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Paul Binding and Branko Ćurgus

Abstract. Form domains are characterized for regular $2n$-th order differential equations subject to general self-adjoint boundary conditions depending affinely on the eigenparameter. Corresponding modes of convergence for eigenfunction expansions are studied, including uniform convergence of the first $n-1$ derivatives.

1 Introduction

We shall consider the eigenvalue problem

$$L(f) = \sum_{j=0}^{n}(-1)^j (p_j f^{(j)})^{(j)} = \lambda r f \quad \text{on} \quad [a,b], \tag{1.1}$$

subject to self-adjoint boundary conditions of the form

$$\begin{bmatrix} D \\ M \end{bmatrix} \mathbf{b}(f) = \lambda \begin{bmatrix} 0 \\ N \end{bmatrix} \mathbf{b}(f), \tag{1.2}$$

where $\left[\begin{smallmatrix} D \\ M \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 \\ N \end{smallmatrix}\right]$ are $2n \times 4n$ matrices and $\mathbf{b}(f)$ is the vector with components

$$f(a), f'(a), \dots, f^{(n-1)}(a), f(b), f'(b), \dots, f^{(n-1)}(b),$$
$$f^{[n]}(a), \dots, f^{[2n-1]}(a), f^{[n]}(b), \dots, f^{[2n-1]}(b).$$

For the definition of the quasi-derivatives $f^{[j]}$ and how they are used to give meaning to the expression $L(f)$ in (1.1), even if the coefficients p_j are not smooth functions, see [18], [20], [24] . Note that by definition $f^{[k]} = f^{(k)}$, $k = 0, 1, \dots, n$ and $L(f) := f^{[2n]}$. The methods of this paper use self-adjoint operators which are bounded from below and the only restrictions on the coefficients are that $1/p_n, p_{n-1}, \dots, p_0, r$ are real integrable on $[a,b]$ and p_n (resp. r) is positive (resp. nonzero) almost everywhere. Many authors have studied problems of this type, the Sturm-Liouville case (with $n = 1$ and separated boundary conditions) being the most commonly treated. See, *e.g.*, [12], [13], [14], [23] and their references.

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It is known that such problems may be cast in the form $Af = \lambda f$ for a suitable self-adjoint operator A in a Hilbert (or Krein, depending on the signs of the coefficients) space $\mathcal{H}$. Usually $\mathcal{H}$ takes the form $L_2([a,b],r) \oplus \mathbb{C}^k$ where $L_2([a,b],r)$ is a Krein space if r changes sign in $[a,b]$ and the (in general indefinite) inner product on $\mathbb{C}^k$ is given by an invertible self-adjoint matrix. Eigenfunction expansions have provided a frequent topic of interest, and several authors have demonstrated for positive r that their $L_2(r)$ convergence can be improved to uniform convergence for elements of $\mathrm{dom}(A)$, *cf.* [12], [14]. For Sturm-Liouville problems Hinton [15] took the further step of showing that uniform convergence holds even in the form domain $\mathrm{fdom}(A)$, (which coincides with $\mathrm{dom}(A^{1/2})$ if $A > 0$).

Hinton also characterized $\mathrm{fdom}(A)$ in a special case, and recently Binding and Browne have completed this characterization for all possible Sturm-Liouville problems with separated boundary conditions [4]. Our goal here is twofold. First, we extend the explicit form domain characterization to general (not necessarily separated) self-adjoint boundary conditions for $2n$-th order problems. For given boundary conditions, we show that the form domain, and its natural topology, depend only on the leading coefficient p_n. In our opinion the methods of the cited authors would lead to considerable complication, and we have instead adopted a unified abstract treatment of the problem. Second, we show in a direct fashion that convergence of the eigenfunction expansions is always stronger than uniform. In fact we characterize this convergence precisely, and we give conditions for it to coincide with convergence in the Sobolev type space $W_2^n \oplus \mathbb{C}^k$. In particular we extend some results of Dijksma [12] (in the case of linear boundary conditions) and Hinton [15] who discusses second order problems with separated boundary conditions. Sobolev space convergence is established for more general problems by Shkalikov [21] and Tretter [22], but they assume smoother coefficients.

Our plan is as follows. The conditions on (1.1) and (1.2) are specified precisely, and the abstract framework is set up for definite and indefinite problems, in Sections 2 and 3 respectively. In particular, an operator A_0 in $\mathcal{H}_0 = L_2(r)$ and an associated operator $\widetilde{A}$ in $\widetilde{\mathcal{H}} = L_2(r) \oplus \mathbb{C}^k$ are constructed from the λ-dependent problem. The operator A_0, which corresponds to a problem with λ-independent boundary conditions, is well understood. Section 4 contains the form domain characterization in terms of p_n and certain "essential" boundary conditions. In the Sturm-Liouville case our formulae are different from (but equivalent to) those of [4]. In Section 5 we discuss the topology of convergence in the form domain generated by A, and uniform convergence of the first $n-1$ derivatives is deduced as a corollary.

2 Hilbert Space Constructions

In this section we present an abstract framework for the problem (1.1)–(1.2). We start with the familiar positive definite case. The indefinite case will then be treated using results from the positive definite case.

Throughout we consider subspaces and operators in direct sums of inner product (Hilbert or Krein) spaces. We consistently use the following notation. If $(\mathcal{H}_j, \langle\cdot,\cdot\rangle_j)$, $j = 1, 2$ are inner product spaces, we denote by $(\widetilde{\mathcal{H}}, \langle\cdot,\cdot\rangle)$ their direct sum, that is

$$\widetilde{\mathcal{H}} := \mathcal{H}_0 \oplus \mathcal{H}_1 \quad \langle f, g\rangle := \langle f_0, g_0\rangle_0 + \langle f_1, g_1\rangle_1,$$
$$f = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}, \quad g = \begin{bmatrix} g_0 \\ g_1 \end{bmatrix} \in \widetilde{\mathcal{H}}, \quad f_j, g_j \in \mathcal{H}_j,\ j = 0, 1.$$

We also use the following convention. If $\mathcal{L} \subset \mathcal{H}_0$, we will also write $\mathcal{L} \subset \widetilde{\mathcal{H}}$, identifying the set $\mathcal{L}$ with the set $\mathcal{L} \oplus 0$. Similarly, we will consider an operator $V : \mathcal{H}_0 \to \mathcal{H}_0$ also as an operator in $\widetilde{\mathcal{H}}$ meaning that

$$V\colon \begin{bmatrix} f_0 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} V f_0 \\ 0 \end{bmatrix}. \tag{2.1}$$

We begin with a closed symmetric operator T_0 with defect index (d, d), $d < +\infty$, in a Hilbert space $(\mathcal{H}_0, \langle\cdot,\cdot\rangle_0)$. By abbreviation we shall say that T_0 has (finite) defect d. Throughout, we assume symmetric operators to be densely defined with finite defects. Let $\mathbf{b}$ be a boundary mapping for T_0, that is $\mathbf{b}\colon\ \mathrm{dom}(T_0^*) \to \mathbb{C}^{2d}$ is a surjective linear mapping and $\ker(\mathbf{b}) = \mathrm{dom}(T_0)$. (As usual, * will denote the adjoint operator). It follows that there exists an invertible self-adjoint $2d \times 2d$ matrix Q (which we shall call the concomitant matrix of $\mathbf{b}$) with d positive and d negative eigenvalues, such that the abstract form of the Lagrange's identity becomes

$$\langle T_0^* f_0, g_0\rangle - \langle f_0, T_0^* g_0\rangle = i\mathbf{b}(g_0)^* Q \mathbf{b}(f_0), \quad f_0, g_0 \in \mathrm{dom}(T_0^*). \tag{2.2}$$

At first we will assume that all boundary conditions include λ so that our abstract eigenvalue problem takes the form

$$T_0^* f_0 = \lambda f_0, \quad f_0 \in \mathrm{dom}(T_0^*), \tag{2.3}$$
$$M\mathbf{b}(f_0) = \lambda N \mathbf{b}(f_0), \tag{2.4}$$

where M and N are $d \times 2d$ matrices. We will study the above eigenvalue problem within the framework of the vector space $\mathcal{H}_0 \oplus \mathbb{C}^d$ with a suitable inner product. Let Δ be an arbitrary $d \times d$ matrix and denote by $\mathbb{C}^d_\Delta$ the space $\mathbb{C}^d$ with the inner product $\langle x, y\rangle_1 := y^* \Delta x$, for $x, y \in \mathbb{C}^d$. We consider this general form of inner product on $\mathbb{C}^d$ only until Proposition 2.1 below. Following Proposition 2.1, we will assume Δ to be an invertible self-adjoint $d \times d$ matrix.

We define an operator $\widetilde{B}$ which is associated with the problem (2.3)–(2.4) in the following way. Let

$$(\widetilde{\mathcal{H}}, \langle\cdot,\cdot\rangle) = (\mathcal{H}_0, \langle\cdot,\cdot\rangle_0) \oplus \mathbb{C}^d_\Delta,$$

that is, with inner product given by

$$\langle f, g\rangle := \langle f_0, g_0\rangle_0 + g_1^* \Delta f_1,$$
$$f = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}, \quad g = \begin{bmatrix} g_0 \\ g_1 \end{bmatrix} \in \widetilde{\mathcal{H}}, \quad f_0, g_0 \in \mathcal{H}_0,\ f_1, g_1 \in \mathbb{C}^d_\Delta.$$

Define

$$\operatorname{dom}(\widetilde{B}) = \left\{ f = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} \in \widetilde{\mathcal{H}} : f_0 \in \operatorname{dom}(T_0^*),\ f_1 = N\mathbf{b}(f_0) \right\} \tag{2.5}$$

and put

$$\widetilde{B}f = \widetilde{B} \begin{bmatrix} f_0 \\ N\mathbf{b}(f_0) \end{bmatrix} := \begin{bmatrix} T_0^* f_0 \\ M\mathbf{b}(f_0) \end{bmatrix}, \quad f \in \operatorname{dom}(\widetilde{B}). \tag{2.6}$$

Note that symmetry of $\widetilde{B}$ (which depends on M, N and Q) in the inner product $\langle \cdot, \cdot \rangle$ (which depends on Δ) forces relations between M, N, Q, and Δ. The following proposition (whose "if" part is known [12], [17] in the case of differential operators) makes these relations precise in the abstract case.

Proposition 2.1 *In the above notation, the operator $\widetilde{B}$ defined by* (2.5)–(2.6) *is symmetric in $(\widetilde{\mathcal{H}}, \langle \cdot, \cdot \rangle)$ if and only if the following conditions are satisfied:*

$$MQ^{-1}M^* = NQ^{-1}N^* = 0, \tag{2.7}$$

$iMQ^{-1}N^*$ is an invertible, self-adjoint matrix and

$$\Delta = \frac{1}{i}(MQ^{-1}N^*)^{-1}. \tag{2.8}$$

Proof First note that if (2.7) is true, then the equality

$$\begin{bmatrix} M \\ N \end{bmatrix} Q^{-1} \begin{bmatrix} M \\ N \end{bmatrix}^* = \begin{bmatrix} 0 & MQ^{-1}N^* \\ (MQ^{-1}N^*)^* & 0 \end{bmatrix} \tag{2.9}$$

holds. Clearly, the matrix on the left hand side of (2.9) is invertible if and only if the matrix $\begin{bmatrix} M & N \end{bmatrix}^T$ is so and the matrix on the right hand side of (2.9) is invertible if and only if $MQ^{-1}N^*$ is so. Therefore, if (2.7) holds, the matrix $MQ^{-1}N^*$ is invertible if and only if the matrix $\begin{bmatrix} M & N \end{bmatrix}^T$ is invertible.

It follows from the definitions of $\mathbf{b}$, Q, $\widetilde{B}$ and $\mathbb{C}^d_\Delta$ that

$$\begin{aligned} &\langle \widetilde{B}f, g \rangle - \langle f, \widetilde{B}g \rangle \\ &\quad = \langle T_0^* f_0, g_0 \rangle_0 - \langle f_0, T_0^* g_0 \rangle_0 + \langle M\mathbf{b}(f_0), N\mathbf{b}(g_0) \rangle_1 - \langle N\mathbf{b}(f_0), M\mathbf{b}(g_0) \rangle_1 \\ &\quad = i\mathbf{b}(g_0)^* Q\mathbf{b}(f_0) + \mathbf{b}(g_0)^* N^* \Delta M\mathbf{b}(f_0) - \mathbf{b}(g_0)^* M^* \Delta N\mathbf{b}(f_0) \\ &\quad = \mathbf{b}(g_0)^* (iQ + N^*\Delta M - M^*\Delta N)\mathbf{b}(f_0). \end{aligned} \tag{2.10}$$

Since $\operatorname{dom}(T_0^*)$ equals the projection of $\operatorname{dom}(\widetilde{B})$ onto $\mathcal{H}_0$ and the mapping $\mathbf{b}$: $\operatorname{dom}(T_0^*) \to \mathbb{C}^{2d}$ is onto, it follows from (2.10) that $\widetilde{B}$ is symmetric if and only if

$$iQ = M^*\Delta N - N^*\Delta M,$$

or, equivalently, if and only if

$$Q = \begin{bmatrix} M \\ N \end{bmatrix}^* \begin{bmatrix} 0 & -i\Delta \\ i\Delta & 0 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}. \tag{2.11}$$

We conclude that if $\widetilde{B}$ is symmetric, then (2.11) holds, and, since Q is invertible all square matrices in (2.11) must be invertible and (2.11) can be rewritten as

$$\begin{bmatrix} 0 & -i\Delta^{-1} \\ i\Delta^{-1} & 0 \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} Q^{-1} \begin{bmatrix} M \\ N \end{bmatrix}^* = \begin{bmatrix} MQ^{-1}M^* & MQ^{-1}N^* \\ NQ^{-1}M^* & NQ^{-1}N^* \end{bmatrix}. \tag{2.12}$$

Clearly (2.12) is equivalent to (2.7) and (2.8).

Conversely, if $MQ^{-1}N^*$ is invertible and (2.7)–(2.8) hold, then (2.12) and (2.11) hold, and therefore $\widetilde{B}$ is symmetric in $(\widetilde{\mathcal{H}}, \langle\cdot,\cdot\rangle)$. ■

Remark 2.2 Proposition 2.1 implies that if $\widetilde{B}$ is symmetric with respect to the inner product $\langle\cdot,\cdot\rangle$ on $\widetilde{\mathcal{H}}$, then $(\widetilde{\mathcal{H}}, \langle\cdot,\cdot\rangle)$ must be a Krein space which is a direct orthogonal sum of a Hilbert space $(\mathcal{H}_0, \langle\cdot,\cdot\rangle_0)$ and the finite dimensional Krein space $\mathbb{C}^d_\Delta$.

Corollary 2.3 *Assume that the operator* $\widetilde{B}$ *defined by* (2.5)–(2.6) *is symmetric in* $(\widetilde{\mathcal{H}}, \langle\cdot,\cdot\rangle)$ *and that*

$$\Delta = \frac{1}{i}(MQ^{-1}N^*)^{-1} \quad \textit{is positive definite.} \tag{2.13}$$

Then $\widetilde{B}$ *is a self-adjoint operator in the Hilbert space* $(\widetilde{\mathcal{H}}, \langle\cdot,\cdot\rangle)$.

Proof By Proposition 2.1 $(\widetilde{\mathcal{H}}, \langle\cdot,\cdot\rangle)$ is a Hilbert space, the equalities (2.7) hold and we can use all the equalities derived in the proof of Proposition 2.1. Assume that $g = \begin{bmatrix} g_0 & g_1 \end{bmatrix}^T \in \widetilde{\mathcal{H}}$ is orthogonal to $\operatorname{dom}(\widetilde{B})$. Then, since $\operatorname{dom}(T_0) \oplus \{0\} \subset \operatorname{dom}(\widetilde{B})$ and $\operatorname{dom}(T_0)$ is dense in $\mathcal{H}_0$, we conclude that $g_0 = 0$. Therefore, $g_1^*\Delta N\mathbf{b}(f_0) = 0$ for all $f_0 \in \operatorname{dom}(T_0^*)$. As $\mathbf{b}\colon \operatorname{dom}(T_0^*) \to \mathbb{C}^{2d}$ is onto, Δ is invertible, and the rows of N are linearly independent, it follows that $g_1 = 0$. Thus, $g = 0$ and therefore $\operatorname{dom}(\widetilde{B})$ is dense in $\widetilde{\mathcal{H}}$.

Since the operator $\widetilde{B}$ defined by (2.5)–(2.6) is evidently closed, to prove that it is self-adjoint we only need to show that $\ker(\widetilde{B} \pm i\widetilde{I}) = \{0\}$. Let $f = \left[\begin{smallmatrix} f_0 \\ N\mathbf{b}(f_0) \end{smallmatrix}\right] \in \ker(\widetilde{B} - i\widetilde{I})$. Then

$$T_0^* f_0 = i f_0, \tag{2.14}$$

$$M\mathbf{b}(f_0) = iN\mathbf{b}(f_0). \tag{2.15}$$

Using (2.14) we have

$$\mathbf{b}(f_0)^*Q\mathbf{b}(f_0) = \frac{1}{i}\big(\langle T_0^* f_0, f_0\rangle_0 - \langle f_0, T_0^* f_0\rangle_0\big) = 2\langle f_0, f_0\rangle_0 \geq 0. \tag{2.16}$$

An application of (2.11) and (2.15) leads to

$$\begin{aligned}\mathbf{b}(f_0)^* Q\mathbf{b}(f_0) &= \mathbf{b}(f_0)^* \begin{bmatrix} M \\ N \end{bmatrix}^* \begin{bmatrix} 0 & -i\Delta \\ i\Delta & 0 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} \mathbf{b}(f_0) \\ &= \begin{bmatrix} iN\mathbf{b}(f_0) \\ N\mathbf{b}(f_0) \end{bmatrix}^* \begin{bmatrix} 0 & -i\Delta \\ i\Delta & 0 \end{bmatrix} \begin{bmatrix} iN\mathbf{b}(f_0) \\ N\mathbf{b}(f_0) \end{bmatrix} \\ &= -2\big(N\mathbf{b}(f_0)\big)^* \Delta N\mathbf{b}(f_0) \le 0. \end{aligned} \tag{2.17}$$

The inequalities (2.16) and (2.17) imply that $f_0 = 0$ and therefore $f = 0$. Similarly, $\ker(\widetilde{B} + i\widetilde{I}) = \{0\}$. ■

Since our original eigenvalue problem (1.1)–(1.2) might include boundary conditions that do not include λ we need the following extension of Corollary 2.3. In this case the symmetric operator T_0 arises as an extension of another symmetric operator T_{m}.

Theorem 2.4 *Let T_{m} be a closed symmetric operator with defect m in a Hilbert space $(\mathcal{H}_0, \langle\cdot,\cdot\rangle_0)$, let $\mathbf{b}\colon \operatorname{dom}(T_{\mathrm{m}}^*) \to \mathbb{C}^{2m}$ be a boundary mapping for T_{m} with concomitant matrix Q.*

Let d be an integer with $0 < d < m$ and let M and N be $d\times 2m$ matrices which satisfy (2.7) and (2.13). Let D be a $(m-d) \times 2m$ matrix such that $DQ^{-1}D^* = DQ^{-1}M^* = DQ^{-1}N^* = 0$ and such that the $(m+d) \times 2m$ matrix $\left[\begin{smallmatrix} D \\ M \\ N \end{smallmatrix}\right]$ has maximal rank $m+d$. Then the operator $\widetilde{B}$ defined by

$$\operatorname{dom}(\widetilde{B}) = \left\{ f = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} \in \widetilde{\mathcal{H}} : f_0 \in \operatorname{dom}(T_{\mathrm{m}}^*), D\mathbf{b}(f_0) = 0, f_1 = N\mathbf{b}(f_0) \right\}, \tag{2.18}$$

and

$$\widetilde{B} f = \widetilde{B} \begin{bmatrix} f_0 \\ N\mathbf{b}(f_0) \end{bmatrix} := \begin{bmatrix} T_{\mathrm{m}}^* f_0 \\ M\mathbf{b}(f_0) \end{bmatrix}, \quad f \in \operatorname{dom}(\widetilde{B}) \tag{2.19}$$

is self-adjoint in the Hilbert space $(\widetilde{\mathcal{H}}, \langle\cdot,\cdot\rangle)$.

Proof It follows from [9, Lemma 3.4] and the assumptions about D, M and N, that the restriction T_0 of T_{m}^* defined on

$$\operatorname{dom}(T_0) = \left\{ f \in \operatorname{dom}(T_{\mathrm{m}}^*) : \begin{bmatrix} D \\ M \\ N \end{bmatrix} \mathbf{b}(f) = 0 \right\} \tag{2.20}$$

is a symmetric extension of T_{m} with defect d and that the domain of its adjoint is

$$\operatorname{dom}(T_0^*) = \{ f \in \operatorname{dom}(T_{\mathrm{m}}^*) : D\mathbf{b}(f) = 0 \}. \tag{2.21}$$

By [9, Lemma 3.5], $\mathbf{b}_0 = \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right] \mathbf{b}|_{\mathrm{dom}(T_0^*)}$ is a boundary mapping for T_0 and its concomitant matrix is

$$\left(\begin{bmatrix} M \\ N \end{bmatrix} Q^{-1} \begin{bmatrix} M \\ N \end{bmatrix}^* \right)^{-1} = \begin{bmatrix} 0 & (NQ^{-1}M^*)^{-1} \\ (MQ^{-1}N^*)^{-1} & 0 \end{bmatrix}.$$

By (2.21), the definition of $\widetilde{B}$ in the present theorem can be rewritten in the form (2.5)–(2.6):

$$\mathrm{dom}(\widetilde{B}) = \left\{ f = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} \in \widetilde{\mathcal{H}} : f_0 \in \mathrm{dom}(T_0^*),\, f_1 = \begin{bmatrix} 0 & I \end{bmatrix} \mathbf{b}_0(f_0) \right\},$$

and

$$\widetilde{B} f = \widetilde{B} \begin{bmatrix} f_0 \\ \begin{bmatrix} 0 & I \end{bmatrix} \mathbf{b}_0(f_0) \end{bmatrix} = \begin{bmatrix} T_0^* f_0 \\ \begin{bmatrix} I & 0 \end{bmatrix} \mathbf{b}_0(f_0) \end{bmatrix}, \quad f \in \mathrm{dom}(\widetilde{B}).$$

Since the matrices $\begin{bmatrix} 0 & I \end{bmatrix}$ (*cf.* N in (2.5)) and $\begin{bmatrix} I & 0 \end{bmatrix}$ (*cf.* M in (2.6)) satisfy all the conditions of Proposition 2.1, Corollary 2.3 implies that $\widetilde{B}$ is self-adjoint in the Hilbert space $\mathcal{H}_0 \oplus \mathbb{C}^d_\Delta$, where

$$\Delta = \frac{1}{i} \left(\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} Q^{-1} \begin{bmatrix} M \\ N \end{bmatrix}^* \begin{bmatrix} 0 & I \end{bmatrix}^* \right)^{-1} = \frac{1}{i} (MQ^{-1}N^*)^{-1}. \qquad \blacksquare$$

Remark 2.5 The operator $\widetilde{B}$ defined by (2.18)–(2.19) is associated with the eigenvalue problem

$$T_{\mathrm{m}}^* f_0 = \lambda f_0, \quad f_0 \in \mathrm{dom}(T_{\mathrm{m}}^*), \tag{2.22}$$

$$\begin{bmatrix} D \\ M \end{bmatrix} \mathbf{b}(f_0) = \lambda \begin{bmatrix} 0 \\ N \end{bmatrix} \mathbf{b}(f_0). \tag{2.23}$$

The following proposition and its corollary are included for the reader's convenience.

Proposition 2.6 *Let $(\widetilde{\mathcal{H}}, \langle \cdot, \cdot \rangle)$ be the direct sum of two Hilbert spaces $(\mathcal{H}_0, \langle \cdot, \cdot \rangle_0)$ and $(\mathcal{H}_1, \langle \cdot, \cdot \rangle_1)$. Assume that $\dim(\mathcal{H}_1) = d < +\infty$. Let T_{m} be a closed symmetric operator in $(\mathcal{H}_0, \langle \cdot, \cdot \rangle_0)$ with defect m. Let B_0 be a self-adjoint extension of T_{m} in $\mathcal{H}_0$. Let $\widetilde{B}$ be a self-adjoint extension of T_{m} in $\widetilde{\mathcal{H}}$. Then the operators B_0 and $\widetilde{B}$ have the same essential spectrum.*

Proof Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Denote by $\widetilde{B_0}$ the direct sum of B_0 and the zero operator on $\mathcal{H}_1$. Then $\widetilde{B_0}$ is self-adjoint in $(\widetilde{\mathcal{H}}, \langle \cdot, \cdot \rangle)$ and $\lambda \in \rho(\widetilde{B_0}) \cap \rho(\widetilde{B})$. The restrictions of $\widetilde{B_0}$ and $\widetilde{B}$ to $\mathrm{dom}(T_{\mathrm{m}}) \oplus \{0\}$ coincide and $\mathrm{dom}(T_{\mathrm{m}}) \oplus \{0\}$ has codimension $d+m < +\infty$ in both $\mathrm{dom}(\widetilde{B_0})$ and $\mathrm{dom}(\widetilde{B})$. Therefore

$$(\widetilde{B_0} - \lambda I)^{-1} - (\widetilde{B} - \lambda I)^{-1}$$

is a finite rank operator. Now the statement of the proposition follows from [16, Theorem IV.5.35]. $\blacksquare$

Corollary 2.7 *Under the conditions of Proposition 2.6, the operator $\widetilde{B}$ is bounded from below in $\widetilde{\mathcal{H}}$ if and only if T_{m} is bounded from below in $\mathcal{H}_0$. The operator $\widetilde{B}$ has discrete spectrum if and only if B_0 has discrete spectrum.*

Next we give an abstract foundation for the characterization of form domains of the operators associated with the eigenvalue problem (1.1)–(1.2) to be presented in Section 4. Let S be a self-adjoint operator which is bounded below in a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and let $\alpha \geq 0$ be such that the operator $S + \alpha I$ is uniformly positive. Then the completion of $\operatorname{dom}(S)$ with respect to the positive definite form $\langle (S+\alpha I)\cdot,\cdot\rangle$ is called the *form domain* of S and it is denoted by $\operatorname{fdom}(S)$. The space $\operatorname{fdom}(S)$ does not depend on the special choice of α and it coincides with $\operatorname{dom}(|S|^{1/2})$. The form $\langle (S+\alpha I)\cdot,\cdot\rangle$ extends to $\operatorname{fdom}(S)$ by continuity to a form which we denote by $\langle\cdot,\cdot\rangle_{S+\alpha I}$. The space $\big(\operatorname{fdom}(S), \langle\cdot,\cdot\rangle_{S+\alpha I}\big)$ is a Hilbert space with $\operatorname{dom}(S)$ as a dense subspace.

Lemma 2.8 *Let T_0 be a closed, symmetric operator with defect d, bounded below in a Hilbert space $(\mathcal{H}_0, \langle\cdot,\cdot\rangle_0)$. Let $(\mathcal{H}_1, \langle\cdot,\cdot\rangle_1)$ be an d-dimensional Hilbert space and denote by $(\widetilde{\mathcal{H}}, \langle\cdot,\cdot\rangle)$ the direct sum of the Hilbert spaces $(\mathcal{H}_j, \langle\cdot,\cdot\rangle_j)$, $j = 1, 2$. Let M and N be $d \times 2d$ matrices which satisfy* (2.7) *and* (2.13). *Let $\widetilde{B}$ be a self-adjoint extension of T_0 in $\widetilde{\mathcal{H}}$ defined by* (2.5) *and* (2.6). *Let B_0 be a self-adjoint extension of T_0 in $\mathcal{H}_0$ defined on*

$$\operatorname{dom}(B_0) := \{f_0 \in \operatorname{dom}(T_0^*) : N\mathbf{b}(f_0) = 0\}.$$

Then $\operatorname{dom}(B_0) \subset \operatorname{dom}(\widetilde{B})$ *and the dimension of the factor space* $\operatorname{dom}(\widetilde{B})/\operatorname{dom}(B_0)$ *is d. There exist $\phi^1, \ldots, \phi^d \in \operatorname{dom}(\widetilde{B})$ which are linearly independent over* $\operatorname{dom}(B_0)$, *such that the form domain* $\operatorname{fdom}(\widetilde{B})$ *of $\widetilde{B}$ is given by*

$$\operatorname{fdom}(\widetilde{B}) = \operatorname{fdom}(B_0) \dot{+} \operatorname{span}\{\phi^1, \ldots, \phi^d\}.$$

Proof Since we assume that T_0 is bounded below by Corollary 2.7 both operators B_0 and $\widetilde{B}$ are bounded below. Therefore, there exists $\alpha > 0$ such that the shifted operators $T_0 + \alpha I$, $B_0 + \alpha I$ and $\widetilde{B} + \alpha \widetilde{I}$ are uniformly positive. Since the domains and form domains remain unchanged under this shift, there is no loss of generality if we assume that all operators $T_0, B_0, \widetilde{B}$ are uniformly positive.

By the definition of B_0, for $f_0 \in \operatorname{dom}(B_0)$ we have $N\mathbf{b}(f_0) = 0$. Therefore, $f_0 = \begin{bmatrix} f_0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_0 \\ N\mathbf{b}(f_0) \end{bmatrix} \in \operatorname{dom}(\widetilde{B})$. In fact, $\operatorname{dom}(\widetilde{B}) \cap \operatorname{dom}(T_0^*) = \operatorname{dom}(B_0)$ (recall the identification (2.1)). The vectors $\begin{bmatrix} g_0^j \\ N\mathbf{b}(g_0^j) \end{bmatrix} \in \operatorname{dom}(\widetilde{B})$, $j = 1, \ldots, q$ are linearly independent modulo $\operatorname{dom}(B_0)$ if and only if the vectors $N\mathbf{b}(g_0^j)$, $j = 1, \ldots, q$, are linearly independent. Since $\mathbf{b}\colon \operatorname{dom}(T_0^*) \to \mathbb{C}^{2d}$ is surjective and since we assume that the $d \times 2d$ matrix N has full rank, there exist $g_0^1, \ldots, g_0^d \in \operatorname{dom}(T_0^*)$ such that $N\mathbf{b}(g_0^j)$, $j = 1, \ldots, q$ are linearly independent. Therefore

$$\dim\big(\operatorname{dom}(\widetilde{B})/\operatorname{dom}(B_0)\big) = d. \tag{2.24}$$

Clearly $\dim\big(\operatorname{dom}(B_0)/\operatorname{dom}(T_0)\big) = d$. As B_0 is one-to-one and onto $\mathcal{H}_0$ and T_0 is one-to-one, the codimension of the range of T_0 in $\mathcal{H}_0$ equals d. Therefore $\dim\big(\ker(T_0^*)\big) = d$. Let $\phi_0^1, \dots, \phi_0^d$ form a basis of $\ker(T_0^*)$. Since the null space of the uniformly positive operator B_0 is trivial, the vectors $N\mathbf{b}(\phi_0^1), \dots, N\mathbf{b}(\phi_0^d)$ are linearly independent. Consequently, the vectors $\phi^j := \begin{bmatrix} \phi_0^j \\ N\mathbf{b}(\phi_0^j) \end{bmatrix}$, $j = 1, \dots, d$ are linearly independent. Together with (2.24) this implies

$$\operatorname{dom}(\widetilde{B}) = \operatorname{dom}(B_0) \dot{+} \operatorname{span}\{\phi^1, \dots, \phi^d\}.$$

The vectors $\phi^1, \dots, \phi^d$ are orthogonal to $\operatorname{dom}(B_0)$ with respect to the form $\langle \widetilde{B}\,\cdot, \cdot\rangle$. Indeed, with $f_0 \in \operatorname{dom}(B_0)$ and $j \in \{1, \dots, d\}$,

$$\left\langle \widetilde{B} \begin{bmatrix} \phi_0^j \\ N\mathbf{b}(\phi_0^j) \end{bmatrix}, \begin{bmatrix} f_0 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} T_0^*\phi_0^j \\ M\mathbf{b}(\phi_0^j) \end{bmatrix}, \begin{bmatrix} f_0 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 0 \\ M\mathbf{b}(\phi_0^j) \end{bmatrix}, \begin{bmatrix} f_0 \\ 0 \end{bmatrix} \right\rangle = 0.$$

Clearly the restriction of the operator $\widetilde{B}$ to $\operatorname{dom}(B_0)$v does not coincide with B_0. In fact, it follows from the invertibility of the matrix $\begin{bmatrix} M & N \end{bmatrix}^T$ that the largest subset of $\operatorname{dom}(B_0)$ on which $\widetilde{B}$ coincides with T_0^* is $\operatorname{dom}(T_0)$. Despite this, it turns out that the form $\langle \widetilde{B}\,\cdot, \cdot\rangle$ when restricted to $\operatorname{dom}(B_0)$ coincides with the form $\langle B_0\,\cdot, \cdot\rangle_0$. Indeed, for $f_0, g_0 \in \operatorname{dom}(B_0)$,

$$\left\langle \widetilde{B} \begin{bmatrix} f_0 \\ 0 \end{bmatrix}, \begin{bmatrix} g_0 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} T_0^* f_0 \\ M\mathbf{b}(f_0) \end{bmatrix}, \begin{bmatrix} g_0 \\ 0 \end{bmatrix} \right\rangle = \langle T_0^* f_0, g_0\rangle_0 = \langle B_0 f_0, g_0\rangle_0.$$

It follows that the completion of $\operatorname{dom}(\widetilde{B})$ with respect to the form $\langle \widetilde{B}\,\cdot, \cdot\rangle$ completes $\operatorname{dom}(B_0)$ to $\operatorname{fdom}(B_0)$ and the finite dimensional space $\operatorname{span}\{\phi^1, \dots, \phi^d\}$ remains unchanged under this completion. Therefore

$$\operatorname{fdom}(\widetilde{B}) = \operatorname{fdom}(B_0) \dot{+} \operatorname{span}\{\phi^1, \dots, \phi^d\}. \qquad \blacksquare$$

3 Krein Space Constructions

In order to study indefinite eigenvalue problems of the form (1.1)–(1.2), the above constructions are carried out instead in a Krein space. Recall that $(\mathcal{K}, [\cdot,\cdot])$ is a Krein space if $\mathcal{K}$ is a complex vector space, $[\cdot,\cdot]$ is an indefinite inner product on $\mathcal{K}$ and there exists a direct, $[\cdot,\cdot]$-orthogonal decomposition $\mathcal{K} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-$ such that $(\mathcal{K}_\pm, \pm[\cdot,\cdot])$ are Hilbert spaces. For such a decomposition the corresponding *fundamental symmetry* is a linear operator J defined by $J(x_+ + x_-) := x_+ - x_-$, and the corresponding Hilbert space inner product on $\mathcal{K}$ is $\langle u, v\rangle := [Ju, v]$, $u, v \in \mathcal{K}$. The topology induced on $\mathcal{K}$ by this Hilbert space inner product is independent of the choice of $\mathcal{K}_\pm$. The definitions of symmetric, self-adjoint and positive operators in a Krein space parallel those in a Hilbert space, see the monographs [1] and [5]. For applications of Krein space operator theory to eigenvalue problems see [13]. We say that a closed symmetric operator S in the Krein space $(\mathcal{K}, [\cdot,\cdot])$

has a (finite) *defect* d if there exists a self-adjoint extension A of S in $\mathcal{K}$ such that $d = \dim\big(\operatorname{dom}(A)/\operatorname{dom}(S)\big) < +\infty$. This is equivalent to the fact that the operator $B := JA$ is a self-adjoint extension of the closed symmetric operator $T := JS$ in the Hilbert space $(\mathcal{K}, \langle\cdot,\cdot\rangle)$, that is, the operator T has defect d, see [10, Section 1.1].

Let S_{m} be a closed symmetric operator with defect m in a Krein space $(\mathcal{K}_0, [\cdot,\cdot]_0)$, let J_0 be a fundamental symmetry on $\mathcal{K}_0$ and let $\langle f, g\rangle_0 = [J_0 f, g]_0$ be the corresponding Hilbert space inner product. Then $T_{\mathrm{m}} = J_0 S_{\mathrm{m}}$ is closed symmetric operator with defect m in the Hilbert space $(\mathcal{K}_0, \langle\cdot,\cdot\rangle_0)$. Clearly $\operatorname{dom}(T_{\mathrm{m}}) = \operatorname{dom}(S_{\mathrm{m}})$. By $S_{\mathrm{m}}^{[*]}$ we denote the adjoint of S_{m} in $(\mathcal{K}_0, [\cdot,\cdot]_0)$, so $S_{\mathrm{m}}^{[*]} = J_0 T_{\mathrm{m}}^*$, where T_{m}^* is the adjoint of T_{m} in $(\mathcal{K}_0, \langle\cdot,\cdot\rangle_0)$. Since $\operatorname{dom}(T_{\mathrm{m}}^*) = \operatorname{dom}(S_{\mathrm{m}}^{[*]})$, any boundary mapping for T_{m} is a boundary mapping for S_{m}. Let $\mathbf{b}$ be a boundary mapping for S_{m}, with Q as the corresponding concomitant matrix, as in Section 2. Our abstract eigenvalue problem is now

$$S_{\mathrm{m}}^{[*]} f_0 = \lambda f_0, \quad f_0 \in \operatorname{dom}(S_{\mathrm{m}}^{[*]}), \tag{3.1}$$

$$\begin{bmatrix} D \\ M \end{bmatrix} \mathbf{b}(f_0) = \lambda \begin{bmatrix} 0 \\ N \end{bmatrix} \mathbf{b}(f_0), \tag{3.2}$$

where M and N are $d \times 2m$ matrices, $0 < d \le m$, and D is a $(m-d) \times 2m$ matrix of full rank. Note that (3.1) is equivalent to $T_{\mathrm{m}}^* f_0 = \lambda J_0 f_0$. Let

$$(\widetilde{\mathcal{K}}, [\cdot,\cdot]) = (\mathcal{K}_0, [\cdot,\cdot]_0) \oplus \mathbb{C}_\Delta^d$$

so the inner product on $\widetilde{\mathcal{K}}$ is given by

$$[f, g] := [f_0, g_0]_0 + g_1^* \Delta^{-1} f_1,$$
$$f = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}, \quad g = \begin{bmatrix} g_0 \\ g_1 \end{bmatrix} \in \widetilde{\mathcal{K}}, \quad f_0, g_0 \in \mathcal{K}_0,\ f_1, g_1 \in \mathbb{C}^d.$$

Then an operator $\widetilde{A}$ can be associated with the problem (3.1)–(3.2) in $\mathcal{K}_0 \oplus \mathbb{C}_\Delta^d$ in the following way. Let

$$\operatorname{dom}(\widetilde{A}) = \left\{ f = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} \in \widetilde{\mathcal{K}} : f_0 \in \operatorname{dom}(S_{\mathrm{m}}^{[*]}), D\mathbf{b}(f_0) = 0, f_1 = N\mathbf{b}(f_0) \right\} \tag{3.3}$$

and define

$$\widetilde{A} f = \widetilde{A} \begin{bmatrix} f_0 \\ N\mathbf{b}(f_0) \end{bmatrix} := \begin{bmatrix} S_{\mathrm{m}}^{[*]} f_0 \\ M\mathbf{b}(f_0) \end{bmatrix}, \quad \text{for } f \in \operatorname{dom}(\widetilde{A}). \tag{3.4}$$

Theorem 3.1 *Let S_{m} be a closed symmetric operator with defect m in a Krein space $(\mathcal{K}_0, [\cdot,\cdot]_0)$. Let $\mathbf{b}\colon \operatorname{dom}(S_{\mathrm{m}}^{[*]}) \to \mathbb{C}^{2m}$ be a boundary mapping for S_{m} with concomitant matrix Q. Let d be an integer with $0 < d \le m$ and assume that M and N are $d \times 2m$ matrices which satisfy* (2.7)*, that the matrix $MQ^{-1}N^*$ is invertible and that*

$$\Delta = \frac{1}{i}(MQ^{-1}N^*)^{-1} \quad \textit{is self-adjoint and invertible.} \tag{3.5}$$

Let D be a $(m-d)\times 2m$ matrix such that $DQ^{-1}D^* = DQ^{-1}M^* = DQ^{-1}N^* = 0$ and such that the $(m+d)\times 2m$ matrix $\begin{bmatrix} D \\ M \\ N \end{bmatrix}$ has maximal rank $m+d$. Then the space $(\widetilde{\mathcal{K}}, [\cdot,\cdot]) = (\mathcal{K}_0, [\cdot,\cdot]_0) \oplus \mathbb{C}^d_\Delta$ is a Krein space and the operator $\widetilde{A}$ defined by (3.3)–(3.4) *is self-adjoint.*

The operator

$$\widetilde{J} := \begin{bmatrix} J_0 & 0 \\ 0 & \operatorname{sgn}(\Delta) \end{bmatrix}, \tag{3.6}$$

where $\operatorname{sgn}(\Delta) = |\Delta|^{-1}\Delta$, *is a fundamental symmetry on the Krein space* $(\widetilde{\mathcal{K}}, [\cdot,\cdot])$.

Proof The claims that $\mathcal{K}_0 \oplus \mathbb{C}^d_\Delta$ is a Krein space and that $\widetilde{J}$ is a fundamental symmetry are clear. The operator $T_{\mathrm{m}} = J_0 S_{\mathrm{m}}$ and the matrices D, N and $\operatorname{sgn}(\Delta)M$ satisfy all the conditions of Theorem 2.4 and the operator $\widetilde{B}$ from that theorem equals the operator $\widetilde{J}\widetilde{A}$ in the present theorem. By Theorem 2.4, $\widetilde{J}\widetilde{A}$ is self-adjoint and consequently $\widetilde{A}$ is self-adjoint. ■

Remark 3.2 The operator $\widetilde{B} := \widetilde{J}\widetilde{A}$ is associated with the positive definite eigenvalue problem

$$J_0 S_{\mathrm{m}}^{[*]} f_0 = \lambda f_0, \quad f_0 \in \operatorname{dom}(S_{\mathrm{m}}^{[*]}), \tag{3.7}$$

$$\begin{bmatrix} D \\ \operatorname{sgn}(\Delta)M \end{bmatrix} \mathbf{b}(f_0) = \lambda \begin{bmatrix} 0 \\ N \end{bmatrix} \mathbf{b}(f_0), \tag{3.8}$$

which is studied in Section 2 in the direct sum of the Hilbert spaces $(\mathcal{K}_0, \langle\cdot,\cdot\rangle_0)$ and $\mathbb{C}^d_{|\Delta|}$. The eigenvalue problem (3.7)–(3.8) will be called the definite (or Hilbert space) eigenvalue problem associated with the indefinite (or Krein space) eigenvalue problem (3.1)–(3.2).

In the next result we collect some spectral properties of the operator $\widetilde{A}$ which can be deduced from special conditions on the symmetric operator T_{m}. These special conditions are fulfilled by symmetric differential operators associated with (1.1) under our basic assumptions on the coefficients. For properties of quasi-uniformly positive operators on Hilbert and Krein spaces we refer to [2] and [11] respectively.

Theorem 3.3 *Let S_{m} be a symmetric operator which is bounded below in the Krein space $(\mathcal{K}_0, [\cdot,\cdot]_0)$. Assume that the operator $T_{\mathrm{m}} = J_0 S_{\mathrm{m}}$ has a self-adjoint extension with discrete spectrum in the Hilbert space $(\mathcal{H}_0, \langle\cdot,\cdot\rangle_0)$. Then*

(a) *The operator $\widetilde{A}$ defined by* (3.3)–(3.4) *is quasi-uniformly positive (and therefore definitizable) with discrete spectrum in the Krein space $(\widetilde{\mathcal{K}}, [\cdot,\cdot])$.*
(b) *Each eigenvalue of $\widetilde{A}$ has finite algebraic multiplicity.*
(c) *The algebraic eigenspaces corresponding to real distinct eigenvalues of $\widetilde{A}$ are mutually orthogonal in the Krein space $(\widetilde{\mathcal{K}}, [\cdot,\cdot])$.*

(d) *All but finitely many eigenvalues of $\widetilde{A}$ are semisimple, real and have the property that $\lambda[\phi, \phi] > 0$ for all nonzero eigenvectors ϕ corresponding to λ.*
(e) *The linear span of the algebraic eigenspaces of $\widetilde{A}$ is dense in $\widetilde{\mathcal{K}}$.*

Proof Let $\widetilde{J}$ be the fundamental symmetry on $(\widetilde{K}, [\cdot,\cdot])$ defined in (3.6) and let B_0 be a self-adjoint extension of T_{m} with discrete spectrum in the Hilbert space $(\mathcal{H}_0, \langle\cdot,\cdot\rangle_0)$. Since T_{m} is bounded below so is B_0. Corollary 2.7 implies that the operator $\widetilde{B}$ is bounded below and has discrete spectrum in $(\widetilde{\mathcal{H}}, \langle\cdot,\cdot\rangle)$. It follows from [11, Proposition 1.4] that $\widetilde{A}$ is quasi-uniformly positive and has discrete spectrum in $(\widetilde{\mathcal{K}}, [\cdot,\cdot])$. The statements (b) and (d) follow from [11, Proposition 1.6]. The statement (c) follows from [5, Theorem II.3.3] and (e) follows from [19, Propositions 5.1 and 5.2]. ■

Remark 3.4 More information about the number of eigenvalues which do not have the three properties listed in Theorem 3.3 (d) and other spectral properties of definitizable operators can be found in [11], [13], [19].

We illustrate the constructions so far with the example of a Sturm-Liouville problem with separated λ-dependent boundary conditions.

Example 3.5 Let $n = 1$ and consider the problem

$$-(py')' + qy = \lambda ry \quad \text{on} \quad [0,1], \tag{3.9}$$

subject to

$$\begin{aligned}(\lambda a_0 + b_0)y(0) &= (\lambda c_0 + d_0)(py')(0)\\ (\lambda a_1 + b_1)y(1) &= (\lambda c_1 + d_1)(py')(1).\end{aligned} \tag{3.10}$$

We assume that $1/p, q, r$ are integrable over $[0,1]$, with $p > 0$ a.e. and that the boundary conditions are nontrivial, that is, $(a_j, b_j, c_j, d_j) \neq (0,0,0,0)$, $j = 0,1$. To distinguish those boundary conditions in (3.10) that are genuinely λ-dependent (corresponding to M and N in (1.2)) we introduce the set $\Lambda \subseteq \{0,1\}$ so that $j \in \Lambda$ if and only if $(a_j, c_j) \neq (0,0)$. For $j \in \Lambda$ we assume

$$\delta_j = \begin{vmatrix} a_j & b_j \\ c_j & d_j \end{vmatrix} \neq 0.$$

We define the operator T_0 in the Hilbert or Krein space $L_2([0,1], r)$ by

$$T_0\colon f \mapsto \ell(f) := \frac{1}{r}\big(-(pf')' + qf\big)$$

for

$$f \in \operatorname{dom}(T_0) := \left\{ f \in \mathcal{D}_{\max} : \begin{matrix} b_j f(j) = d_j (pf')(j),\, j = 0,1, \\ a_j f(j) = c_j (pf')(j),\, j \in \Lambda \end{matrix} \right\},$$

where

$$\mathcal{D}_{\max} = \{f \in L_2([0,1],r) : f, pf' \in AC[0,1], \ell(f) \in L_2([0,1],r)\}.$$

It follows from the methods of [20] (or [24]) that T_0 is a closed symmetric operator with defect index (μ,μ), that is with defect μ, as required for the abstract problem (2.3)–(2.4). Here μ is the number of elements in Λ. It turns out that the domain of the adjoint of T_0 is given by

$$\operatorname{dom}(T_0^*) := \{f \in \mathcal{D}_{\max} : b_j f(j) = d_j (pf')(j), j \notin \Lambda\},$$

so the operator T_0^* includes only the λ-independent boundary conditions in (3.10), corresponding to D in (1.2). If Λ is empty, then T_0 is self-adjoint. Otherwise T_0 is a symmetric operator in $L_2([0,1],r)$. For the problem (3.9)–(3.10) the boundary mapping is defined by $\mathbf{b}(f) = \begin{bmatrix} f(0) & f(1) & (pf')(0) & (pf')(1)\end{bmatrix}^T$ defined for all $f \in \mathcal{D}_{\max}$. From (2.2) we calculate the concomitant matrix of $\mathbf{b}$ to be

$$Q = i\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

The boundary conditions (3.10) then take the form (1.2) if we set

$$M = \begin{bmatrix} b_0 & 0 & -d_0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} -a_0 & 0 & c_0 & 0 \end{bmatrix} \quad \text{if } \Lambda = \{0\},$$

$$M = \begin{bmatrix} 0 & b_1 & 0 & -d_1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & -a_1 & 0 & c_1 \end{bmatrix} \quad \text{if } \Lambda = \{1\},$$

$$M = \begin{bmatrix} b_0 & 0 & -d_0 & 0 \\ 0 & b_1 & 0 & -d_1 \end{bmatrix}, \quad N = \begin{bmatrix} -a_0 & 0 & c_0 & 0 \\ 0 & -a_1 & 0 & c_1 \end{bmatrix} \quad \text{if } \Lambda = \{0,1\}.$$

We easily verify (2.7) and that

$$\Delta = (-1)^{j-1}\delta_j \quad \text{if } \Lambda = \{j\} \quad \text{and}$$

$$\Delta = \begin{bmatrix} -\delta_0 & 0 \\ 0 & \delta_1 \end{bmatrix} \quad \text{if } \Lambda = \{0,1\}$$

satisfies (3.5). Thus $\widetilde{\mathcal{K}} = L_2([0,1],r) \oplus \mathbb{C}^{\mu}_{\Delta}$ is a Hilbert space if r and Δ are positive. If not, then $\widetilde{\mathcal{K}}$ is in general a Krein space and the definite problem associated with (3.9)–(3.10) is

$$-(py')' + qy = \lambda |r| y \quad \text{on } [0,1], \tag{3.11}$$

subject to

$$\begin{aligned} \big(\lambda a_0 - \operatorname{sgn}(\delta_0) b_0\big)\, y(0) &= \big(\lambda c_0 - \operatorname{sgn}(\delta_0) d_0\big)\,(py')(0) \\ \big(\lambda a_1 + \operatorname{sgn}(\delta_1) b_1\big)\, y(1) &= \big(\lambda c_1 + \operatorname{sgn}(\delta_1) d_1\big)\,(py')(1) \end{aligned} \tag{3.12}$$

which is considered in the Hilbert space $\widetilde{\mathcal{H}} = L_2([a,b],|r|) \oplus \mathbb{C}^{\mu}_{|\Delta|}$. It follows from well known spectral properties of the problem (3.11)–(3.12) (see *e.g.*, [15]) that Theorem 3.3 applies to the operator $\widetilde{A}$ associated with the indefinite problem (3.9)–(3.10) in the Krein space $L_2([0,1],r) \oplus \mathbb{C}^{\mu}_{\Delta}$.

4 Form Domains of Operators in $L_2([a,b],r) \oplus \mathbb{C}^d_\Delta$

Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space, let J be a fundamental symmetry on $\mathcal{K}$, $\langle x, y\rangle := [Jx, y]$ the Hilbert space inner product associated with J and let $\|x\| := \langle x, x\rangle^{1/2}$ be a norm on $\mathcal{K}$. Let A be a self-adjoint operator which is bounded below in a Krein space $(\mathcal{K}, [\cdot,\cdot])$, so there exists a real number α such that $[Ax, x] \geq \alpha\|x\|^2$ for all $x \in \operatorname{dom}(A)$. Clearly, A is bounded below in $(\mathcal{K}, [\cdot,\cdot])$ if and only if the operator $B := JA$ is bounded below in the Hilbert space $(\mathcal{K}, \langle\cdot,\cdot\rangle)$. The form $[A\,\cdot,\cdot] = \langle B\,\cdot,\cdot\rangle$ is in a natural way associated with A. Therefore the natural form domain associated with A in the Krein space $\mathcal{K}$ is identical with the form domain of B in the Hilbert space $(\mathcal{K}, \langle\cdot,\cdot\rangle)$, and we define

$$\operatorname{fdom}(A) := \operatorname{fdom}(JA). \tag{4.1}$$

This definition is independent of the choice of a fundamental symmetry J (see [7, Section 1]). The natural form on $\operatorname{fdom}(A)$ is $[\cdot,\cdot]_A := \langle\cdot,\cdot\rangle_B$.

The goal of this section is to characterize the form domains of the operators associated with the problem (1.1)–(1.2) in the Krein space $L_2([a,b],r) \oplus \mathbb{C}^d_\Delta$. Besides (1.1) we shall consider the equation

$$L(f) = \lambda|r|f \quad \text{on } [a,b]. \tag{4.2}$$

A natural setting in which operators are associated with (1.1) (and (4.2)) is the Krein space $L_2([a,b],r)$, that is the space of all (equivalence classes of) measurable functions f defined on $[a,b]$ for which $\int_a^b |f|^2\,|r| < +\infty$. The indefinite and definite inner products on $L_2([a,b],r)$ are

$$[f,g] := \int_a^b f\overline{g}r \quad \text{and} \quad \langle f,g\rangle := \int_a^b f\overline{g}|r|, \tag{4.3}$$

respectively. Clearly, the fundamental symmetry connecting the two inner products in (4.3) is

$$(J_0 f)(t) := \big(\operatorname{sgn} r(t)\big)\, f(t), \quad t \in [a,b], \tag{4.4}$$

and $L_2([a,b],r) = L_2([a,b],|r|)$. Recall that the maximal operator $T_{\max}$ associated with (4.2) is defined in $L_2([a,b],|r|)$ by $T_{\max}(f) := \frac{1}{|r|}L(f)$ for all f in

$$\operatorname{dom}(T_{\max}) = \Big\{ f \in L_2([a,b],r) : f^{[k]} \in AC[a,b], k = 0,\dots,2n-1, \\ \text{and } \frac{1}{|r|}f^{[2n]} = \frac{1}{|r|}L(f) \in L_2([a,b],r) \Big\}.$$

The minimal operator $T_{\min}$ associated with (4.2) is the restriction of $T_{\max}$ to the set of all $f \in \operatorname{dom}(T_{\max})$ such that $f^{[k]}(a) = f^{[k]}(b) = 0$ for $k = 0,1,\dots,2n-1$. It is a closed symmetric operator with defect $2n$ and $T_{\min}^* = T_{\max}$ in the Hilbert space

$L_2([a,b],|r|)$. For properties of $T_{\min}$ and $T_{\max}$ and the conditions for a $2n$-th order symmetric differential expression $L(f)$ under which these operators are studied see [18], [20] or [24, Section 3]. The maximal and minimal operators associated with (1.1) are now defined as $S_{\max} := J_0 T_{\max}$ and $S_{\min} := J_0 T_{\min}$, respectively, with J_0 defined in (4.4), see [10, Section 2]. Since $T_{\min}$ is a closed symmetric operator with defect $2n$ in $L_2([a,b],|r|)$, the operator $S_{\min}$ is closed symmetric and has defect $2n$ in the Krein space $L_2([a,b],r)$. An operator S_0 is a closed symmetric extension of $S_{\min}$ in $L_2([a,b],r)$ if and only if $T_0 := J_0 S_0$ is a closed symmetric extension of $T_{\min}$ in $L_2([a,b],|r|)$. It is well known that the domain of a closed symmetric extension T_0 of $T_{\min}$, and hence the domain of a closed symmetric extension S_0 of $S_{\min}$, is determined as a subspace of $\operatorname{dom}(T_{\max}) = \operatorname{dom}(S_{\max})$ by a set of $2n+d$ boundary conditions. In the rest of this paper, for the closed symmetric operator $T_{\min}$, and hence for $S_{\min}$ as well, we use the boundary mapping $\mathbf{b}\colon \operatorname{dom}(T_{\max}) \to \mathbb{C}^{4n}$ for which the column vector $\mathbf{b}(f)$ is

$$\begin{aligned}[t] [\,f(a)\ f'(a)\ \ldots\ f^{(n-1)}(a)\ f(b)\ f'(b)\ \ldots\ f^{(n-1)}(b) \\ f^{[n]}(a)\ \ldots\ f^{[2n-1]}(a)\ f^{[n]}(b)\ \ldots\ f^{[2n-1]}(b)]^T. \end{aligned}$$

Integrating by parts in (2.2) we calculate the concomitant matrix of $\mathbf{b}$ to be

$$Q = i\begin{bmatrix} 0 & 0 & -R & 0 \\ 0 & 0 & 0 & R \\ R & 0 & 0 & 0 \\ 0 & -R & 0 & 0 \end{bmatrix} \quad \text{with} \quad R = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where the matrix R and all the other blocks in Q are $n \times n$ matrices. It is convenient to introduce a self-adjoint $2n \times 2n$ matrix $Q_e := \left[\begin{smallmatrix} R & 0 \\ 0 & -R \end{smallmatrix}\right]$ so that Q can be written as $Q = i\left[\begin{smallmatrix} 0 & -Q_e \\ Q_e & 0 \end{smallmatrix}\right]$. Since R is a self-adjoint matrix and $RR = I$, the matrices Q and Q_e are self-adjoint and $QQ = I$ and $Q_e Q_e = I$.

A $1 \times 4n$ row vector u is called a *boundary condition*. A function f in $\operatorname{dom}(T_{\max})$ *satisfies* the boundary condition u if $u\mathbf{b}(f) = 0$. The boundary condition u is called *essential* if the final $2n$ components of u are all 0. Essential boundary conditions are discussed by M. G. Krein in [18, §7], see also [3], [6]. Let $0 \le k \le 4n$. A convenient way to write a set of k linearly independent boundary conditions is to use a $k \times 4n$ matrix D of maximal rank. Then a function f in $\operatorname{dom}(T_{\max})$ satisfies all k boundary conditions in D if $D\mathbf{b}(f) = 0$. We will study restrictions of the operator $T_{\max}$ to domains of the form

$$\{f \in \operatorname{dom}(T_{\max}) : D\mathbf{b}(f) = 0\}. \tag{4.5}$$

Denote by T_0 the adjoint (in $L_2([a,b],r)$) of the restriction of $T_{\max}$ onto the domain in (4.5). By [9, Lemma 3.5] the operator T_0 is symmetric in $L_2([a,b],r)$ if and only if $DQ^{-1}D^* = 0$.

Clearly the domain in (4.5) will not change if we row reduce the matrix D. Since essential boundary conditions play an important role in what follows we will use the following construction. Given a $k \times 4n$ matrix X, of maximal rank k, let Y be its reduced row echelon form (starting the reduction at the bottom right corner). Write

$$Y = \begin{bmatrix} X_e & 0 \\ X_{21} & X_{22} \end{bmatrix}, \tag{4.6}$$

where 0 is a $p \times 2n$ matrix and the $(k-p) \times 2n$ matrix X_{22} and $p \times 2n$ matrix X_e are of maximal ranks. We allow for the possibility that $p = 0$. In that case we consider X_e to be an "empty" matrix, that is, all formulae involving X_e can be dropped.

Let $\mathcal{F}_{\max}$ be the set of all functions f in $L_2([a,b], r)$ such that $f, f', \ldots, f^{(n-1)}$ are absolutely continuous on $[a,b]$ and $\int_a^b p_n |f^{(n)}|^2 < +\infty$. For functions $f \in \mathcal{F}_{\max}$ we introduce the *essential boundary mapping* $\mathbf{b}_e \colon \mathcal{F}_{\max} \to \mathbb{C}^{2n}$ by

$$\mathbf{b}_e(f) := \begin{bmatrix} f(a) & f'(a) & \cdots & f^{(n-1)}(a) & f(b) & f'(b) & \cdots & f^{(n-1)}(b) \end{bmatrix}^T.$$

Clearly $\mathbf{b}_e$ is surjective. It follows from [18, §7] that the kernel $\ker(\mathbf{b}_e)$ equals the form domain of the Friedrichs extension of $T_{\min}$.

The next two theorems deal with a self-adjoint operator in the Krein space $L_2([a,b], r) \oplus \mathbb{C}^d_\Delta$ which can be associated with the problem (1.1)–(1.2). Since the operator $T_{\min} = J_0 S_{\min}$ is bounded below and has a self-adjoint extension with discrete spectrum, the first theorem is an immediate consequence of Theorems 3.1 and 3.3.

Theorem 4.1 *Let D be a $(2n-d) \times 4n$ matrix and let M and N be $d \times 4n$ matrices such that $DQ^{-1}D^* = MQ^{-1}M^* = NQ^{-1}N^* = DQ^{-1}M^* = DQ^{-1}N^* = 0$, the $(2n+d) \times 4n$ matrix $\left[\begin{smallmatrix} D \\ M \\ N \end{smallmatrix}\right]$ has maximum rank $2n+d$ and such that $\Delta^{-1} := iMP^{-1}N^*$ is self-adjoint and invertible. Let $\widetilde{A}$ be the extension of $S_{\min}$ in $\widetilde{\mathcal{K}} = L_2([a,b], r) \oplus \mathbb{C}^d_\Delta$ defined by* (3.3)–(3.4), *where the operator S_{m} in* (3.3)–(3.4) *is replaced by $S_{\min}$.*

Then $\widetilde{A}$ is a self-adjoint operator in the Krein space $\widetilde{\mathcal{K}} = L_2([a,b], r) \oplus \mathbb{C}^d_\Delta$ and it has all the properties listed in Theorem 3.3.

Theorem 4.2 *In addition to the assumptions of Theorem 4.1, also assume that the matrices D and N are of the form given in* (4.6) *and that the matrix N_e is of size $k \times 2n$. Let $\widetilde{A}$ be the self-adjoint extension of S_0 in Theorem 4.1. Then the form domain* $\mathrm{fdom}(\widetilde{A})$ *of $\widetilde{A}$ is given by*

$$\mathrm{fdom}(\widetilde{A}) = \left\{ \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} \in \widetilde{\mathcal{K}} : f_0 \in \mathcal{F}_{\max}, D_e \mathbf{b}_e(f_0) = 0, f_1 = \begin{bmatrix} N_e \mathbf{b}_e(f_0) \\ x \end{bmatrix}, x \in \mathbb{C}^{d-k} \right\}. \tag{4.7}$$

Proof By definition, $\mathrm{fdom}(\widetilde{A}) = \mathrm{fdom}(\widetilde{B})$, where $\widetilde{B} = \widetilde{J}\widetilde{A}$, with $\widetilde{J}$ as in (3.6). Therefore, without loss of generality, we can assume that $r > 0$ and that Δ is positive definite. Thus we shall replace $S_{\min}$, $S_{\max}$ and $\widetilde{A}$ by the Hilbert space notation $T_{\min}$, $T_{\max}$ and $\widetilde{B}$ from Section 2.

It follows from [9, Lemma 3.4] and the assumptions about D, M and N, that the restriction T_0 of $T_{\max}$ defined on

$$\operatorname{dom}(T_0) = \left\{ f \in \operatorname{dom}(T_{\max}) : \begin{bmatrix} D \\ M \\ N \end{bmatrix} \mathbf{b}(f) = 0 \right\} \tag{4.8}$$

is a symmetric extension of $T_{\min}$ with defect d and that the domain of its adjoint is

$$\operatorname{dom}(T_0^*) = \{ f \in \operatorname{dom}(T_{\mathrm{m}}^*) : D\mathbf{b}(f) = 0 \}.$$

Denote the right hand side of (4.7) by $\mathcal{F}_1$. First we prove that $\operatorname{fdom}(\widetilde{B}) \subset \mathcal{F}_1$. Since the $2n \times 4n$ matrix $\left[\begin{smallmatrix} D \\ N \end{smallmatrix}\right]$ has maximum rank $2n$ and satisfies

$$\begin{bmatrix} D \\ N \end{bmatrix} Q^{-1} \begin{bmatrix} D \\ N \end{bmatrix}^* = 0,$$

[9, Lemma 3.4] implies that the restriction B_0 of $T_{\max}$ to the domain

$$\operatorname{dom}(B_0) := \{ f_0 \in \operatorname{dom}(T_{\max}) : D\mathbf{b}(f_0) = N\mathbf{b}(f_0) = 0 \}$$

is a self-adjoint operator in the Hilbert space $L_2([a,b],r)$. The operator B_0 is exactly the self-adjoint extension of T_0 used in Lemma 2.8. Using the notation and results of Lemma 2.8, to complete the proof of $\operatorname{fdom}(\widetilde{B}) \subset \mathcal{F}_1$, we only need to prove that

$$\operatorname{fdom}(B_0) \subset \mathcal{F}_1 \quad \text{and} \quad \operatorname{span}\{\phi^1, \ldots, \phi^d\} \subset \mathcal{F}_1. \tag{4.9}$$

Since $\phi^1, \ldots, \phi^d \in \operatorname{dom}(\widetilde{B}) \subset \mathcal{F}_1$, the second inclusion in (4.9) follows directly from the definition of $\operatorname{dom}(\widetilde{B})$. To prove the first inclusion in (4.9), we use the description of the form domain

$$\operatorname{fdom}(B_0) = \{ f_0 \in \mathcal{F}_{\max} : D_e\mathbf{b}_e(f_0) = N_e\mathbf{b}_e(f_0) = 0 \}$$

which is given in [18, §7]. Therefore,

$$\operatorname{fdom}(B_0) \ni \begin{bmatrix} f_0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_0 \\ N_e\mathbf{b}_e(f_0) \\ 0 \end{bmatrix} \in \mathcal{F}_1.$$

A proof of $\mathcal{F}_1 \subset \operatorname{fdom}(\widetilde{B})$ follows. Since the vectors $\phi^1, \ldots, \phi^d \in \operatorname{dom}(\widetilde{B})$ are linearly independent over $\operatorname{dom}(B_0)$, it follows that the $d \times 1$ vectors $N\mathbf{b}(\phi_0^1), \ldots, N\mathbf{b}(\phi_0^d)$ are linearly independent. Let f be an arbitrary element in $\mathcal{F}_1$. Our goal is to find $g_0 \in \operatorname{fdom}(B_0)$ and $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$ such that

$$f = \begin{bmatrix} f_0 \\ N_e\mathbf{b}_e(f_0) \\ x \end{bmatrix} = \begin{bmatrix} g_0 \\ 0 \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^d \alpha_j \phi_0^j \\ \sum_{j=1}^d \alpha_j N\mathbf{b}(\phi_0^j) \end{bmatrix}. \tag{4.10}$$

Thus, we have to find $\alpha_1, \dots, \alpha_d \in \mathbb{C}$ such that

$$f_0 - \sum_{j=1}^{d} \alpha_j \phi_0^j \in \mathrm{fdom}(B_0), \tag{4.11}$$

$$\begin{bmatrix} N_e \mathbf{b}_e(f_0) \\ x \end{bmatrix} = \sum_{j=1}^{d} \alpha_j N\mathbf{b}(\phi_0^j). \tag{4.12}$$

Clearly the function on the left hand side of (4.11) satisfies the smoothness condition to be in $\mathrm{fdom}(B_0)$. To make sure that this function is in $\mathrm{fdom}(B_0)$ we need to check the essential boundary conditions:

$$D_e\mathbf{b}_e(f_0) = \sum_{j=1}^{d} \alpha_j D_e \mathbf{b}_e(\phi_0^j) \quad \text{and} \quad N_e\mathbf{b}_e(f_0) = \sum_{j=1}^{d} \alpha_j N_e \mathbf{b}_e(\phi_0^j). \tag{4.13}$$

The definition of $\mathcal{F}_1$ and the fact that $\phi_0^j \in \mathrm{dom}(T_0^*)$, $j = 1, \dots, d$, imply that the left hand equality in (4.13) is satisfied. The right hand equality in (4.13) is in fact a part of the equality (4.12). Thus, to complete the proof we only need to note that the system (4.12) has a solution for $\alpha_1, \dots, \alpha_d \in \mathbb{C}$ since the vectors $N\mathbf{b}(\phi_0^1), \dots, N\mathbf{b}(\phi_0^d)$ are linearly independent. ■

5 Expansions in the Form Domain

In this section we will consider various norms on form domains of differential operators in the space $\widetilde{\mathcal{K}} = L_2([a,b], r) \oplus \mathbb{C}^d_\Delta$. Let $f \in \mathcal{F}_{\max}$. Put

$$|||f||| := \left(\int_a^b p_n^{1/2} |f^{(n)}|^2 \right)^{1/2} + \sum_{k=0}^{n-1} \max\{|f^{(k)}(x)| : x \in [a,b]\}.$$

Lemma 5.1 *The space $(\mathcal{F}_{\max}, |||\cdot|||)$ is a Banach space. The essential boundary mapping $\mathbf{b}_e \colon (\mathcal{F}_{\max}, |||\cdot|||) \to \mathbb{C}^{2n}$ is continuous.*

Proof Let $\{f_j\}$ be a Cauchy sequence in $(\mathcal{F}_{\max}, |||\cdot|||)$. Then $f_j^{(k)} \to g_k$ as $j \to +\infty$ in $\mathcal{C}([a,b])$ for $k = 0, 1, \dots, n-1$ and $p_n^{1/2} f_j^{(n)} \to p_n^{1/2} g_n$ in $L_2([a,b])$ (or, $f_j^{(n)} \to g_n$ in $L_2([a,b], p_n)$).

Since strong convergence in L_2 implies weak convergence, it follows that

$$\int_a^x f_j^{(n)} = \int_a^b p_n^{-1/2} \chi_{[a,x]} p_n^{1/2} f_j^{(n)} \to \int_a^b p_n^{-1/2} \chi_{[a,x]} p_n^{1/2} g_n = \int_a^x g_n \quad \text{as } j \to +\infty.$$

Therefore,

$$f_j^{(n-1)}(x) - f_j^{(n-1)}(a) \to \int_a^x g_n \quad \text{as } j \to +\infty \text{ for all } x \in [a,b],$$

and, consequently $g_{n-1}(x) - g_{n-1}(a) = \int_a^x g_n$. Since the function $g_n = p_n^{-1/2} \cdot p_n^{1/2} g_n$, as a product of two L_2 functions, is integrable, we conclude that g_{n-1} is absolutely continuous on $[a, b]$ and $g'_{n-1} = g_n$ a.e. on $[a, b]$. Similarly, we conclude that all the functions g_k, $k = 0, 1, \ldots, n-1$ are absolutely continuous on $[a, b]$ and that $g'_{k-1} = g_k$, $k = 1, \ldots, n$. Since the function $p_n^{1/2} g_n$ is integrable over $[a, b]$, it follows that $g_0 \in \mathcal{F}_{\max}$. Also, $f_j^{(k)} \to g_0^{(k)}$ as $j \to +\infty$ in $\mathcal{C}([a, b])$ for $k = 0, 1, \ldots, n-1$ and $p_n^{1/2} f_j^{(n)} \to p_n^{1/2} g_0^{(n)}$ in $L_2([a, b])$. Therefore, $\{f_j\}$ converges to g_0 in $(\mathcal{F}_{\max}, |||\cdot|||)$.

The essential mapping $\mathbf{b}_e$ is continuous since clearly for each $f \in \mathcal{F}_{\max}$ the maximum of the moduli of the components of $\mathbf{b}_e(f)$ is less or equal to $|||f|||$. ■

Consider the direct sum $\mathcal{F}_{\max} \oplus \mathbb{C}^d$ of Banach spaces $(\mathcal{F}_{\max}, |||\cdot|||)$ and $\mathbb{C}^d$ with the norm of $x \in \mathbb{C}^d$ given by the maximum of the moduli of its components. This norm will also be denoted by $|||\cdot|||$. Let $\widetilde{A}$ be the self-adjoint operator of Theorem 4.1, and let α be chosen so that $\widetilde{A} + \alpha\widetilde{J}$ is uniformly positive in $L_2([a, b], r) \oplus \mathbb{C}^d_\Delta$.

Our main result about form domain convergence is as follows.

Theorem 5.2 *The vector space* $\operatorname{fdom}(\widetilde{A})$ *is a Hilbert space* $\mathcal{H}$ *under* $[\cdot, \cdot]_{\widetilde{A}+\alpha\widetilde{J}}$ *and a Banach space* $\mathcal{B}$ *under* $|||\cdot|||$, *and the corresponding two norms are equivalent.*

Proof The first contention follows by the definitions and the second follows from Theorem 4.2 and Lemma 5.1. For the third, let $\imath$ denote the inclusion mapping from $\mathcal{H}$ to $\mathcal{B}$.

Suppose that $x_n \to x$ in $\mathcal{H}$ and $\imath x_n \to y$ in $\mathcal{B}$. Since the topologies in $\mathcal{H}$ and $\mathcal{B}$ are stronger than that of $\widetilde{\mathcal{H}}$, we have $x_n \to x$ and $\imath x_n = x_n \to y$ in $\widetilde{\mathcal{H}}$, whence $x = y$. But $\imath x = x$, so $\imath x = y$ and we conclude that $\imath$ is closed, hence bounded by the closed graph theorem [16, Theorem III.5.20].

By the same reasoning, the inclusion mapping from $\mathcal{B}$ to $\mathcal{H}$ is bounded, and the proof is complete. ■

Remark 5.3 If $1/p_n \in L_\infty[a, b]$, then the norms in Theorem 5.2 dominate the Sobolev norm of $W_2^n \oplus \mathbb{C}^d$. If in addition $p_n \in L_\infty[a, b]$, then all three norms are equivalent.

As a consequence of Theorem 5.2, classical results from the spectral theory of self-adjoint operators in Hilbert spaces which are given in terms of convergence in $\mathcal{H}$ are equivalent to results about (uniform) convergence in $\mathcal{B}$. We are now ready for our main result on series convergence in the form domain of $\widetilde{A}$, which we recall is quasi-uniformly positive in the Krein space $\widetilde{\mathcal{K}} = L_2([a, b], r) \oplus \mathbb{C}^d_\Delta$.

Theorem 5.4 *Let* λ_j, $j = 1, 2, \ldots$ *be precisely those eigenvalues of* $\widetilde{A}$ *(each repeated according to its multiplicity) which are semisimple, real and have the property that* $\lambda_j[\phi, \phi] > 0$ *for all nonzero eigenvectors corresponding to* λ_j. *Let*

$$\phi_j = \begin{bmatrix} \phi_{j0} \\ N\mathbf{b}(\phi_{j0}) \end{bmatrix}, \quad j \in \mathbb{N}, \tag{5.1}$$

be corresponding nontrivial eigenvectors of $\widetilde{A}$ which are mutually orthogonal in $\widetilde{\mathcal{K}}$. For an element $f = \left[\begin{smallmatrix} f_0 \\ f_1 \end{smallmatrix}\right] \in \widetilde{\mathcal{K}}$, the following are equivalent

(a) *f belongs to* $\operatorname{fdom}(\widetilde{A})$.

(b) *The series (with positive terms)* $\sum_{j=1}^{+\infty} \lambda_j \frac{|[f,\phi_j]|^2}{[\phi_j,\phi_j]}$ *converges in* $\mathbb{R}$.

(c) *The series*

$$\sum_{j=1}^{+\infty} \frac{[f,\phi_j]}{[\phi_j,\phi_j]} \begin{bmatrix} \phi_{j0}^{(k)} \\ N\mathbf{b}(\phi_{j0}) \end{bmatrix} \tag{5.2}$$

converges in the uniform norm in $\mathcal{C}[a,b] \oplus \mathbb{C}^d$ if $k = 0, 1, \ldots, n-1$ and in $L_2([a,b], p_n) \oplus \mathbb{C}^d$ if $k = n$.

Proof Denote by $\mathcal{L}_\infty$ the closed linear span of the eigenspaces corresponding to the eigenvalues λ_j, $j = 1, 2, \ldots$. We proved in Theorem 3.3 that there are only finitely many eigenvalues of $\widetilde{A}$ that are not included among λ_j, $j = 1, 2, \ldots$ and that their algebraic eigenspaces are finite dimensional. Denote by $\mathcal{L}_0$ the closed linear span of the eigenspaces corresponding to these eigenvalues. Both subspaces $\mathcal{L}_\infty$ and $\mathcal{L}_0$ are invariant under $\widetilde{A}$ and Theorem 3.3 implies that

$$L_2([a,b], r) \oplus \mathbb{C}^d_\Delta = \mathcal{L}_0 [\dotplus] \mathcal{L}_\infty, \tag{5.3}$$

a direct and orthogonal sum. Consequently, $(\mathcal{L}_\infty, [\cdot,\cdot])$ is a Krein space. Denote by P_0 and P_∞ the orthogonal projections corresponding to (5.3).

Denote by $\widetilde{A}_\infty$ the restriction of $\widetilde{A}$ onto $\mathcal{L}_\infty$. The operator $\widetilde{A}_\infty$ is uniformly positive in $(\mathcal{L}_\infty, [\cdot,\cdot])$ and $\sigma(\widetilde{A}_\infty) = \{\lambda_j, j = 1, 2, \ldots\}$. It follows from [8, Proposition 3.1] that $g \in \mathcal{L}_\infty$ belongs to $\operatorname{fdom}(\widetilde{A}_\infty)$ if and only if the series

$$\sum_{j=1}^{+\infty} \lambda_j \frac{|[g,\phi_j]|^2}{[\phi_j,\phi_j]} \tag{5.4}$$

converges. Clearly, $\mathcal{L}_0 \subset \operatorname{dom}(\widetilde{A})$. It was shown in the proof of [8, Theorem 3.4] that

$$\operatorname{fdom}(\widetilde{A}) = \mathcal{L}_0 [\dotplus] \operatorname{fdom}(\widetilde{A}_\infty). \tag{5.5}$$

Let f be an arbitrary vector in $L_2([a,b], r) \oplus \mathbb{C}^d_\Delta$. Then $f = h + g$ with $h = P_0 f \in \mathcal{L}_0$ and $g = P_\infty f \in \mathcal{L}_\infty$. The equality (5.5) implies that $f \in \operatorname{fdom}(\widetilde{A})$ if and only if $P_\infty f = g \in \operatorname{fdom}(\widetilde{A}_\infty)$, which is equivalent to the convergence of the series in (5.4). Since $h \in \mathcal{L}_0$ is orthogonal to each ϕ_j, $j = 1, 2, \ldots$ we have $[g,\phi_j] = [f,\phi_j]$, and consequently

$$\sum_{j=1}^{+\infty} \lambda_j \frac{|[g,\phi_j]|^2}{[\phi_j,\phi_j]} = \sum_{j=1}^{+\infty} \lambda_j \frac{|[f,\phi_j]|^2}{[\phi_j,\phi_j]}.$$

This equality, and the previously established equivalences, prove that (a) is equivalent to (b).

To prove the equivalence of (b) and (c), note that the series in (b) coincides term by term with the series

$$\sum_{j=1}^{+\infty} \frac{|[f,\phi_j]_{\widetilde{A}_\infty}|^2}{[\phi_j,\phi_j]_{\widetilde{A}_\infty}}.$$

Since the vectors $\phi_j/[\phi_j,\phi_j]_{\widetilde{A}_\infty}^{1/2}$ form an orthonormal basis of the Hilbert space $\big(\operatorname{fdom}(\widetilde{A}_\infty),[\cdot,\cdot]_{\widetilde{A}_\infty}\big)$ it follows that (b) is equivalent to convergence of the series

$$\sum_{j=1}^{+\infty} \frac{[f,\phi_j]}{[\phi_j,\phi_j]}\phi_j \tag{5.6}$$

in the Hilbert space $\big(\operatorname{fdom}(\widetilde{A}_\infty),[\cdot,\cdot]_{\widetilde{A}_\infty}\big)$. As $\widetilde{A}_\infty$ is uniformly positive in the Krein space $(\mathcal{L}_\infty,[\cdot,\cdot])$, the norm of the Hilbert space $\big(\operatorname{fdom}(\widetilde{A}_\infty),[\cdot,\cdot]_{\widetilde{A}_\infty}\big)$ is stronger than the norm of the original Krein space $L_2([a,b],r)\oplus\mathbb{C}^d_\Delta$, that is the norm of the Hilbert space $L_2([a,b],|r|)\oplus\mathbb{C}^d_{|\Delta|}$. Consequently, if we equip the finite dimensional space $\mathcal{L}_0$ with the norm of $L_2([a,b],|r|)\oplus\mathbb{C}^d_{|\Delta|}$ and $\operatorname{fdom}(\widetilde{A}_\infty)$ with the norm of the Hilbert space $\big(\operatorname{fdom}(\widetilde{A}_\infty),[\cdot,\cdot]_{\widetilde{A}_\infty}\big)$, then the direct sum $\|\cdot\|_s$ of these norms on

$$\operatorname{fdom}(\widetilde{A}) = \mathcal{L}_0[\dot{+}]\operatorname{fdom}\big(\widetilde{A}_\infty\big)$$

will be stronger than the norm on $L_2([a,b],|r|)\oplus\mathbb{C}^d_{|\Delta|}$. Reasoning in the same way as in the proof of Theorem 5.2, we see that the norm $\|\cdot\|_s$ is equivalent to $|||\cdot|||$ on $\operatorname{fdom}(\widetilde{A})$. Consequently, the norm of $[\cdot,\cdot]_{\widetilde{A}_\infty}$ is equivalent to $|||\cdot|||$ on $\operatorname{fdom}(\widetilde{A}_\infty)$. Therefore, (b) is equivalent to convergence of the series (5.6) in $|||\cdot|||$ and hence to (c). ■

Remark 5.5 If the conditions of Theorem 5.4 hold and if we put $g = \begin{bmatrix} g_0 & g_1\end{bmatrix}^T = P_\infty f$ (*i.e.*, the projection of f onto $\mathcal{L}_\infty$ of (5.3)), then the series in (5.2) converge to $\begin{bmatrix} g_0^{(k)} & g_1\end{bmatrix}^T$, in the corresponding topologies. In order to get expansions that will converge to $\begin{bmatrix} f_0^{(k)} & f_1\end{bmatrix}^T$, finitely many generalized eigenfunctions of $\widetilde{A}$ must be appended to (5.1).

It turns out that in the Hilbert space case Theorem 5.4 can be slightly strengthened and a simpler proof can be given. We state it as the following:

Theorem 5.6 *Assume that $\widetilde{\mathcal{H}} = L_2([a,b],r)\oplus\mathbb{C}^d_\Delta$ is a Hilbert space and let $\widetilde{A}=\widetilde{B}$ be the operator introduced in Theorem 4.2. Let λ_j, $j\in\mathbb{N}$ be eigenvalues of $\widetilde{B}$, each repeated according to its multiplicity, and let*

$$\phi_j = \begin{bmatrix} \phi_{j0} \\ N\mathbf{b}(\phi_{j0})\end{bmatrix}, \quad j\in\mathbb{N}, \tag{5.7}$$

be corresponding eigenfunctions of $\widetilde{B}$, orthogonal in $\widetilde{\mathcal{H}}$.

For an element $f = \left[\begin{smallmatrix} f_0 \\ f_1 \end{smallmatrix}\right] \in \widetilde{\mathcal{H}}$, the statements (a), (b) and (c) of Theorem 5.4 are equivalent.

If the conditions (a), (b) and (c) of Theorem 5.4 hold, the series in (5.2) *converge to $\left[\begin{smallmatrix} f_0^{(k)} \\ f_1 \end{smallmatrix}\right]$ in the corresponding topologies. In particular the series for f converges in $W_2^n \oplus \mathbb{C}^d$ if $1/p_n \in L_\infty[a, b]$.*

Proof Since the operator $\widetilde{B}$ is self-adjoint in the Hilbert space $\widetilde{\mathcal{H}}$ each $f \in \widetilde{\mathcal{H}}$ can be expanded in terms of the eigenfunctions (5.7) in the norm of $\widetilde{\mathcal{H}}$ and the series $\sum_{j=1}^{+\infty} \frac{|\langle f, \phi_j \rangle|^2}{\langle \phi_j, \phi_j \rangle}$ converges in $\mathbb{R}$. Also, $\operatorname{fdom}(\widetilde{B}) = \operatorname{fdom}(\widetilde{B} + \alpha \widetilde{I})$ for each $\alpha \in \mathbb{R}$. Therefore, without loss of generality, we can assume that $\widetilde{B}$ is uniformly positive.

The membership $f \in \operatorname{fdom}(\widetilde{B})$ is equivalent to $\widetilde{B}^{1/2} f \in \widetilde{\mathcal{H}}$, which is the same as

$$\sum_{j=1}^{+\infty} \lambda_j^{1/2} \frac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle} \phi_j = \sum_{j=1}^{+\infty} \lambda_j^{1/2} \frac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle^{1/2}} \frac{\phi_j}{\langle \phi_j, \phi_j \rangle^{1/2}} \in \widetilde{\mathcal{H}},$$

after we expand f in terms of the $\widetilde{\mathcal{H}}$-orthogonal basis (5.7). Since $\phi_j / \langle \phi_j, \phi_j \rangle^{1/2}$ form an orthonormal basis of $\widetilde{\mathcal{H}}$, the equivalence between (a) and (b) follows.

Since ϕ_j also form an orthogonal basis of $\mathcal{H}$, we can rewrite the above equivalence as

$$f = \sum_{j=1}^{+\infty} \frac{\langle f, \phi_j \rangle_{\widetilde{B}}}{\langle \phi_j, \phi_j \rangle_{\widetilde{B}}} \phi_j$$

with convergence in $\mathcal{H}$, that is,

$$\left\| f - \sum_{j=1}^{n} \frac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle} \phi_j \right\|_{\mathcal{H}} \to 0.$$

The equivalence with (c) now follows from Theorem 5.2.

The last statement follows from Remark 5.3. ∎

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