_0)=\tilde{g}(0)$, to obtain

$$2\pi g(u_0)=f(u_0)+\int_0^{\frac{\pi}{2}} \tilde{f}'(\varphi)\sec\varphi\,d\varphi.$$

Substituting for $\tilde{f}$, and changing the order of integration, we arrive at the formula

$$(5)\qquad 2\pi g(u_0)=f(u_0)+\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\frac{\partial f(\theta,\varphi)}{\partial\varphi}\sec\varphi\,d\varphi\,d\theta.$$

It seems that (5) was familiar to Blaschke; see [2, p. 155].

4. A new inversion formula

THEOREM 4.1. *Let K be a centered, strictly convex body in $\mathbb{E}^3$ with $\rho_K\in C^1_{\mathrm{e}}(\mathbb{S}^2)$, and suppose that $\rho_K=Rg$ for some $g\in C_{\mathrm{e}}(\mathbb{S}^2)$. Denote by u_0 the north pole in $\mathbb{S}^2$. Then*

$$g(u_0)=-\frac{1}{4\pi^2}\int_0^{\infty}\frac{A'_K(z)}{z}\,dz,$$

where $A_K(z_0)$ denotes the area of the intersection of K with the plane $z=z_0$.

Proof. For each θ_0, let K_{θ_0} be the intersection of K with the quarter-plane $\{(\rho,\theta,\varphi)\colon\ \theta=\theta_0, 0\le\varphi\le\pi/2\}$ containing the z-axis for $z\ge 0$, where (ρ,θ,φ) are spherical polar coordinates (with $\rho\ge 0$ as usual). For each fixed θ, there is a unique point $x^*=x^*(\theta)$ in the boundary of K_θ which has maximum height $z^*=z^*(\theta)$. If $x^*=(\rho^*,\theta,\varphi^*)$, then x^* divides the part of the boundary of K_θ contained in the boundary of K into two arcs; namely, that for which $0\le\varphi\le\varphi^*$ (which may be degenerate), and that for which $\varphi^*\le\varphi\le\pi/2$. These arcs can be described in cylindrical polar coordinates (r,θ,z) by

$$r_1=r_1(\theta,z),\quad \rho_K(u_0)\le z\le z^*,$$

and

$$r_2=r_2(\theta,z),\quad 0\le z\le z^*.$$

For $j = 1, 2$, extend the domain of r_j by defining $r_j(\theta, z) = 0$ for all $z \geq 0$ outside these intervals. Note that

$$A_K(z) = \begin{cases} \frac{1}{2}\int_0^{2\pi} r_2^2(\theta, z)\, d\theta & \text{if } 0 \leq z \leq \rho_K(u_0) \\ \frac{1}{2}\int_0^{2\pi} \left(r_2^2(\theta, z) - r_1^2(\theta, z)\right) d\theta & \text{if } \rho_K(u_0) \leq z. \end{cases}$$

By (5), we have

$$2\pi g(u_0) = \rho_K(u_0) + \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{\partial \rho_K(\theta, \varphi)}{\partial \varphi} \sec\varphi\, d\varphi\, d\theta. \tag{6}$$

We transform this expression to cylindrical polar coordinates. Let us again fix θ. For $0 \leq \varphi \leq \varphi^*$, we have $\rho^2 = r_1^2 + z^2$ and $\tan\varphi = r_1/z$, while for $\varphi^* \leq \varphi \leq \pi/2$, we have $\rho^2 = r_2^2 + z^2$ and $\tan\varphi = r_2/z$. Therefore

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\partial \rho_K}{\partial \varphi} \sec\varphi\, d\varphi &= \int_0^{\varphi^*} \frac{\partial \rho_K}{\partial \varphi} \sec\varphi\, d\varphi + \int_{\varphi^*}^{\frac{\pi}{2}} \frac{\partial \rho_K}{\partial \varphi} \sec\varphi\, d\varphi \\ &= \int_{\rho_K(u_0)}^{\infty} \left(\frac{1}{2z}\frac{\partial}{\partial z}(r_1^2) + 1\right) dz - \int_0^{\infty} \left(\frac{1}{2z}\frac{\partial}{\partial z}(r_2^2) + 1\right) dz \\ &= -\int_0^{\rho_K(u_0)} \frac{1}{2z}\frac{\partial}{\partial z}(r_2^2)\, dz \\ &\quad - \int_{\rho_K(u_0)}^{\infty} \frac{1}{2z}\frac{\partial}{\partial z}(r_2^2 - r_1^2)\, dz - \rho_K(u_0). \end{aligned}$$

Substituting into (6), we now obtain

$$\begin{aligned} 2\pi g(u_0) = -\frac{1}{2\pi}\int_0^{2\pi} \Bigg(&\int_0^{\rho_K(u_0)} \frac{1}{2z}\frac{\partial}{\partial z}\left(r_2^2(\theta, z)\right) dz \\ &+ \int_{\rho_K(u_0)}^{\infty} \frac{1}{2z}\frac{\partial}{\partial z}\left(r_2^2(\theta, z) - r_1^2(\theta, z)\right) dz \Bigg)\, d\theta. \end{aligned}$$

The theorem follows from a change in the order of integration and the above expression for $A_K(z)$. □

5. The main results

The following theorem is a slight improvement of a similar result (see [11, Theorem 5.1]).

THEOREM 5.1. *Let L be a centered axis-convex body of revolution in $\mathbb{E}^3$ such that $\rho_L \in C_e^1(\mathbb{S}^2)$ and the curvature of L exists on the intersection of the boundary of L with the plane through the origin orthogonal to the axis of L. Then L is the intersection body of a star body.*

Proof. We have to prove the existence of a nonnegative $g \in C_{\mathrm{e}}(\mathbb{S}^2)$ such that $\rho_L = Rg$. From Section 3 we know that this is equivalent to the existence of a nonnegative $g \in C_{\mathrm{e}}(\mathbb{S}^2)$ which satisfies (1), with f replaced by ρ_L. We may assume that the axis of revolution of L is the vertical axis, so that $\rho_L = \rho_L(\varphi)$, where φ is the vertical spherical polar coordinate. Let $x = \sin\varphi$, and define h by $h(x) = \rho_L(\sin^{-1} x)$, for $0 \le x \le 1$. Then h is continuously differentiable on $[0, 1)$. Furthermore,

$$\lim_{x\to 1-} h'(x) = \lim_{x\to 1-} \frac{\rho_L'(\sin^{-1} x)}{\sqrt{1-x^2}} = -\rho_L''(\pi/2),$$

which exists by our assumptions, so $h \in C^1([0,1])$. From Section 3, we know that this implies that a $g \in C_{\mathrm{e}}(\mathbb{S}^2)$ exists of the form given by (2), so that

$$g(\cos^{-1} t) = \frac{1}{2\pi}\frac{d}{dt}\int_0^t \frac{x\rho_L(\sin^{-1} x)}{\sqrt{t^2-x^2}}\,dx,$$

for $0 < t \le 1$, and $g(\cos^{-1} 0) = \rho_L(\sin^{-1} 0)/2\pi$.

We have to show that g is nonnegative, and this will follow if the integral increases with t. As in [11, Theorem 5.1], we substitute $s = x/t$, and the integral becomes

$$\int_0^1 \frac{st\rho_L\left(\sin^{-1}(st)\right)}{\sqrt{1-s^2}}\,ds.$$

But the axis-convexity of L means that $\sin\varphi\rho_L(\varphi)$ increases with φ, so $x\rho_L(\sin^{-1} x)$ increases with x, and the result follows. $\square$

THEOREM 5.2. *Let K be a centered convex body in $\mathbb{E}^3$, such that K has everywhere positive Gaussian curvature and $\rho_K \in C_{\mathrm{e}}^{\infty}(\mathbb{S}^2)$. Then K is the intersection body of a star body.*

Proof. Since $\rho_K \in C_{\mathrm{e}}^{\infty}(\mathbb{S}^2)$, Proposition 2.1 guarantees the existence of a $g \in C_{\mathrm{e}}(\mathbb{S}^2)$ such that $\rho_K = Rg$. We have to show that $g \ge 0$.

Let $u_0 \in \mathbb{S}^2$. Suppose $\bar{K}$ is the Schwarz symmetral of K in the direction u_0. We claim that $\rho_{\bar{K}} \in C_{\mathrm{e}}^1(\mathbb{S}^2)$. To see this, let u_0 be the north pole for coordinates on $\mathbb{S}^2$, and let $\bar{r} = \bar{r}(z)$ give (in cylindrical polar coordinates) the boundary of $\bar{K}$. Denote by Φ the horizontal shear transformation which takes the point x^* in the boundary of K with maximal height z^* to the point $(0,0,z^*)$; then the body ΦK has the same Schwarz symmetral in the direction u_0 as K. The transformation rule

$$\rho_{\Phi K}(x) = \rho_K(\Phi^{-1}x),$$

for the change in the extended radial function under the linear map Φ, follows from its definition. From this, we see that the usual radial function of

ΦK satisfies $\rho_{\Phi K} \in C_{\mathrm{e}}^{\infty}(\mathbb{S}^2)$. Since the boundary of ΦK can be expressed in cylindrical polar coordinates as a single function $s = s(\theta, z)$, we have

$$\pi \bar{r}^2(z) = \frac{1}{2} \int_0^{2\pi} s^2(\theta, z)\, d\theta,$$

for all z. For any z_0 with $0 < z_0 < z^*$, this shows that $\bar{r}$ is a C^{∞} function on $[-z_0, z_0]$, and hence that $\rho_{\bar{K}}$ is a C^{∞} function except possibly at $\pm u_0$. The assumptions that K has everywhere positive Gaussian curvature and $\rho_K \in C_{\mathrm{e}}^{\infty}(\mathbb{S}^2)$ allow us to apply Blaschke's rolling ball theorem (see, for example, [30, Corollary 3.2.10]) to conclude that K has $\varepsilon\mathbb{B}$ as a Minkowski summand for some $\varepsilon > 0$. Therefore there is a translate of $\varepsilon\mathbb{B}$ which contains x^* and which is contained in K. It follows that there is also a translate of $\varepsilon\mathbb{B}$ which contains $(0, 0, z^*)$ and which is contained in $\bar{K}$. So $\bar{K}$ has a unique tangent plane at $(0, 0, \pm z^*)$, and this shows that $\rho_{\bar{K}} \in C_{\mathrm{e}}^{1}(\mathbb{S}^2)$, as claimed.

Theorem 5.1 implies that $\bar{K}$ is the intersection body of a star body, so that $\rho_{\bar{K}} = R\bar{g}$, for some nonnegative $\bar{g} \in C_{\mathrm{e}}(\mathbb{S}^2)$. By the definition of $\bar{K}$, we have $A_{\bar{K}} = A_K$, in the notation of Theorem 4.1. The latter theorem implies that $g(u_0) = \bar{g}(u_0)$, and therefore $g(u_0) \geq 0$. Since $u_0 \in \mathbb{S}^2$ was arbitrary, the proof is complete. □

COROLLARY 5.3. *Every centered convex body in $\mathbb{E}^3$ is an intersection body.*

Proof. It is shown in [14] that the class of intersection bodies is closed under uniform limits. The corollary follows from Theorem 5.2 and the fact that the class of centered convex bodies whose Gaussian curvature is everywhere positive and whose radial functions are infinitely differentiable is dense in the class of all centered convex bodies (see [29]). □

The results of [35] show that Theorem 5.2 and Corollary 5.3 are false in four or more dimensions. In particular, as we remarked in the introduction, Zhang shows that a centered cube in $\mathbb{E}^n$, $n \geq 4$, is not an intersection body.

COROLLARY 5.4. *The Busemann-Petty problem has a positive answer in $\mathbb{E}^3$.*

Proof. This follows immediately, either from Theorem 2.4 and Theorem 5.2, or from Theorem 2.5 and Corollary 5.3. □

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(Received May 29, 1993)